Keuzefuncties als onzekerheidsmodellen

Choice Functions as a Tool to Model Uncertainty

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PREFACE

I am pleased that you are about to read this dissertation. For me, writing this preface marks the end of an intense period of writing and doing research on choice functions. It was sometimes hard, but I have really enjoyed it. The past period feels like a journey, in which I gained experience about how to do research. I learned a lot about fascinating aspects of mathematics, and other interesting things.

The person who has taught me the most is without any doubt Gert de Cooman. He has been my supervisor since 'the beginning' in 2011, but he is also so much more than that. I cannot find words to express my gratitude—maybe "thank you very much" is the best summary of it—towards him. He succeeds in creating an environment that consists of people with a passion for research and mathematics. As a member of this environment, I feel constantly stimulated to actively do research, and work and think together at the blackboard. Furthermore, all this happens in a relaxed and friendly atmosphere. I am convinced that such an environment is rare, and I am very grateful to Gert for being the creator of it.

Gert took me with him to Oviedo to work together with Enrique Miranda. I returned three times on my own, for a cumulated period of 4 months. Enrique became my second supervisor. He took care of me, and did this in an exceptional way: he made me feel at home in Spain, was available every day to discuss (he even let me work in his own office) and every weekend he took me out for a trip, very often to the beautiful mountains of the wonderful region of Asturias. Enrique, you taught me the useful 'art' of constructing counterexamples, and how looking at examples might provide insight into the problem at hand. I enjoyed being and working with you immensely. Thank you very much! Also, I am very grateful to the members of his research group, in particular Ignacio, Susana D. and Susana M., for guiding me around in Asturias.

Gert and Enrique, I wish you time to fulfil your passion of doing research.

I have mentioned the environment already, and how important it is to me. The members of the SYSTeMS research group of Ghent University made my time in the office so much more enjoyable. I especially enjoy the open and kind atmosphere there, and I am grateful that I have been privileged to witness this since the start of my research back in 2011. One person has made the entire journey together with me: Jasper De Bock. We share many experiences, such as our first conference, and the road trips together with Gert. Thank you for always being available to discuss, for bringing life to the party, and for being such a good friend! There are two other close colleagues (or, actually, friends) that I would like to express my gratitude to: Stavros and Simon. They are always ready to assist with all kinds of problems. I enjoy especially our free moments in which we discuss important, or less important, subjects, or just have fun. When I started as a PhD student, Erik and Filip, two alumni members of the SYSTeMS research group, showed me around and helped me understand some aspects of imprecise probabilities. I will always remember Filip's relaxed character, and his willingness to discuss and teach me about credal networks. Later on, after they had left the SYSTeMS research group, Erik let me stay in his apartment in Amsterdam for three days, to explore together the connection between choice functions and sets of desirable gambles. Filip and Erik, thank you for being such kind and warm people. In a broader context, I would like to thank all the members of SIPTA, the Society for Imprecise Probability: Theories and Applications. I wish all of you the best in your future endeavours.

I also wish to thank the 'Gentse Roei- en Sportvereniging' and all its members, for providing me with a healthy hobby: rowing. I find it hard to explain why eight (nine with the coxswain) people are willing to train so hard every day in an attempt to win "the 8", but for some reason, everybody, including me, seems to do it with total devotion. Dear team mates, thank you for the nice and intense moments, and dear coaches and volunteers, thank you for making this possible. I wish that we will win "the 8" in many more years to come!

There are two more people that I would like to thank: Machteld en Falk, for checking parts of this dissertation for language mistakes, and for being the warm friends they are.

At last I would like to thank my parents and my brother, Felix, just for being so important to me.

I hope you enjoy your reading.

SUMMARY

This dissertation presents a study of choice functions as a tool to model uncertainty. Choice functions constitute a very general uncertainty model that is capable of representing a rich variety of types of beliefs. In particular, it includes as a special case *sets of desirable gambles*, the most general binary—or pairwise—choice model in imprecise probabilities [82]. So the field of this dissertation is 'imprecise probabilities', which is an umbrella term for mathematical models that are meant to be used in situations of imprecise or incomplete information, where it may not be possible (or advisable) to use (precise) probabilities. With our coherent choice functions, we want to be able to do everything that can be done within imprecise probabilities, such as conservative reasoning, updating and conditioning, and also to cope with structural judgements such as independence and exchangeability.

Suppose we have a subject whose beliefs we want to model. A direct way of doing this, is by looking at the choices he makes between uncertain options. Such choices can be captured by means of his choice function, which is a function that maps any set of options (also called *option set*) to the subset of those elements that are the chosen or preferred ones. Equivalently, we can consider rejection functions instead of choice functions, which return the rejected (non-chosen) options from within any given option set. Rejection functions are often easier and more intuitive to work with.

In order for a choice function to reflect rational behaviour, it needs to satisfy some rationality criteria. Choice functions that satisfy them are called *coherent*. The first thing we do in this dissertation, is identify a set of axioms that models rational behaviour, and is at the same time weak enough to allow for coherent choice functions to be sufficiently general and versatile. The set of rationality axioms we consider, is based on Kadane et al.'s [45] and Seidenfeld et al.'s [67] account of coherent choice functions. These authors introduce a theory of coherent choice functions that is capable of describing choices that are not necessarily determined by pairwise comparison. Our resulting theory differs from Seidenfeld et al.'s [67] in a number of respects. First of all, mainly for technical reasons, we only consider *finite* option sets, while Seidenfeld et al. [67] also allow for possibly infinite but closed option sets. Secondly, Seidenfeld et al. [67] require that their choice functions should

satisfy an Archimedean (continuity) axiom, which prevents them from having sets of desirable gambles as a special case. As we are particularly interested in the connection with sets of desirable gambles, we omit their Archimedean axiom. Furthermore, since their 'mixtures' or 'convexity' axiom is not compatible with Walley–Sen maximality [71, 82] as a decision rule, we disregard 'convexity' as an axiom but merely regard it as an additional property that choice functions may or may not satisfy. A final difference between Seidenfeld et al.'s [67] approach and ours lies in the uncertain options on which the choice functions are defined. While they use horse lotteries—which do not constitute a linear space—we use elements of an abstract pre-determined vector space as options.

As a first result, we show that under some mild conditions, the choice functions of Seidenfeld et al. [67] can be embedded into our framework. Since this connection is done through so-called *vector-valued gambles* [86], every result we prove for coherent choice functions on vector-valued gambles can also be related to choice functions on horse lotteries that satisfy the corresponding rationality axioms.

In order to be able to *reason conservatively* with choice functions, we introduce a partial order on them that has the interpretation of being 'at most as informative as'. The partially ordered set of all coherent choice functions forms a complete infimum-semilattice under this partial order: the infimum $\inf \mathcal{C}$ of any collection $\mathcal{C}$ of coherent choice functions exists, and is itself coherent. This is crucial: it is this property that allows for conservative reasoning with choice functions, at least in principle. For instance, it allows us to consider the important device of *natural extension*: the unique least informative coherent extension of a partially specified choice function. We characterise the circumstances under which the natural extension is coherent. If it is—we say then that the partially specified choice function *avoids complete rejection*—we obtain an explicit expression for it. The idea of natural extension has an important role in this dissertation.

Our motivation for using choice functions instead of sets of desirable gambles—which are those uncertain options that the subject strictly prefers to the status quo in a pairwise comparison—is that choice functions are versatile enough to represent choices that are not necessarily based on pairwise comparisons of the options only. We relate both models (choice functions and sets of desirable gambles) to one another through a compatibility relation: an option u belongs to a set of desirable gambles if and only if the status quo represented by 0 is rejected from the option set $\{0, u\}$ consisting of u and 0. Interestingly, given a coherent choice function, there is exactly one coherent set of desirable gambles that is compatible with it, but conversely, given a coherent set of desirable gambles, there are in general multiple compatible coherent choice functions. This shows that choice functions are indeed more general than sets of desirable gambles. We will illustrate this with a type of belief that can be modelled satisfactorily using choice functions, but not using sets of desirable gambles.

Seidenfeld et al.'s [67] set of rationality axioms for choice allows for a representation theorem of coherent choice functions: every coherent choice function can be written as an infimum of (choice functions corresponding to) probability–utility pairs belonging to an arbitrary set. This is a very powerful result, that immediately also shows that their maximal choice functions—the maximal elements of the partially ordered set of all coherent choice functions under the partial order of being 'at most as informative as'—are those that correspond to a single probability-utility pair. Our axiomatisation does not allow for such a representation: we show by means of a counterexample that even restricting ourselves to the class of coherent choice functions that also satisfy the additional 'convexity' property considered in [67], does not suffice.

Once the basic properties of our coherent choice functions, and the natural extension of local assessments are in place, we move to the study of structural assessments. By 'structural' assessment, we mean an assessment about the general properties or structure of the choice function. This contrasts with 'direct' or 'local' assessments, of which a partially specified choice function constitutes an example. The first kind of structural assessment we study is that of *indifference*. Consider a set of options I that the subject assesses as being indifferent to the option that represents the status quo. I is then called the subject's set of indifferent options. To be indifferent between two options means that these two options are considered to be equivalent to each other, in the sense that the subject is willing to swap one for another. We introduce a compatibility relation between coherent choice functions and sets of indifferent options, and identify the least informative coherent choice function that is compatible with a given set of indifferent options. As it turns out, choice functions that describe indifference are simpler, in the sense that they correspond to a unique representing choice function on a lower dimensional domain. We connect our account of indifference with an earlier characterisation by Seidenfeld [64].

As mentioned, with our theory of choice we want to be able to deal with *conditioning or updating*. If we have a choice function describing the subject's beliefs about an uncertain variable X, we need a procedure that takes new information such as 'X assumes a value in the non-empty set E' into account. Our proposed procedure for conditioning preserves coherence, and, due to the connection of our choice functions with sets of desirable gambles, it has no problems with conditioning on events of probability (in the set of probabilities associated with a choice function) zero. We take this one step further and consider a multivariate context, where we have a finite number of uncertain variables assuming values in non-empty finite possibility spaces. We introduce an obvious procedure for marginalisation—given a coherent choice function describing the subject's beliefs of a set of variables, how can we derive a choice function about a subset of them?—and some kind of inverse operation—given a coherent choice function describing the subject's beliefs of a set of variables, how can we extend it to a coherent choice function about a superset of vari-

ables, in a least informative manner? Our previously introduced conditioning rule can be made to deal with a multivariate context. Finally, we investigate another type of structural assessment: that of irrelevance, which is an asymmetric variant of independence. We characterise choice functions that satisfy such assessments, and find the least informative coherent one, called the *irrelevant natural extension*.

This dissertation culminates in our study of *exchangeability*, where we find occasion to use most of the previously introduced concepts. Exchangeability is a structural assessment about a sequence of variables that is important for statistical inference purposes. Loosely speaking, making a judgement of exchangeability means that the order in which the variables are observed is not relevant. We derive de Finetti-like representation theorems for finite exchangeable sequences. We take also this one step further and consider a countable sequence of variables, for which we also find a de Finetti-like representation theorem. We consider conditioning on observing a finite number of these variables, show that this conserves exchangeability, and show that counting occurrences provides a sufficient statistic for this type of inference.

We conclude by looking back at what has been achieved, and looking ahead at a number of interesting problems that still remain.

SAMENVATTING

Dutch summary

Dit proefschrift legt een studie voor van keuzefuncties als een manier om onzekerheid te modelleren. Keuzefuncties vormen een zeer algemeen onzekerheidsmodel dat ons in staat stelt om een rijke verscheidenheid aan types van overtuigingen te modelleren. Ze omvatten, als bijzonder geval, *verzamelingen van begeerlijke gokken*, het meest algemene binaire—of paarsgewijze—keuzemodel in de imprecieze waarschijnlijkheden [82]. Dit proefschrift situeert zich dus in het veld van de 'imprecieze waarschijnlijkheden', wat een kapstokbenaming is voor wiskundige modellen die kunnen worden gebruikt bij imprecieze of onvolledige informatie, waar het niet mogelijk (of raadzaam) is om (precieze) waarschijnlijkheden te gebruiken. We willen met onze coherente keuzefuncties alles kunnen doen wat ook met precieze en imprecieze waarschijnlijkheden kan, zoals conservatief redeneren, conditioneren en updaten, en ook omgaan met structurele aannames zoals onafhankelijkheid en uitwisselbaarheid.

Neem aan dat we iemands onzekerheid willen modelleren. Een directe aanpak bestaat erin te kijken naar de keuzes die hij maakt tussen onzekere opties. Zulke keuzes kunnen worden vastgelegd door zijn keuzefunctie, die een functie is die elke verzameling van opties (ook wel optieverzameling genoemd) afbeeldt op de deelverzameling die bestaat uit die elementen die verkozen of geprefereerd zijn. Equivalent hiermee kunnen we ook verwerpingsfuncties beschouwen in plaats van keuzefuncties, die de verworpen (niet-verkozen) opties weergeven in elke optieverzameling. Verwerpingsfuncties zijn vaak makkelijker en intuïtiever om ermee te werken.

Opdat een keuzefunctie rationeel gedrag zou weerspiegelen, moet het aan een aantal rationaliteitscriteria voldoen. Keuzefuncties die hieraan voldoen noemen we *coherent*. Het eerste wat we in dit proefschrift doen, is een verzameling van axioma's identificeren die rationeel gedrag modelleren, en tegelijkertijd zwak genoeg zijn om het werken met keuzefuncties voldoende algemeen en veelzijdig te maken. De verzameling van rationaliteitscriteria die we beschouwen is gebaseerd op de beschijving van coherente keuzefuncties van Kadane et al. [45] en Seidenfeld et al. [67] . Deze auteurs introduceren een theorie van coherente keuzefuncties die in staat is om keuzes te modelleren die niet noodzakelijk bepaald worden door paarsgewijze vergelijkingen. De theorie die wij bouwen, verschilt in een aantal aspecten van die van Seidenfeld et al. [67]. Om te beginnen beschouwen we enkel *eindige* optieverzamelingen, vooral omwille van technische redenen, terwijl Seidenfeld et al. [67] ook oneindige maar gesloten optieverzamelingen toelaten. Ten tweede eisen Seidenfeld et al. [67] dat hun keuzefuncties aan een Archimedisch (continuïteits-)axioma voldoen, wat maakt dat hun keuzefuncties verzamelingen van begeerlijke gokken niet als een bijzonder geval omvatten. Omdat wij bijzonder geïnteresseerd zijn in het verband tussen keuzefuncties en verzamelingen van begeerlijke gokken, laten we hun Archimedisch axioma in dit proefschrift achterwege. Bovendien is hun 'mengings-' of 'convexiteitsaxioma' niet verenigbaar met Walley–Sen-maximaliteit [71, 82] als beslissingsregel, en daarom behandelen we deze 'convexiteit' niet als een axioma, maar liever als een bijkomende eigenschap waaraan keuzefuncties al dan niet kunnen voldoen. Een laatste verschil tussen de aanpak van Seidenfeld et al. [67] en de onze ligt in de onzekere opties waartussen de keuzefuncties kiezen. Terwijl zij paardenloterijen—die geen lineaire ruimte vormen—gebruiken, werken wij met elementen van een abstracte vooraf bepaalde vectorruimte als opties.

Als eerste resultaat tonen we aan dat, onder zwakke voorwaarden, de keuzefuncties van Seidenfeld et al. [67] kunnen ingebed worden in ons kader. Omdat deze inbedding gebeurt met behulp van zogeheten *vectorwaardige gokken* [86], kan elk resultaat dat wij bewijzen voor coherente keuzefuncties op vectorwaardige gokken ook gerelateerd worden aan keuzefuncties op paardenloterijen die aan de overeenkomstige rationaliteitscriteria voldoen.

Om *conservatief te kunnen redeneren* met keuzefuncties introduceren we een partiële ordening op de keuzefuncties, die we kunnen interpreteren als 'is ten hoogste zo informatief als'. De partieel geordende verzameling van alle coherente keuzefuncties vormt een complete infimum-semitralie onder deze partiële ordening: het infimum $\inf \mathcal{C}$ van elke verzameling $\mathcal{C}$ van coherente keuzefuncties bestaat, en is zelf weer coherent. Dat is cruciaal: het is deze eigenschap die ons in staat stelt, tenminste in principe, conservatief te redeneren met keuzefuncties. Het maakt het ons bijvoorbeeld mogelijk om het belangrijke concept van *natuurlijke uitbreiding* te beschouwen: de unieke minst informatieve coherente uitbreiding van een onvolledig gespecificeerde keuzefunctie. We karakteriseren de omstandigheden waaronder die natuurlijke uitbreiding coherent is. Als ze dat is—we zeggen dan dat de onvolledig gespecificeerde keuzefunctie *volledige verwerping vermijdt*—leiden we er een expliciete uitdrukking voor af. Het idee van natuurlijke uitbreiding speelt in dit proefschrift een belangrijke rol.

Onze motivering om keuzefuncties te gebruiken in plaats van verzamelingen van begeerlijke gokken—begeerlijke gokken zijn die gokken die door een persoon strikt verkozen worden boven het status quo in een paarsgewijze vergelijking—is dat keuzefuncties veelzijdig genoeg zijn om keuzes voor te stellen die niet noodzakelijk alleen maar gegrond zijn op paarsgewijze vergelijkingen tussen de opties. We brengen de beide modellen (keuzefuncties en verzamelingen van begeerlijke gokken) met elkaar in verband met behulp van een verenigbaarheidsrelatie: een optie u behoort tot een verzameling van begeerlijke gokken als en slechts als het status quo, voorgesteld door 0, verworpen wordt in de optieverzameling $\{0, u\}$ die bestaat uit u en 0. Voor een coherente keuzefunctie is er exact één coherente verzameling van begeerlijke gokken die ermee verenigbaar is, maar omgekeerd zijn er voor een coherente verzameling van begeerlijke gokken in het algemeen meerdere verenigbare coherente keuzefuncties. Dat toont aan dat keuzefuncties inderdaad algemener zijn dan verzamelingen van begeerlijke gokken.

De rationaliteitscriteria voor keuze van Seidenfeld et al. [67] stellen hun in staat om een representatieresultaat voor hun coherente keuzefuncties te bewijzen: elke coherente keuzefunctie kan geschreven worden als een infimum van (keuzefuncties overeenstemmend met) waarschijnlijkheids-utiliteitsparen. Dit is een zeer krachtig resultaat, dat ook onmiddellijk aantoont dat hun maximale keuzefuncties—de maximale elementen van de partieel geordende verzameling van alle coherente keuzefuncties onder de partiële ordening die 'ten hoogste zo informatief als' voorstelt—die keuzefuncties zijn die overeenstemmen met een enkel waarschijnlijkheids-utiliteitspaar. Onze axiomatisering laat zulke representatie niet toe: we tonen door middel van een tegenvoorbeeld aan dat er geen zulke representatie is, zelfs als we ons beperken tot de klasse van coherente keuzefuncties die ook aan de bijkomende 'convexiteitseigenschap' voldoen, beschouwd in Referentie [67].

Als eenmaal de basiseigenschappen van onze coherente keuzefuncties, en de natuurlijke uitbreiding van lokale aannames, vastgelegd zijn, gaan we verder met de studie van structurele aannames. Met 'structurele' aannames bedoelen we aannames over de algemene eigenschappen of structuur van de keuzefunctie. Ze staan tegenover 'directe' of 'lokale' aannames, waarvan onvolledig gespecificeerde keuzefuncties een voorbeeld zijn. De eerste soort van structurele aanname die we bestuderen is die van *onverschilligheid*. Beschouw een verzameling van opties I die een subject beoordeelt als onverschillend van het status quo. We noemen I dan zijn verzameling van onverschillige opties. Onverschillig zijn tussen twee opties betekent dat deze twee opties worden beschouwd als equivalent met elkaar, in de zin dat ons subject een van de twee opties wil ruilen voor de andere. We stellen een verenigbaarheidsrelatie tussen coherente keuzefuncties en verzamelingen van onverschillige opties voor, en identificeren de minst informatieve coherente keuzefunctie die verenigbaar is met een gegeven verzameling van onverschillige opties. Het blijkt dat keuzefuncties die onverschilligheid uitdrukken eenvoudiger zijn, in de zin dat ze overeenstemmen met een unieke representerende keuzefunctie op een domein met een lagere dimensie. We leggen een verband tussen onze aanpak van indifferentie en een eerdere karakterisatie door Seidenfeld [64].

Zoals gezegd willen we met onze theorie van keuzefuncties kunnen omgaan met *conditioneren of updaten*. Als we een keuzefunctie hebben die de overtuigingen van een subject over een onzekere veranderlijke X modelleert, dan moeten we een methode hebben die nieuwe informatie zoals 'X neemt een waarde aan in de niet-lege verzameling E' in rekening brengt. De methode die we voorstellen bewaart coherentie, en, door het verband tussen onze keuzefuncties en verzamelingen van begeerlijke gokken, heeft ze geen problemen met conditioneren op gebeurtenissen met waarschijnlijkheid (in de verzameling van waarschijnlijkheden geassocieerd met een keuzefunctie) nul. We nemen deze ideeën mee naar een multivariate context, waar we een eindig aantal onzekere veranderlijken hebben die waarden aannemen in niet-lege mogelijkhedenruimtes. We stellen een voor de hand liggende methode om te marginaliseren voor: gegeven een coherente keuzefunctie die de overtuigingen van een subject beschrijft over een verzameling van veranderlijken, hoe kunnen we dan een keuzefunctie afleiden over een deelverzameling? Daarnaast bekijken we ook een soort inverse operatie: gegeven een keuzefunctie die de overtuigingen van een subject beschrijft over een verzameling van veranderlijken, hoe kunnen we die dan uitbreiden naar een coherente keuzefunctie over een grotere verzameling van veranderlijken, op een zo weinig informatief mogelijke manier? Onze eerder voorgestelde methode om te conditioneren kunnen we ook gebruiken in zo'n context met meerdere veranderlijken. Ten slotte onderzoeken we een ander type structurele aanname: irrelevantie, een asymmetrische variant van onafhankelijkheid. We karakteriseren keuzefuncties die aan deze aanname voldoen, en we vinden de minst informatieve onder hen: de *irrelevante natuurlijke uitbreiding*.

Dit proefschrift culmineert in onze studie van uitwisselbaarheid, waarbij we de meeste van de voorheen voorgestelde concepten kunnen gebruiken. uitwisselbaarheid is een structurele aanname over een rij van veranderlijken, die belangrijk is voor statistische gevolgtrekkingen. Een aanname van uitwisselbaarheid te maken betekent ruwweg dat de volgorde waarin we de veranderlijken observeren niet relevant is. We leiden de Finetti-achtige representatieresultaten af voor eindige uitwisselbare rijen van veranderlijken die waarden aannemen in een eindige verzameling van categorieën. We nemen deze ideeën mee naar een bredere context, waarbij we een aftelbare rij van veranderlijken beschouwen, waarvoor we ook een de Finetti-achtig representatieresultaat vinden. We bekijken ook conditioneren op het waarnemen van een eindig aantal veranderlijken, tonen aan dat het proces van conditioneren uitwisselbaarheid bewaart, en bewijzen dat het tellen van hoeveel keer elke categorie optreedt een toereikende statistiek oplevert voor dit type van gevolgtrekkingen.

We besluiten dit proefschrift door terug te kijken naar wat we hebben bereikt, en vooruit te kijken naar een aantal interessante open problemen.

List of symbols and terminology

This list of symbols is ordered per topic. We provide a short description, and a reference to the page, section, definition, proposition or theorem where they are first introduced. For symbols that have many variations, we list a generic version. Symbols that are only used locally are not included in this list.

Number sets

Symbol	Meaning	Location
$\mathbb{R}$	Set of real numbers	Page 9
$\mathbb{R}_{>0}$	Set of positive real numbers	Page 9
$\mathbb{R}_{\geq 0}$	Set of non-negative real numbers	Page 9
$\mathbb{N}$	Set of natural numbers: $\{1,2,\dots\}$	Page 10
$\mathbb{Z}_{\geq 0}$	Set of natural numbers with zero: $\mathbb{N}\cup\{0\}$	Page 10

Option space

Symbol	Meaning	Location
$\mathcal{V}$	Linear space of all options	Section 2.1$_{9}$
u, v, w	Option: element of $\mathcal{V}$	Section 2.1$_{9}$
$\operatorname{span}(A)$	Linear hull of the set A	Section 2.1$_{9}$
$\operatorname{posi}(A)$	Positive hull of the set A	Section 2.1$_{9}$
$\operatorname{conv}(A)$	Convex hull of the set A	Section 2.1$_{9}$
$\leq$	Vector ordering	Section 2.1$_{9}$
$\prec$	Irreflexive part of $\leq$	Section 2.1$_{9}$
$\mathcal{V}_{\geq 0}$	Options u in $\mathcal{V}$ such that $0 \leq u$	Section 2.1$_{9}$
$\mathcal{V}_{\succ 0}$	Options u in $\mathcal{V}$ such that $0 \prec u$	Section 2.1$_{9}$

X	Uncertain variable	Section 2.1$_{9}$
$\mathcal{X}$	Possibility space or state part of the domain	Section 2.1$_{9}$
x, y, z	Generic element of $\mathcal{X}$	Section 2.1$_{9}$
E	Event: subset of $\mathcal{X}$	Section 2.1$_{9}$
f, g, h	Gamble or vector-valued gamble	Section 2.1$_{9}$
$\mathcal{L}(\mathcal{X})$, $\mathcal{L}$	Set of all gambles (on $\mathcal{X}$)	Section 2.1$_{9}$
$\mathbb{I}_E$	Indicator of E	Section 2.1$_{9}$
$\mathcal{R}$	Reward set or reward part of the domain	Section 2.1$_{9}$
$\mathcal{P}(\mathcal{V})$	Power set of $\mathcal{V}$: the set of all the subsets of $\mathcal{V}$	Section 2.2$_{14}$
A, A'	Option set: non-empty finite subset of $\mathcal{V}$	Section 2.2$_{14}$
$\mathcal{Q}(\mathcal{V}), \mathcal{Q}$	Set of all option sets: the set of all non-empty finite subsets of $\mathcal{V}$	Section 2.2$_{14}$
$\mathcal{Q}_0(\mathcal{V}), \mathcal{Q}_0$	Set of all option sets that include 0	Section 2.2$_{14}$
$\mathcal{Q}_{\overline{0}}(\mathcal{V}), \mathcal{Q}_{\overline{0}}$	Set of all option sets that exclude 0	Section 2.2$_{14}$
$\mathcal{L}(\mathcal{X}\times\mathcal{R})$	Set of all vector-valued gambles on $\mathcal{X}$	Definition 9$_{28}$
H	Horse lottery	Definition 10$_{28}$
$\mathcal{H}(\mathcal{X},\mathcal{R})$, $\mathcal{H}$	Set of all horse lotteries	Definition 10$_{28}$
$\preccurlyeq$	Ordering of option sets	Definition 13$_{43}$
$\Sigma_{\mathcal{X}}$	Unit simplex in $\mathbb{R}^{\mathcal{X}}$	Page 73
$\mathrm{id}_{\mathcal{A}}$	Identity map on $\mathcal{A}$	Page 19

UNCERTAINTY MODELS

Symbol	Meaning	Location
C	Choice function	Definition 1$_{14}$
R	Rejection function	Definition 2$_{14}$
$\lhd$	Choice relation	Definition 3$_{15}$
$\mathbf{C}(\mathcal{V})$, $\mathbf{C}$	Set of all choice functions (on $\mathcal{V}$)	Definition 1$_{14}$
$\mathbf{R}(\mathcal{V})$, $\mathbf{R}$	Set of all rejection functions (on $\mathcal{V}$)	Definition 2$_{14}$
$\mathbf{S}(\mathcal{V})$, $\mathbf{S}$	Set of all choice relations (on $\mathcal{V}$)	Definition 3$_{15}$
$\overline{\mathbf{C}}(\mathcal{V})$, $\overline{\mathbf{C}}$	Set of all coherent choice functions (on $\mathcal{V}$)	Definition 6$_{20}$
$\overline{\mathbf{R}}(\mathcal{V})$, $\overline{\mathbf{R}}$	Set of all coherent rejection functions (on $\mathcal{V}$)	Definition 7$_{20}$

$\overline{\mathbf{S}}(\mathcal{V}), \overline{\mathbf{S}}$	Set of all coherent choice relations (on $\mathcal{V}$)	Definition 8_{20}
$\sqsubseteq$	At most as informative as	Definition 14_{46}
$\hat{\mathbf{C}}$	Set of maximal coherent choice functions	Section 2.6_{46}
$\hat{\mathbf{R}}$	Set of maximal coherent rejection functions	Section 2.6_{46}
C_{v}	The vacuous choice function	Section 2.6_{46}
R_{v}	The vacuous rejection function	Section 2.6_{46}
$\triangleleft_{\mathrm{v}}$	The vacuous choice relation	Section 2.6_{46}
D	Set of desirable options	Definition 19_{56}
$\prec$	Preference relation	Page 57
$\mathbf{D}(\mathcal{V}), \mathbf{D}$	Set of all sets of desirable options	Definition 19_{56}
$\overline{\mathbf{D}}(\mathcal{V}), \overline{\mathbf{D}}$	Set of all coherent sets of desirable options	Definition 20_{57}
$\overline{\mathbf{P}}(\mathcal{V}), \overline{\mathbf{P}}$	Set of all coherent preference relations	Definition 21_{57}
$\hat{\mathbf{D}}(\mathcal{V}), \hat{\mathbf{D}}$	Set of all maximal coherent sets of desirable options	Page 59
D_{v}	The vacuous set of desirable options	Proposition 50_{58}
D_C	Set of desirable options compatible with C	Proposition 53_{61}
C_D	Least informative choice function compatible with D	Proposition 54_{62}
$\underline{P}$	Lower prevision	Page 71
p	Probability mass function: element of $\Sigma_{\mathcal{X}}$	Page 73
E	Linear prevision	Page 73
$\mathbb{P}_{\mathcal{X}}$	Set of all linear previsions on $\mathcal{L}(\mathcal{X})$	Page 73
$\mathcal{K}$	Set of linear previsions	Page 74
$\mathcal{M}$	Set of probability mass functions	Page 74
$\mathbb{K}$	Rejection set	Definition 26_{76}
C^{M}	M-admissible choice function	Definition 27_{82}
C^{E}	E-admissible choice function	Definition 28_{84}
K	Coordinate rejection set	Definition 36_{148}
$D\rfloor E$	Set of desirable gambles conditional on E	Section 6.1_{206}
$\mathbb{I}_E f$	Vector-valued gamble that is equal to f on $E \times \mathcal{R}$ and to 0 outside $E \times \mathcal{R}$	Section 6.1_{206}
$\prec\rfloor E$	Preference relation conditional on E	Section 6.1_{206}
$C\rfloor E$	Choice function conditional on E	Definition 44_{208}
$R\rfloor E$	Rejection function conditional on E	Page 210
$\triangleleft\rfloor E$	Choice relation conditional on E	Definition 45_{210}

LEXICOGRAPHIC MODELS

Symbol	Meaning	Location
$\overline{\mathbf{D}}_{\mathrm{L}}$	Set of lexicographic sets of desirable gambles	Definition 33_{128}
$\overline{\mathbf{C}}_{\mathrm{L}}$	Set of lexicographic choice functions	Section 4.3_{143}
$<_{\mathrm{L}}$	Lexicographic ordering	Page 130
$(p_1,\dots,p_\ell)$, p	Lexicographic probability system	Definition 34_{130}
$<_p$	Preference relation based on lexicographic probability system p	Definition 34_{130}

INDIFFERENCE AND SYMMETRY

Symbol	Meaning	Location
I	Set of indifferent options	Section 5.1_{175}
$\mathbf{I}$	Set of all sets of indifferent options	Definition 38_{176}
$[u]$	Equivalence class of option u: $[u] = \{u\} + I$	Section 5.2_{177}
$\mathcal{V}/I$	Quotient space: set of all equivalence classes	Section 5.2_{177}
A/I	Option set of equivalence classes $[u]$ associated with the options u in the option set A	Section 5.2_{177}
D/I	Representing set of desirable options	Section 5.3_{178}
C/I	Representing choice function	Section 5.4_{179}
π	Permutation of an index set	Section 5.8_{191}
$\mathcal{P}$	Set of all permutations	Section 5.8_{191}
π^t	Permutation lifted to the option space of vector-valued gambles	Section 5.8_{191}
$[x]_{\mathcal{P}}$	Permutation invariant atom	Section 5.8_{191}

DIFFERENT TYPES OF NATURAL EXTENSIONS

Symbol	Meaning	Location
$\mathcal{B}$	Direct assessment: subset of $\mathcal{Q}_0$	Section 3.1_{90}
$\mathcal{E}(\mathcal{B})$	Natural extension of the direct assessment $\mathcal{B}$	Definition 31_{91}
$R_{\mathcal{B}}$	Rejection function based on $\mathcal{B}$	Equation $(3.1)_{92}$

$\mathcal{E}^{\mathbf{D}}(B)$	Natural extension for desirability of the direct binary assessment B	Theorem 85_{100}
$\mathcal{E}_I(\mathcal{B})$	Natural extension of $\mathcal{B}$ under the set of indifferent options I	Definition 41_{194}
$R_{\mathcal{B},I}$	Rejection function based on $\mathcal{B}$ and I	Equation $(5.6)_{194}$
$\mathcal{E}_I^{\mathbf{D}}(B)$	Natural extension for desirability of B under indifference	Theorem 145_{201}
$\mathrm{ext}_{1:n}(R_O)$	Weak extension of the rejection function R_O	Proposition 161_{227}
$\mathrm{ext}_{1:n}^{\mathbf{D}}(D_O)$	Weak extension of the set of desirable gambles D_O	Equation $(7.5)_{230}$
$\mathcal{B}_{R_O}^{I\to O}$	Assessment of epistemic irrelevance	Page 237
$\mathcal{E}_{\mathrm{ex}}^n(\mathcal{B})$	Exchangeable natural extension	Equation $(8.9)_{255}$

EXCHANGEABILITY

Symbol	Meaning	Location
$\mathcal{P}_n$	Set of all permutations of $\{1,\dots,n\}$	Section 8.1_{247}
$I_{\mathcal{P}_n}$	Set of indifferent options corresponding to finite exchangeability	Section 8.1_{247}
T	Counting map	Section 8.1_{247}
$T(x)$, m	Count vector (of x)	Section 8.1_{247}
$\mathcal{N}^n$	Set of all count vectors	Section 8.1_{247}
H_n	Linear map, connected with the hyper-geometric distribution	Section 8.1_{247}
$\tilde{\mathrm{H}}_n$	Variant of H_n that is defined on $\mathcal{L}(\mathcal{X}^n\times\mathcal{R})/I_{\mathcal{P}_n}$	Equation $(8.5)_{251}$
$\check{x}$	Observed sample	Page 257
$\mathcal{V}(\Sigma_{\mathcal{X}}\times\mathcal{R})$	Polynomial vector-valued gambles	Page 259
$\mathcal{V}^n(\Sigma_{\mathcal{X}}\times\mathcal{R})$	Polynomial vector-valued gambles whose degree is not higher than n	Page 259
B_m	Bernstein gamble	Definition 50_{259}
M_n	Linear map, connected with the multinomial distribution	Page 260

$\tilde{\mathrm{M}}_n$	Variant of M_n that is defined on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$	Equation (8.10)$_{260}$
$\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$	Gambles of finite structure	Definition 51$_{264}$
$I_{\mathcal{P}}$	Set of indifferent options corresponding to countable exchangeability	Section 8.2$_{263}$

1

INTRODUCTION

1.1 WHAT IS THIS DISSERTATION ABOUT

This dissertation is about *choice between uncertain options*. We want our notion of 'choice', and our theory resulting from it, to be sufficiently general, so that it (i) allows for 'imprecise choice', and (ii) is capable of describing choices that are not necessarily determined by pairwise comparisons of the options.

Let us first explain what we mean by 'imprecise choice'. In general, very often there will be multiple chosen (or preferred) options amongst the set of available options: the subject (or expert) whose choices we model is undecided between these chosen options, and we then say that these options are incomparable to him. With 'imprecise choice', we mean that the subject *does not need to be indifferent between the incomparable options*—to be indifferent between two options means that these two options are considered equivalent to each other, in the sense that the subject is equally willing to swap one for another. If some options are indifferent to the subject, this implies that they are incomparable to him in the technical sense described above, but the converse implication does not necessarily hold. In other words, with 'imprecise choice' we mean that the subject can have *multiple* reasons to be undecided between incomparable options: one of them is that he is indifferent between them, but there might be other reasons. Indeed, the indecisiveness may for instance be due to incomplete information (for instance, due to lack of time or resources to gather more information), or due to cautiousness of the subject towards specifying too specific choices, etcetera.

One theory that takes the difference between indecisiveness and indifference seriously, is that of *imprecise probabilities* [82]. 'Imprecise probabilities' is an umbrella term for mathematical models that are meant to be used in situations of imprecise or incomplete information, where it may not be possible (or advisable) to use (precise) probabilities. In particular, it covers sets of probability measures and various types of non-additive measures and functionals, such as coherent lower previsions [51], belief functions [35, 70] and possibility measures [12, 22]. All of these models can be expressed in terms of coherent sets of desirable gambles [57, 64, 82, 83], which collect the gambles that a subject, whose beliefs we want to model, strictly prefers to the status quo. Sets of desirable gambles are not only the most general imprecise-probabilistic binary choice model, they constitute moreover arguably the most elegant model to work with, and definitively the model with the clearest and most direct interpretation. They can be—and have been—used to replace probabilities in Bayesian networks, for predictive inference, and so on [13, 18, 19, 24, 31, 55].

Often, choice cannot be reduced to pairwise (or binary) comparison of the available options. To make this more specific, for three options u, v and w, it can happen that u is chosen in the pairwise comparisons with v—so u is chosen from within $\{u,v\}$—and with w—so u is chosen from within $\{u,w\}$—but, when considering all three options u, v and w, that u is rejected (not chosen). An illustration of this will be given in Example 10_{85}. Choices that are not necessarily determined by pairwise comparisons of the options can be modelled using *choice functions*: functions that map any set of options (we will call a set of options an *option set*) to a subset whose elements are the chosen or preferred options. Choice functions are related to the fundamental problem in decision theory: how to make a choice from within a set of available options. In their book, von Neumann and Morgenstern [81] provide an axiomatisation of choice based on a pairwise comparison between options only. Later on, as choice functions gained popularity, many authors, such as Arrow [4], Uzuwa [73] and Rubin [59], generalised the idea and proposed a theory of choice functions based on choice between more than two elements.

We have seen that the theory of imprecise probabilities allows for imprecise choice, and that the theory of choice functions allows for the choice to be not necessarily determined by pairwise comparisons of the options only. But, on the one hand, most of the imprecise-probabilistic models, such as the very useful model of sets of desirable gambles, reduce to pairwise comparisons between the options. On the other hand, most of the choice functions, such as those in Rubin's [59] theory, are precise—do not distinguish between indifference and incomparability. We want to combine the advantages of both theories (imprecise probabilities and choice functions): we want a theory of choice functions that has imprecise probabilities, and more specifically sets of desirable gambles, as a particular case. In order to get there, some of the axioms of the choice functions that Rubin [59] considers must be weakened. Furthermore, we want our resulting theory of choice functions to have a clear interpretation and to be operational.

There already exists at least one such theory of choice functions: Kadane et al. [45] and Seidenfeld et al. [67] generalise Rubin's [59] axioms to allow for incomparability. They introduce an axiomatisation of choice that is weaker than Rubin's [59] and allows for imprecise choice. However, sets of desirable gambles are no particular case of their choice functions, something that we consider to be important for our choice functions. We will allow ourselves to be inspired by their axiomatisation, and will drop those axioms that prevent sets of desirable gambles to be a particular case.

One important difference between our approach and the approach considered by Seidenfeld et al. [67] is that they use *horse lotteries* to represent options, while we use abstract vectors that belong to some pre-determined vector space. As the horse lotteries do not form a linear space, at first sight this seems a crucial difference. However, as we will see in Chapter 2_9, we can embed the choice functions considered by Seidenfeld et al. [67] into our framework, under some mild conditions. The advantage of our use of more abstract vectors is that they are useful for dealing with indifference in a more directly constructive way than Seidenfeld's [63] treatment, as we will see in Chapter 5_{175}. Furthermore, since our options belong to a vector space, they can be added to each other, and multiplied with constants, which is something that will turn out to be very useful.

All these considerations lead to a theory of choice, presented in this dissertation, that has sets of desirable gambles as a special case. We impose a system of four rationality axioms—based on those of Seidenfeld et al. [67]—on the choice functions, that is weak enough in order to have desirability as a special case. We call choice functions that satisfy the rationality axioms *coherent*. With our coherent choice functions, we want to be able to do everything that can be done with imprecise-probabilistic models, such as conservative reasoning, updating and conditioning, and coping with structural judgements such as independence and exchangeability.

To be able to do *conservative reasoning*, we impose a specific partial order on the set of all coherent choice functions, having the interpretation of being 'at most as informative as'. With this partial order, we are able to distinguish more informative choice functions from less informative ones, and (partially) order them accordingly. As we will see, the partially ordered set of all coherent choice functions under this partial order, forms a complete infimum-semilattice: the infimum $\inf \mathcal{C}$ (under this partial order) of any collection $\mathcal{C}$ of coherent choice functions exists, and is coherent itself. This is crucial: using these concepts allows us to reason conservatively, at least in principle. Given a partially specified choice function, we collect all the coherent choice functions that are compatible with it in $\mathcal{C}$, and $\inf \mathcal{C}$ will be the least informative coherent choice function that is compatible with the partially specified choice function we started with. This choice function $\inf \mathcal{C}$ is important throughout this dissertation: it is the unique coherent extension of the partially specified choice function that uses the *rationality axioms only*. It is called the *natural extension* of the partially specified choice function, which is the counterpart of the deductive closure of classical propositional logic. Furthermore, we need to have an expression for $\inf \mathcal{C}$, as this makes our theory *operational*. The operational aspect of our theory exists in offering a number of option sets to the subject. In every option set offered, he then can make statements regarding the options he rejects (does not find preferable). This corresponds to partially specifying a choice function, and with our expression for $\inf \mathcal{C}$, we are then able to find its natural extension.

As mentioned, with our theory of choice we want to be able to deal with *conditioning or updating*. Assume that we have a choice function describing the subject's beliefs about an uncertain variable X, and suppose that new information, in the form of 'X assumes a value in the non-empty set E' becomes available. This new information can be taken into account by conditioning the initial choice function.

An incompletely specified choice function is an example of a direct or local assessment: it only requires that the choice function that describes the choices (or preferences) of the subject, identifies certain choices from within certain sets of options. Besides these direct assessments, our theory of coherent choice functions also needs to be able to cope with *structural* assessments: judgements that a subject makes about global properties of the choice function. Such structural judgements can for instance be an assessment of indifference, looking like 'I am indifferent between the options in some set I'. Special cases of such assessments are symmetry and exchangeability. Other examples of structural judgements are related to the irrelevance or independence of two or more variables. It turns out that our account of choice functions is able to deal with local and some types of structural assessments. At a later stage, not treated in this dissertation because outside of its scope, this should allow us to lay the foundation for a theory of statistical inference with choice functions.

1.2 INFORMATION ABOUT REFERENCES

In this dissertation you will find both external and internal references. External references are bibliographic ones. They are enumerated, and listed in the Bibliography$_{277}$. Internal references are used for chapters, sections, equations, theorems, propositions, corollaries, lemmas and remarks. They have an index which refers to the page number where the reference can be found. For instance, Theorem 81$_{97}$ can be found on page 97. The explicit reference to the page number is omitted when we refer to something on the same double-page spread, and recto and verso pages are referenced by the symbols ↜ and ↝, respectively. I would like to thank Gert de Cooman and Matthias Troffaes for providing me with the LaTeX code for the internal reference system used in their book [72]. The idea of this goes back to Erik Quaeghebeur's dissertation [56].

1.3 OVERVIEW OF THE CHAPTERS

This dissertation consists of 9 chapters. Apart from this introduction and the conclusions in Chapter 9_{273}, the main results can be found in Chapters 2_9–8_{247}, of which we give a concise overview here. The DAG in Figure 1.1 shows the relation between the chapters. An arrow departing in Chapter i and arriving in Chapter j means that Chapter j uses concepts introduced in Chapter i or its ancestors.

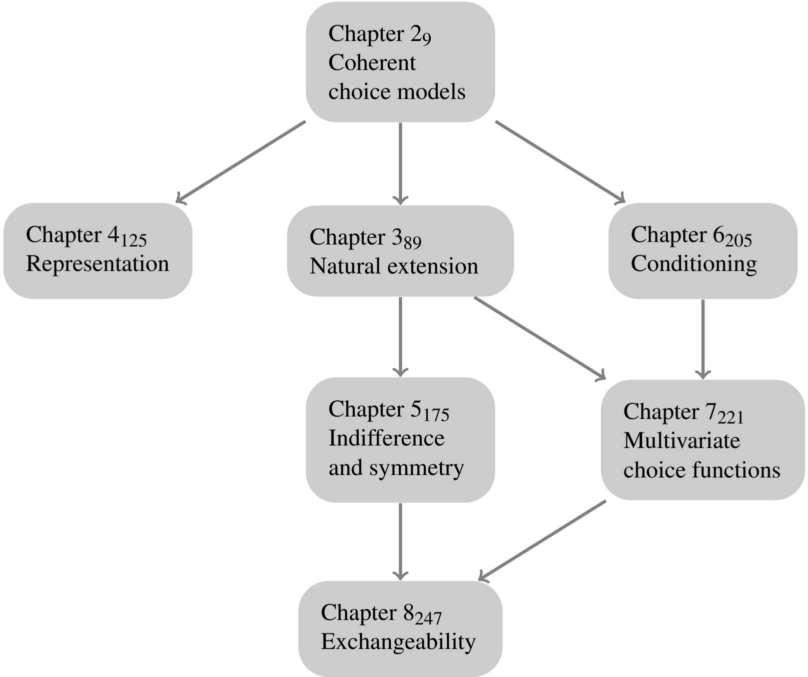


Figure 1.1: Relation between the chapters. This relation is transitive.

Chapter 2_9 introduces choice functions and equivalent models, such as rejection functions and choice relations, which are often more elegant to work with. These models are commonly referred to as choice models. We introduce the rationality axioms for each of the three equivalent choice models, and establish their connection. At this point, we already have enough tools to connect our notion of choice functions with the choice functions considered by Seidenfeld et al. [67]. Once this is established, we investigate order-theoretic properties of the coherent choice models, a crucial step for doing conservative reasoning with them. As a special class of coherent choice functions, we take a closer look at purely binary choice functions: choice functions that are completely determined by pairwise comparisons only. These choice functions are in a one-to-one correspondence with sets of desirable gambles. As it turns out, there is a fourth equivalent choice model, which we call *rejection set*. It reveals the connection with desirability in a more direct way, and it has very nice properties in two-dimensional option spaces. We end this chapter by giving some examples of coherent choice functions, and link it with the choice functions that appear in the literature.

In many cases, it will be very difficult for a subject to completely specify a coherent choice function that describes his beliefs. Instead, mostly he will resort to a partial (incomplete) specification of his choice function. In Chapter 3_{89} we consider such a situation, and we look for the least informative coherent choice function that extends this partially specified choice function. This least informative coherent choice function is what we call the *natural extension*. We will characterise those partial specifications that have a coherent extension. Furthermore, we will find an explicit expression for the natural extension. Interestingly, this expression allows us to discover a special class of coherent choice functions: choice functions that are no infimum of purely binary ones.

Ideally, we would like coherent choice functions to satisfy the following two desirable properties. First, we want every coherent choice function to be an infimum of its dominated maximal (under the partial order of being 'not more informative than') coherent choice functions. Second, we would like these maximal coherent choice functions to have an easy description, and, since the only choice functions with easy description we know are the purely binary ones, we would ideally want the maximal coherent choice functions to be purely binary ones. However, none of these two properties is established, and it is an interesting question whether they hold. In Chapter 4_{125} we are concerned with such questions. We build on the discovery in the previous chapter of coherent choice functions that are no infima of purely binary ones, which already shows that at least one of the two aforementioned desirable properties does not hold. This inspires us to consider an extra property (called 'convexity', or 'mixtures') as a rationality axiom, which helps Seidenfeld et al. [67] to establish that their coherent choice functions satisfy the two aforementioned desirable properties. In the first part of Chapter 4_{125}, we investigate some of the consequences of this extra axiom. As it turns out, there is a connection between such choice functions and lexicographic probability systems. In the second part of this chapter, we prove the negative result that this extra axiom is not sufficient to guarantee that our choice functions satisfy the two desirable properties mentioned at the beginning of this paragraph. In the remainder of this dissertation, we will therefore pay no extra attention to this additional property.

Chapter 5_{175} investigates the interplay between an assessment of indifference and a choice function. We give a characterisation of indifferent choice functions in terms of (indifferent) *equivalence classes* of options, and link it with the earlier definition of indifference by Seidenfeld [63]. We retrieve the established treatment of indifference with sets of desirable gambles as a special case. Eventually, we find the natural extension of a direct assessment (as in Chapter 3_{89}) combined with an indifference assessment.

Chapter 6_{205} shows how to condition a choice function. We consider a variable X whose outcome is uncertain. The options are no longer arbitrary vectors, but instead vector-valued gambles, which is sufficiently general to guarantee the connection with Seidenfeld et al.'s [67] choice functions. Suppose that we have a choice function describing the subject's beliefs about X, and that new information, in the form of 'X assumes a value in the non-empty set E' becomes available. This can be taken into account by conditioning our initial choice function.

In Chapter 7_{221} we take this one step further. We consider a finite number of variables whose outcomes are uncertain, and consider choice functions that describe the subject's beliefs about all these variables at once. This leads to choice functions in a multivariate context. We generalise the concepts of marginalisation, weak extension and irrelevant natural extension for sets of desirable gambles in Reference [29] to choice models. Interestingly, as it turns out, these concepts are not much more involved for coherent choice functions.

Chapter 8_{247} is the final main chapter of my dissertation. It brings together most of the concepts of the previous chapters to study exchangeability with choice functions. Exchangeability is a structural assessment on a sequence of variables that is important for inference purposes. Loosely speaking, making a judgement of exchangeability means that the order in which the variables are observed, is considered irrelevant. This irrelevancy is typically modelled through an indifference assessment. In the first part of this chapter, we derive de Finetti-like representation theorems for finite exchangeable sequences. In the second part we take this one step further, and consider a countable sequence of variables.

1.4 PUBLICATIONS

This dissertation is the product of research on choice functions, which has led to six publications. Two of them have been published, or are accepted for publication, in international journals [77,78]; two of them have been published, or are accepted for publication, as book chapters [53, 80]; the other two are published in the proceedings of international conferences [76,79]:

- Arthur Van Camp, Gert de Cooman and Enrique Miranda. Lexicographic choice functions. Accepted for publication in the International Journal of Approximate Reasoning [77].

- Arthur Van Camp, Gert de Cooman, Enrique Miranda and Erik Quaeghebeur. Coherent choice functions, desirability and indifference.

Accepted for publication in Fuzzy Sets and Systems [78].

- Arthur Van Camp, Gert de Cooman, Enrique Miranda and Erik Quaeghebeur. Modelling indifference with choice functions. Published in the proceedings of ISIPTA 2015 [79].

- Arthur Van Camp and Gert de Cooman. Exchangeable choice functions. Published in the proceedings of ISIPTA 2017 [76].

- Arthur Van Camp, Enrique Miranda and Gert de Cooman. Lexicographic choice functions without Archimedeanicity. Published as a book chapter in Soft Methods for Data Science. Advances in Intelligent Systems and Computing, vol 456. Springer, Cham [80].

- Enrique Miranda, Arthur Van Camp and Gert de Cooman. Choice functions and rejection sets. Accepted for publication as a book chapter in The Mathematics of the Uncertain, 2018 [53].

These publications constitute the core of Chapters 2, 4_{125}, 5_{175} and 8_{247}. The results in the other chapters are rather more recent and have therefore not been published yet.

Besides the references mentioned above, I have been involved in a number of other publications [20,21,25,36,74,75]. They are related to the subject, but only in an indirect way, and therefore I have decided to not include them in this dissertation.

- Jasper De Bock, Arthur Van Camp, Márcio Alves Diniz and Gert de Cooman. Representation theorems for partially exchangeable random variables. Published in Fuzzy Sets and Systems [20].

- Márcio Alves Diniz, Jasper De Bock and Arthur Van Camp. Characterizing Dirichlet priors. Published in The American Statistician [36].

- Cedric De Boom, Jasper De Bock, Arthur Van Camp and Gert de Cooman. Robustifying the Viterbi algorithm. Published in the proceedings of PGM 2014 [21].

- Arthur Van Camp and Gert De Cooman. Modelling practical certainty and its link with classical propositional logic. Published in the proceedings of ISIPTA 2013 [75].

- Arthur Van Camp and Gert De Cooman. A new method for learning imprecise hidden Markov models. Published in the proceedings of IPMU 2012 [74].

- Gert de Cooman, Jasper De Bock and Arthur Van Camp. Recent advances in imprecise-probabilistic graphical models. Published in the proceedings of ECAI 2012 [25].

2

Coherent choice models

This thesis is concerned with choice. The subject's choice will be captured using a choice function. Before we can introduce such choice functions, we first need to know what the objects the subject chooses between will look like.

2.1 What do we choose between?

As we will see, a choice function identifies from within every set of options, those options that are not rejected by a subject. We will collect all the options in $\mathcal{V}$.

Assumption 2.1. *We will assume that the set of all options form a real vector space $\mathcal{V}$, provided with the vector addition $(+)$ and scalar multiplication.*

Let us introduce some basic concepts for vector spaces. Denote the additive identity of $\mathcal{V}$ by 0. For any subsets A_1 and A_2 of $\mathcal{V}$ and any λ in $\mathbb{R}$,[1] we let $\lambda A_1 := \{\lambda u : u \in A_1\}$ and

$$A_1 + A_2 := \{u + v : u \in A_1, v \in A_2\},$$

called the *Minkowski sum* of A_1 and A_2.

[1] $\mathbb{R}$ is the set of real numbers. We use $\mathbb{R}_{>0}$ as a shorthand notation for $\{\alpha \in \mathbb{R} : \alpha > 0\}$, and $\mathbb{R}_{\geq 0}$ for $\{\alpha \in \mathbb{R} : \alpha \geq 0\} = \mathbb{R}_{>0} \cup \{0\}$.

Given any subset A of $\mathcal{V}$, we define its *linear hull* $\operatorname{span}(A)$ as the set of all finite linear combinations of elements of A:[2]

$$\operatorname{span}(A) := \left\{\sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}, u_k \in A\right\} \subseteq \mathcal{V},$$

its *positive hull* $\operatorname{posi}(A)$ as the set of all positive finite linear combinations of elements of A:

$$\operatorname{posi}(A) := \left\{\sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, u_k \in A\right\} \subseteq \operatorname{span}(A) \subseteq \mathcal{V},$$

and its *convex hull* $\operatorname{conv}(A)$ as the set of convex combinations of elements of A:

$$\operatorname{conv}(A) := \left\{\sum_{k=1}^{n} \alpha_k u_k : n \in \mathbb{N}, \alpha_k \in \mathbb{R}_{\geq 0}, \sum_{k=1}^{n} \alpha_k = 1, u_k \in A\right\} \subseteq \operatorname{posi}(A) \subseteq \mathcal{V}.$$

A subset A of $\mathcal{V}$ is called a *convex cone* if it is closed under positive finite linear combinations, i.e. if $\operatorname{posi}(A) = A$. A convex cone $\mathcal{K}$ is called *proper* if $\mathcal{K} \cap -\mathcal{K} = \{0\}$. With any proper convex cone $\mathcal{K} \subseteq \mathcal{V}$, we can associate an ordering $\preceq_{\mathcal{K}}$ on $\mathcal{V}$, defined for all u and v in $\mathcal{V}$ as follows:

$$u \preceq_{\mathcal{K}} v \Leftrightarrow v - u \in \mathcal{K}.$$

We also write $u \succeq_{\mathcal{K}} v$ for $v \preceq_{\mathcal{K}} u$. The ordering $\preceq_{\mathcal{K}}$ is actually a *vector ordering*: it is a partial order—reflexive, antisymmetric and transitive—that satisfies the following two characteristic properties:

$$u_1 \preceq_{\mathcal{K}} u_2 \Leftrightarrow u_1 + v \preceq_{\mathcal{K}} u_2 + v; \tag{2.1}$$

$$u_1 \preceq_{\mathcal{K}} u_2 \Leftrightarrow \lambda u_1 \preceq_{\mathcal{K}} \lambda u_2, \tag{2.2}$$

for all u_1, u_2 and v in $\mathcal{V}$, and all λ in $\mathbb{R}_{>0}$. Observe, by the way, that as a consequence

$$u \preceq_{\mathcal{K}} v \Leftrightarrow 0 \preceq_{\mathcal{K}} v - u \Leftrightarrow u - v \preceq_{\mathcal{K}} 0$$

for all u and v in $\mathcal{V}$.

Conversely, given any vector ordering $\preceq$, the proper convex cone $\mathcal{K}$ from which it is derived can always be retrieved by $\mathcal{K} = \{u \in \mathcal{V} : u \succeq 0\}$.

Finally, with any vector ordering $\preceq$, we associate the strict partial ordering $\prec$ as follows:

$$u \prec v \Leftrightarrow (u \preceq v \text{ and } u \neq v) \Leftrightarrow v - u \in \mathcal{K} \setminus \{0\}, \text{ for all } u \text{ and } v \text{ in } \mathcal{V}.$$

[2] We define the natural numbers $\mathbb{N} := \{1, 2, \ldots\}$ as the set of all positive integers. We use $\mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$ to denote the non-negative integers.

We call u *positive* if $u \succ 0$, and collect all positive options in the convex cone $\mathcal{V}_{\succ 0} \coloneqq \mathcal{K} \setminus \{0\} = \{u \in \mathcal{V} : u \succ 0\}$. We use similar notations $\mathcal{V}_{\geq 0} \coloneqq \mathcal{K} = \{u \in \mathcal{V} : u \geq 0\} = \mathcal{V}_{\succ 0} \cup \{0\}$ for the *non-negative* vectors, $\mathcal{V}_{\prec 0} \coloneqq -\mathcal{K} \setminus \{0\} = \{u \in \mathcal{V} : u \prec 0\} = -\mathcal{V}_{\succ 0}$ for the *negative* vectors, and $\mathcal{V}_{\leq 0} \coloneqq -\mathcal{K} = \{u \in \mathcal{V} : u \leq 0\} = -\mathcal{V}_{\geq 0} = \mathcal{V}_{\prec 0} \cup \{0\}$ for the *non-positive* vectors.

Assumption 2.2. *From now on, we assume that $\mathcal{V}$ is an* ordered vector space*, with a generic but fixed vector ordering $\leq_{\mathcal{K}}$. We will refrain from explicitly mentioning the actual proper convex cone $\mathcal{K}$ we are using, and simply write $\mathcal{V}$ to mean the ordered vector space, and use $\leq$ as a generic notation for the associated vector ordering.*

Elements of $\mathcal{V}$ are intended as abstract representations of options amongst which a subject can express his preferences, by specifying, as we will see below, choice functions. We will call such a real vector space an *option space*. We will motivate our decisions to use vectors as options further on, in Section 2.1.2$_{\curvearrowright}$.

2.1.1 *Technical lemmas about option spaces*

In this section, we collect basic technical lemmas about option spaces, needed to prove a number of results in this thesis.

Lemma 1. *Consider two arbitrary subsets A and A' of $\mathcal{V}$. Then* $\operatorname{posi}(A \cup A') = \operatorname{posi}(A) \cup \operatorname{posi}(A') \cup (\operatorname{posi}(A) + \operatorname{posi}(A'))$.

Proof. We first show that $\operatorname{posi}(A \cup A') \subseteq \operatorname{posi}(A) \cup \operatorname{posi}(A') \cup (\operatorname{posi}(A) + \operatorname{posi}(A'))$. Consider any u in $\operatorname{posi}(A \cup A')$. Then $u = \sum_{k=1}^{m} \lambda_k v_k + \sum_{\ell=1}^{n} \mu_\ell w_\ell$ for some m and n in $\mathbb{Z}_{\geq 0}$ such that $\max\{m,n\} \geq 1$, $\lambda_1, \dots, \lambda_m$, $\mu_1, \dots, \mu_n$ in $\mathbb{R}_{>0}$, $v_1, \dots, v_m$ in A, and $w_1, \dots, w_n$ in A'. If $m = 0$ then $u \in \operatorname{posi}(A')$, and similarly, if $n = 0$ then $u \in \operatorname{posi}(A)$. Finally, if $m > 0$ and $n > 0$, then $u \in \operatorname{posi}(A) + \operatorname{posi}(A')$. So in any of the three cases, u belongs indeed to $\operatorname{posi}(A) \cup \operatorname{posi}(A') \cup (\operatorname{posi}(A) + \operatorname{posi}(A'))$.

We now prove that $\operatorname{posi}(A) \cup \operatorname{posi}(A') \cup (\operatorname{posi}(A) + \operatorname{posi}(A')) \subseteq \operatorname{posi}(A \cup A')$. Consider any u in $\operatorname{posi}(A) \cup \operatorname{posi}(A') \cup (\operatorname{posi}(A) + \operatorname{posi}(A'))$. If $u \in \operatorname{posi}(A)$ or $u \in \operatorname{posi}(A')$, then also $u \in \operatorname{posi}(A \cup A')$. If $u \in \operatorname{posi}(A) + \operatorname{posi}(A')$, then $u = \sum_{k=1}^{m} \lambda_k v_k + \sum_{\ell=1}^{n} \mu_\ell w_\ell$ for some m and n in $\mathbb{N}$, $\lambda_1, \dots, \lambda_m$, $\mu_1, \dots, \mu_n$ in $\mathbb{R}_{>0}$, $v_1, \dots, v_m$ in A, and $w_1, \dots, w_n$ in A'. Then $u = \sum_{i=1}^{m+n} \lambda_i v_i$, where we let $\lambda_{n+1} \coloneqq \mu_1, \dots, \lambda_{m+n} \coloneqq \mu_n$, $v_{n+1} \coloneqq w_1, \dots, v_{m+n} \coloneqq w_n$, and therefore $u \in \operatorname{posi}(A \cup A')$. So in any of the three cases, u indeed belongs to $\operatorname{posi}(A \cup A')$. □

Lemma 2 ([58, Lemma 1])**.** *Consider two arbitrary subsets A and A' of $\mathcal{V}$. Then $0 \notin A + A' \Leftrightarrow A' \cap -A = \emptyset$.*

Proof. We prove $0 \in A + A' \Leftrightarrow A' \cap -A \neq \emptyset$. For necessity, assume that $0 \in A + A'$. Then there are u in A and u' in A' such that $u + u' = 0$, so $u' = -u$ belongs to $A' \cap -A$, which is therefore indeed non-empty.

For sufficiency, assume that $A' \cap -A \neq \emptyset$, then there is some u in A' such that $u \in -A$ or, equivalently, $-u \in A$. Therefore indeed $0 = u - u \in A' + A$. □

Lemma 3 (See Reference [58, Lemma 2])**.** *Consider three arbitrary subsets A, A' and A'' of $\mathcal{V}$. Then $(A''+A')\cap A=\varnothing \Leftrightarrow A'\cap(A-A'')=\varnothing$.*

Proof. Apply Lemma 2 twice: $(A''+A')\cap A=\varnothing \Leftrightarrow 0\notin A''+A'-A=A'+(A''-A) \Leftrightarrow A'\cap(A-A'')=\varnothing$. □

Lemma 4. *Consider any u and w in $\mathcal{V}$. Then $u\in\operatorname{posi}(\mathcal{V}_{>0}\cup\{w\})$ if and only if $0<u$ or $\mu w\leq u$ for some μ in $\mathbb{R}_{>0}$.*

Proof. For necessity, assume that $u\in\operatorname{posi}(\mathcal{V}_{>0}\cup\{w\})$. Then $u\in\mathcal{V}_{>0}\cup\operatorname{posi}(\{w\})\cup(\mathcal{V}_{>0}+\operatorname{posi}(\{w\}))$ using Lemma 1, and therefore $0<u$, or $\mu w=u$ for some μ in $\mathbb{R}_{>0}$, or $\mu' w<u$ for some μ' in $\mathbb{R}_{>0}$. Then indeed $0<u$ or $\mu w\leq u$ for some μ in $\mathbb{R}_{>0}$.

For sufficiency, assume that $0<u$ or $\mu w\leq u$ for some μ in $\mathbb{R}_{>0}$. Then $u\in\mathcal{V}_{>0}$ or $(\exists\mu\in\mathbb{R}_{>0})(\mu w=u$ or $\mu w<u)$, so $u\in\mathcal{V}_{>0}\cup\operatorname{posi}(\{w\})\cup(\mathcal{V}_{>0}+\operatorname{posi}(\{w\}))$, and therefore indeed $u\in\operatorname{posi}(\mathcal{V}_{>0}\cup\{w\})$, using Lemma 1. □

Lemma 5. *Consider any u and w in $\mathcal{V}$. Then $u\in\mathcal{V}_{>0}+\operatorname{posi}\{0,w\}$ if and only if $\mu w<u$ for some μ in $\mathbb{R}_{\geq0}$.*

Proof. Infer the following equivalences. $u\in\mathcal{V}_{>0}+\operatorname{posi}\{0,w\}$ is equivalent to $v<u$ for some v in $\operatorname{posi}\{0,w\}$. Since by Lemma 1 [with $A:=\{0\}$ and $A':=\{w\}$] $\operatorname{posi}\{0,w\}=\operatorname{posi}\{0\}\cup\operatorname{posi}\{w\}\cup(\operatorname{posi}\{0\}+\operatorname{posi}\{w\})=\{0\}\cup\operatorname{posi}\{w\}=\{\mu w:\mu\in\mathbb{R}_{\geq0}\}$, this is indeed equivalent to $\mu w<u$ for some μ in $\mathbb{R}_{\geq0}$. □

2.1.2 *Example of option spaces*

We will often consider the special case of an uncertain variable X that assumes values in a *possibility space* $\mathcal{X}$ of mutually exclusive elementary events, one of which is guaranteed to occur. Here, options will be real-valued maps f on $\mathcal{X}$:

$$f\colon\mathcal{X}\to\mathbb{R}\colon x\mapsto f(x),$$

typically bounded[3] if $\mathcal{X}$ is infinite. We interpret them as *uncertain rewards* or *risky transactions*, and therefore call them *gambles*. These maps take a central position in the theory of imprecise probabilities—they are especially important for coherent lower previsions and coherent sets of desirable gambles, see References [51,64,72,82]. We collect the *set of all bounded gambles on $\mathcal{X}$* in $\mathcal{L}(\mathcal{X})$, also denoted as $\mathcal{L}$ when it is clear from the context what domain $\mathcal{X}$ the gambles are defined on. The interpretation is as follows: If a subject has ownership of some gamble f, then, if the actual outcome x in $\mathcal{X}$ of X has been determined, his capital is changed by the—possibly negative—pay-off $f(x)$, described in a linear utility scale.

[3]We say that a map $f\colon\mathcal{X}\to\mathbb{R}\colon x\mapsto f(x)$ is bounded if it is bounded above—its supremum $\sup f:=\sup\{f(x):x\in\mathcal{X}\}$ is finite—and bounded below—its infimum $\inf f:=\inf\{f(x):x\in\mathcal{X}\}$ is finite.

The order $\leq$ on the real numbers induces a natural order on $\mathcal{L}$, also denoted by $\leq$: for all f and g in $\mathcal{L}$,

$$f \leq g \Leftrightarrow f(x) \leq g(x) \text{ for all } x \text{ in } \mathcal{X}.$$

Its strict variant $<$ on $\mathcal{L}$ is given by $f < g \Leftrightarrow (f \leq g \text{ and } f \neq g)$ for f and g in $\mathcal{L}$. The order $\leq$ on $\mathcal{L}$ is clearly a partial order, that furthermore satisfies the characteristic Properties $(2.1)_{10}$ and $(2.2)_{10}$ of a vector ordering. As usual, we write $f > g \Leftrightarrow g < f$ and $f \geq g \Leftrightarrow g \leq f$ for all f and g in $\mathcal{L}$, and we denote $\mathcal{L}_{\leq 0} \coloneqq \{f \in \mathcal{L} : f \leq 0\}$, $\mathcal{L}_{<0} \coloneqq \{f \in \mathcal{L} : f < 0\}$, $\mathcal{L}_{\geq 0} \coloneqq \{f \in \mathcal{L} : f \geq 0\}$, and $\mathcal{L}_{>0} \coloneqq \{f \in \mathcal{L} : f > 0\}$.

As in Reference [72], we will make no distinction between constant gambles $a(x) \coloneqq a$ for all x in $\mathcal{X}$, and the real number a. This allows us to write down $f \leq a$ to mean $f(x) \leq a$ for all x in $\mathcal{X}$. Furthermore, we extend any binary operation $\star$ on $\mathbb{R}$ to a binary operation on gambles, as follows:

$$(f \star g)(x) \coloneqq f(x) \star g(x) \text{ for } f \text{ and } g \text{ in } \mathcal{L}, \text{ and } x \text{ in } \mathcal{X}.$$

This allows us to consider $f + g$, $f - g$, fg, and $-f \coloneqq 0 - f$, and define scalar multiplication with λ, by viewing λ as a constant gamble: $(\lambda f)(x) = \lambda f(x)$ for all f and in $\mathcal{L}$, λ in $\mathbb{R}$, and x in $\mathcal{X}$. This guarantees that $\mathcal{L}$ is a linear space with additive identity 0, that therefore can serve as option space $\mathcal{V}$, whose order $\preceq$ will typically be the point-wise order $\leq$, but need not be. We also define $|f|$ as the *absolute value* of a gamble f, given by $|f|(x) \coloneqq |f(x)|$ for all x in $\mathcal{X}$.

We now introduce a particular class of gambles. With any subset E of $\mathcal{X}$, we associate its indicator $\mathbb{I}_E$ of E, which is the $\{0,1\}$-valued gamble given by

$$\mathbb{I}_E(x) \coloneqq \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \text{ for all } x \text{ in } \mathcal{X}.$$

We call $\mathbb{I}_E$ the *indicator of* E.

More often than not in the literature, choice functions choose between *horse lotteries* rather than the vectors in an option space. It will be useful to note already that horse lotteries do *not* form a linear space. As we will see in Section 2.4_{28} however, under some mild conditions, those choice functions can be embedded into the choice functions on options. This connection uses the more general notion of *vector-valued gambles* [86]: vector-valued maps on the possibility space. The domain of those gambles is $\mathcal{X} \times \mathcal{R}$ with $\mathcal{R}$ an arbitrary set, and hence the partial map $f(x, \bullet)$ is an element of the vector space $\mathcal{L}(\mathcal{R})$.

Let us conclude this section with a small discussion about why we do not always work with gambles, but instead use the arbitrary vector space $\mathcal{V}$ as our option space. One reason for our working with the more abstract notion of options is that they are better suited for dealing with indifference: as we will see in Chapter 5_{175}, this involves working with equivalence classes of options, which again constitute a vector space. These equivalence classes can no longer be interpreted easily or directly as gambles, or horse lotteries for that matter. Another reason for using options that are more general than real-valued gambles is that recent work by Zaffalon and Miranda [86] has shown that a very general theory of binary preference can be constructed using vector-valued gambles, rather than horse lotteries. Such vector-valued gambles again constitute a real vector, or option, space. In Section 2.428, we will show that the conclusions of this work [86, Section 4] can be extended from binary preferences to choice functions.

2.2 CHOICE MODELS

Now that we know what is chosen between, and what the option space looks like, we are ready to describe *what* choice between options means. This choice will be captured by a *choice function*. Next to choice functions—which we will define in Definition 1—there are two alternative equivalent models: rejection functions (see Definition 2) and choice relations (see Definition 3). We call these functions *choice models*.

We denote by $\mathcal{Q}(\mathcal{V}) \coloneqq \{A \Subset \mathcal{V} : A \neq \emptyset\}$ the set of all *option sets*: all non-empty *finite* subsets of $\mathcal{V}$, where $A \Subset \mathcal{V}$ is taken to mean that A is a finite subset of $\mathcal{V}$. $\mathcal{Q}(\mathcal{V})$ a strict subset of the power set $\mathcal{P}(\mathcal{V}) \coloneqq \{A : A \subseteq \mathcal{V}\}$ of $\mathcal{V}$. We will use the notation $\mathcal{Q}_0(\mathcal{V}) \coloneqq \{A \in \mathcal{Q}(\mathcal{V}) : 0 \in A\}$ to denote those option sets that include the additive identity 0, and $\mathcal{Q}_{\overline{0}}(\mathcal{V}) \coloneqq \{A \in \mathcal{Q}(\mathcal{V}) : 0 \notin A\} = \mathcal{Q}(\mathcal{V}) \setminus \mathcal{Q}_0(\mathcal{V})$ to denote those option sets that exclude 0. When it is clear what option space $\mathcal{V}$ we are considering, we will also use the simpler notation $\mathcal{Q}$, $\mathcal{Q}_0$, and $\mathcal{Q}_{\overline{0}}$. $\mathcal{Q}$ is the domain of any choice function: elements A of $\mathcal{Q}$ are the *option sets*: sets amongst whose members a subject can indicate his preferred ones.

2.2.1 *Choice and rejection functions*

Definition 1 (Choice function)**.** *A* choice function *C on an option space $\mathcal{V}$ is a map*

$$C\colon \mathcal{Q} \to \mathcal{Q} \cup \{\emptyset\} \colon A \mapsto C(A) \text{ such that } C(A) \subseteq A.$$

We collect all the choice functions on $\mathcal{V}$ in $\mathbf{C}(\mathcal{V})$, often simply denoted as $\mathbf{C}$ *when it is clear from the context what the option space is.*

The idea underlying this definition is that a choice function C selects the set $C(A)$ of options in the option set A that are not rejected. $C(A)$ is then called the *choice set* of A. Our definition resembles the one commonly used in the literature [1, 67, 69], except perhaps for a restriction to *finite* option sets [42, 62, 68].

Equivalently to a choice function, we may consider a rejection function.

Definition 2 (Rejection function)**.** *A* rejection function *R on an option space $\mathcal{V}$ is a map*

$$R\colon \mathcal{Q} \to \mathcal{Q} \cup \{\emptyset\} \colon A \mapsto R(A) \text{ such that } R(A) \subseteq A.$$

We collect all the rejection functions on $\mathcal{V}$ in $\mathbf{R}(\mathcal{V})$, often simply denoted as $\mathbf{R}$ when it is clear from the context what the option space is.

Choice functions and rejection functions can correspond with one another: given any rejection function R, its corresponding choice function C_R is defined by $C_R(A) \coloneqq A \setminus R(A)$ for all A in $\mathcal{Q}$. It returns the options $C_R(A)$ that are not rejected by R. Similarly, given any choice function C, its corresponding rejection function R_C is defined by $R_C(A) \coloneqq A \setminus C(A)$ for all A in $\mathcal{Q}$.

They are trivially in a one-to-one correspondence:

Proposition 6. *Consider any choice function C and any rejection function R. Then $C_{R_C} = C$ and $R_{C_R} = R$.*

As we will see, many concepts are more easily described using rejection functions rather than choice functions. Since they are in a one-to-one correspondence, we use them interchangeably, and call any choice function C and rejection function R *corresponding* when

$$R = R_C, \text{ or by Proposition 6 equivalently, } C = C_R. \tag{2.3}$$

2.2.2 *Choice relations*

Another equivalent notion is that of a *choice relation*.

Definition 3 (Choice relation)**.** *A* choice relation $\lhd$ *on an option space $\mathcal{V}$ is a binary relation on $\mathcal{Q}$ that satisfies the following two properties for all A, A_1 and A_2 in $\mathcal{Q}$:*

(i) *if $A_1 \cup A_2 = A_1 \cup A$ then $A_1 \lhd A_2 \Leftrightarrow A_1 \lhd A$;*

(ii) *if $A_1 \cup A_2 \subseteq A$ then $(A_1 \lhd A$ and $A_2 \lhd A) \Leftrightarrow A_1 \cup A_2 \lhd A$.*

We collect all the choice relations on $\mathcal{V}$ in $\mathbf{S}(\mathcal{V})$, often simply denoted as $\mathbf{S}$ when it is clear from the context what the option space is.

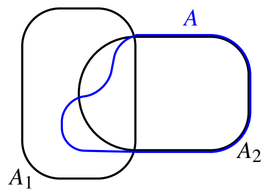


Figure 2.1: Illustration of $A_1 \cup A_2 = A_1 \cup A$ in Definition 3(i)

Choice relations can correspond to a choice function or a rejection function, as in Reference [67]:

Definition 4 ([67, Section 3])**.** *Given any choice function C, define the binary relation $\lhd_C$ on $\mathcal{Q}$ by*

$$A_1 \lhd_C A_2 \Leftrightarrow C(A_1 \cup A_2) \subseteq A_2 \setminus A_1, \textit{ for all } A_1 \textit{ and } A_2 \textit{ in } \mathcal{Q}. \tag{2.4}$$

Given any rejection function R, define the binary relation $\lhd_R$ on $\mathcal{Q}$ by

$$A_1 \lhd_R A_2 \Leftrightarrow A_1 \subseteq R(A_1 \cup A_2), \textit{ for all } A_1 \textit{ and } A_2 \textit{ in } \mathcal{Q}. \tag{2.5}$$

It turns out that $\lhd_C$ and $\lhd_R$ are well defined in the sense that it is irrelevant whether they are specified using a choice function or its corresponding rejection function.

Proposition 7. *Consider any choice function C and rejection function R. If C and R correspond (satisfy Equation* (2.3)$_{\curvearrowleft}$*), then $\lhd_C = \lhd_R$.*

Proof. Consider any A_1 and A_2 in $\mathcal{Q}$, and consider the following equivalences:

$$\begin{aligned} A_1 \lhd_R A_2 &\Leftrightarrow A_1 \subseteq R(A_1 \cup A_2) && \text{by Definition 4} \\ &\Leftrightarrow A_1 \subseteq (A_1 \cup A_2) \setminus C(A_1 \cup A_2) && \text{because } C \text{ and } R \text{ correspond} \\ &\Leftrightarrow A_2 \setminus A_1 = (A_1 \cup A_2) \setminus A_1 \supseteq C(A_1 \cup A_2) && \\ &\Leftrightarrow A_1 \lhd_C A_2 && \text{by Definition 4.} \end{aligned}$$

□

So Equations (2.4) and (2.5) are consistent, in the sense that when C and R correspond, then $\lhd_C$ and $\lhd_R$ are equal. This clarifies the intuition behind a choice relation $\lhd$ derived from a rejection function, or, for that matter, a choice function: $A_1 \lhd A_2$ whenever every option in A_1 is rejected when presented with the options in $A_1 \cup A_2$.

However, it is not yet guaranteed that $\lhd_C$ and $\lhd_R$ *are* actually choice relations: they need to satisfy the two requirements of Definition 3$_{\curvearrowleft}$. Because of Proposition 7, it suffices to show this for $\lhd_R$ alone.

Proposition 8. *For any rejection function R, the corresponding binary relation $\lhd_R$ is a choice relation.*

Proof. We have to check that $\lhd_R$ satisfies the two conditions of Definition 3$_{\curvearrowleft}$. For (i)$_{\curvearrowleft}$, consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $A_1 \cup A = A_1 \cup A_2$. This sets off the following cascade of equivalences:

$$A_1 \lhd_R A_2 \Leftrightarrow A_1 \subseteq R(A_1 \cup A_2) \Leftrightarrow A_1 \subseteq R(A_1 \cup A) \Leftrightarrow A_1 \lhd_R A,$$

where the first and the last equivalences follow from Equation (2.5). For (ii)$_{\curvearrowleft}$, consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $A_1 \cup A_2 \subseteq A$. Then $A_1 \cup A = A_2 \cup A = A$, which sets off the following cascade of equivalences:

$$\begin{aligned} A_1 \lhd_R A \text{ and } A_2 \lhd_R A &\Leftrightarrow A_1 \subseteq R(A_1 \cup A) \text{ and } A_2 \subseteq R(A_2 \cup A) \\ &\Leftrightarrow A_1 \subseteq R(A) \text{ and } A_2 \subseteq R(A) \\ &\Leftrightarrow A_1 \cup A_2 \subseteq R(A) = R(A_1 \cup A_2 \cup A) \Leftrightarrow A_1 \cup A_2 \lhd_R A. \end{aligned}$$

□

Therefore, due to Proposition 7, the binary relation $\triangleleft_C$ that corresponds to a choice function C also is a choice relation. We call $\triangleleft_C$ the choice relation corresponding to the choice function C, and, similarly, $\triangleleft_R$ the choice relation corresponding to the rejection function R.

Now we know how to go from choice functions—or rejection functions for that matter—to choice relations. Let us look for some kind of inverse operation: given a choice relation, can we associate a choice function with it, in such a way that its corresponding choice relation is equal to the original one?

Definition 5. *Given any choice relation $\triangleleft$, define the corresponding choice function $C_\triangleleft$ and corresponding rejection function $R_\triangleleft$ by*

$$C_\triangleleft(A) \coloneqq \bigcap\{A' \subseteq A : A \setminus A' \triangleleft A\} \text{ for all } A \text{ in } \mathcal{Q}$$

and

$$R_\triangleleft(A) \coloneqq \bigcup\{A' \subseteq A : A' \triangleleft A\} \text{ for all } A \text{ in } \mathcal{Q}.$$

$C_\triangleleft$ and $R_\triangleleft$ are well defined in the sense that they correspond:

Proposition 9. *For any choice relation $\triangleleft$, its corresponding choice function $C_\triangleleft$ and rejection function $R_\triangleleft$ correspond to one another.*

Proof. Due to Proposition 6_{15}, it suffices to show that $C_{R_\triangleleft} = C_\triangleleft$. Consider any A in $\mathcal{Q}$, then

$$\begin{aligned} C_{R_\triangleleft}(A) = A \setminus R_\triangleleft(A) = A \setminus \left(\bigcup\{A' \subseteq A : A' \triangleleft A\}\right) \\ &= \bigcap\{A \setminus A' : A' \subseteq A \text{ and } A' \triangleleft A\} \\ &= \bigcap\{A'' : A'' \subseteq A \text{ and } A \setminus A'' \triangleleft A\} = C_\triangleleft(A), \end{aligned}$$

where the second and last equalities are due to Definition 5, the third one to De Morgan's laws, and the fourth one because set complement relative to A preserves the property of being a subset of A. □

We know now that Definition 4 is consistent, but there is no guarantee yet that choice relations and choice functions correspond: do we retrieve the original choice relation?

Proposition 10. *Consider any choice function C, any rejection function R and any choice relation $\triangleleft$. Then $C_{\triangleleft_C} = C$, $R_{\triangleleft_R} = R$, and $\triangleleft_{C_\triangleleft} = \triangleleft_{R_\triangleleft} = \triangleleft$.*

Proof. It suffices to prove that $R_{\triangleleft_R} = R$ and $\triangleleft_{R_\triangleleft} = \triangleleft$. To see that this suffices, if $R_{\triangleleft_R} = R$, by Proposition 9 then $C_{\triangleleft_R} = C_R$, whence by choosing R to be the rejection function corresponding to C, by Proposition 7 indeed $C_{\triangleleft_C} = C$. If $\triangleleft_{R_\triangleleft} = \triangleleft$, since by Proposition 9 we have that $R_\triangleleft = C_\triangleleft$, therefore indeed $\triangleleft_{C_\triangleleft} = \triangleleft$.

To show that $R_{\triangleleft_R} = R$, consider any rejection function R and any A in $\mathcal{Q}$. Use Definitions 5 and 4 to find that then indeed

$$R_{\triangleleft_R}(A) = \bigcup\{A' \subseteq A : A' \triangleleft_R A\} = \bigcup\{A' \subseteq A : A' \subseteq R(A)\} = R(A).$$

To show that $\lhd_{R_\lhd} = \lhd$, consider any choice relation $\lhd$ and any A_1 and A_2 in $\mathcal{Q}$. Use Definitions 4_{16} and $5_{\curvearrowleft}$ to find that then

$$A_1 \lhd_{R_\lhd} A_2 \Leftrightarrow A_1 \subseteq R_\lhd(A_1 \cup A_2) \Leftrightarrow A_1 \subseteq \bigcup\{A' \subseteq A_1 \cup A_2 : A' \lhd A_1 \cup A_2\} =: A_3. \quad (2.6)$$

Since $A_3 = \bigcup\{A' \subseteq A_1 \cup A_2 : A' \lhd A_1 \cup A_2\}$ is a finite union, infer from Definition 3_{15}(ii) that $A_3 \lhd A_1 \cup A_2$. Now if $A_1 \subseteq A_3$, then $A_3 \lhd A_1 \cup A_2$ implies that $A_1 \lhd A_1 \cup A_2$, again by Definition 3_{15}(ii). Conversely, if $A_1 \lhd A_1 \cup A_2$ then, trivially, $A_1 \subseteq A_3$. Hence

$$A_1 \subseteq A_3 \Leftrightarrow A_1 \lhd A_1 \cup A_2 \Leftrightarrow A_1 \lhd A_2,$$

where the last equivalence follows from Definition 3_{15}(i). Combining this with the equivalences in Equation (2.6) completes the proof. □

As we will see, Propositions 6_{15} and $9_{\curvearrowleft}$ imply that choice functions, rejection function and choice relations are in a one-to-one correspondence. This implies statements as $C_{\lhd_R} = C_R$, $\lhd_{C_R} = \lhd_R$, and so on, where C is a choice function, R a rejection function, and $\lhd$ a choice relation. We call any choice function C, rejection function R and choice relation $\lhd$ *corresponding* to each other when

$$C_R = C_\lhd = C, \text{ or equivalently, } R_C = R_\lhd = R, \text{ and also } \lhd_C = \lhd_R = \lhd. \quad (2.7)$$

2.2.3 *Connection between the choice models*

Let us focus on the interplay between the three different choice models and introduce the following maps between the choice models:

$$\begin{aligned}
&\rho\colon \mathbf{C} \to \mathbf{R}\colon C \mapsto \rho(C) \coloneqq R_C,\\
&\rho'\colon \mathbf{R} \to \mathbf{C}\colon R \mapsto \rho'(R) \coloneqq C_R,\\
&\sigma\colon \mathbf{R} \to \mathbf{S}\colon R \mapsto \sigma(R) \coloneqq \lhd_R,\\
&\sigma'\colon \mathbf{S} \to \mathbf{R}\colon \lhd \mapsto \sigma'(\lhd) \coloneqq R_\lhd,\\
&\kappa\colon \mathbf{S} \to \mathbf{C}\colon \lhd \mapsto \kappa(\lhd) \coloneqq C_\lhd,\\
&\kappa'\colon \mathbf{C} \to \mathbf{S}\colon C \mapsto \kappa'(C) \coloneqq \lhd_C.
\end{aligned}$$

The importance of the different maps ρ, σ, κ, ρ', σ' and κ' lies in the fact that they are bijections:

Proposition 11. *The functions ρ, σ and κ are bijections, with respective inverses ρ', σ' and κ'.*

Proof. The proof has the following structure: we first prove (i) that ρ is a bijection whose inverse is ρ', then (ii) that σ is a bijection whose inverse is σ', and finally (iii) that κ is a bijection whose inverse is κ'.

For (i), ρ is one-to-one (or injective) since for all C_1 and C_2 in $\mathbf{C}$, if $R_{C_1} = \rho(C_1) = \rho(C_2) = R_{C_2}$, by Proposition 6_{15} then indeed $C_1 = C_2$. It is onto (or surjective) since for all R in $\mathbf{R}$ there is some C in $\mathbf{C}$—namely, using Proposition 6_{15}, $C_R = \rho'(R)$—such

$$\begin{tikzcd}
& \mathbf{C} \arrow[dl, bend right, "\kappa'"'] \arrow[dr, "\rho"] & \\
\mathbf{S} \arrow[ur, "\kappa"] \arrow[rr, bend right, "\sigma'"'] & & \mathbf{R} \arrow[ll, "\sigma"'] \arrow[ul, bend right, "\rho'"']
\end{tikzcd}$$

Figure 2.2: The maps ρ, ρ', σ, σ', κ and κ'

that indeed $R = \rho(C)$. To show that ρ' is the inverse of ρ, it suffices to show that $\rho \circ \rho' = \mathrm{id}_{\mathbf{R}}$,[4] or equivalently, that $R = \rho(\rho'(R)) = R_{C_R}$ for all R in $\mathbf{R}$. Proposition 6_{15} guarantees that this is indeed true.

For (ii), σ is one-to-one since for all R_1 and R_2 in $\mathbf{R}$, if $\lhd_{R_1} = \sigma(R_1) = \sigma(R_2) = \lhd_{R_2}$, by Proposition 10_{17} then indeed $R_1 = R_2$. It is onto since for all $\lhd$ in $\mathbf{S}$ there is some R in $\mathbf{R}$—namely, using Proposition 10_{17}, $R_\lhd = \sigma'(\lhd)$—such that indeed $\lhd = \sigma(R)$. To show that σ' is the inverse of σ, it suffices to show that $\sigma \circ \sigma' = \mathrm{id}_{\mathbf{S}}$, or equivalently, that $\lhd = \lhd_{R_\lhd}$ for all $\lhd$ in $\mathbf{S}$. Proposition 10_{17} guarantees that this is indeed true.

For (iii), κ is one-to-one since for all $\lhd_1$ and $\lhd_2$ in $\mathbf{S}$, if $C_{\lhd_1} = \kappa(\lhd_1) = \kappa(\lhd_2) = C_{\lhd_2}$, by Proposition 10_{17} then indeed $\lhd_1 = \lhd_2$. It is onto since for all C in $\mathbf{C}$ there is some $\lhd$ in $\mathbf{S}$—namely, using Proposition 10_{17}, $\lhd_C = \kappa'(C)$—such that indeed $C = \kappa(\lhd)$. To show that κ' is the inverse of κ, it suffices to show that $\kappa \circ \kappa' = \mathrm{id}_{\mathbf{C}}$, or equivalently, that $C = C_{\lhd_C}$ for all C in $\mathbf{C}$. Proposition 10_{17} guarantees that this is indeed true. $\square$

This means that all the types of choice models are in one-to-one correspondences with each other. Therefore, from now on we can focus on any of them. In any section of this thesis, we will typically choose to work with the type of choice model that is best suited to what we are doing there.

2.3 RATIONALITY AXIOMS

In order for choice functions to reflect rational behaviour, they should satisfy some rationality requirements. If a choice function satisfies these requirements, we call it *coherent*. To give an example of irrational behaviour, consider the special option space of gambles $\mathcal{V} = \mathcal{L}$, whose interpretation is explained in Section 2.1_9. Consider the choice between the two constant gambles 0 and 1. If some choice function C identifies 0 as the choice between 0 and 1—in other words, if $\{0\} = C(\{0,1\})$—then C reflects a choice of the *status quo* (the gamble 0) above the certain reward of 1 units of utility, irrespective of the outcome. This is not considered rational behaviour as it implies a sure loss of 1, and we need to rule out choice functions that reflect this.

[4] For arbitrary set $\mathcal{A}$, we let $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} : A \mapsto \mathrm{id}_{\mathcal{A}}(A) := A$ of $\mathcal{A}$ be the identity map on $\mathcal{A}$.

Seidenfeld et al. [67, Section 3] call a choice function C coherent if there is a non-empty set $\mathcal{T}$ of probability-utility pairs such that $C(A)$ is the set of options in A that maximise expected utility for some element of $\mathcal{T}$. They also provide an axiomatisation for this type of coherence, based on the one for binary preferences [3]. In Section 2.4$_{28}$ we will comment on their axiomatisation, and connect it with ours.

We prefer to define coherence directly in terms of axioms, without reference to probability and utility. For each of the three choice models, they are:

Definition 6 (Coherent choice function)**.** *We call a choice function C on $\mathcal{V}$* coherent *if for all A, A_1 and A_2 in $\mathcal{Q}$, all u and v in $\mathcal{V}$, and all λ in $\mathbb{R}_{>0}$:*
C1. $C(A) \neq \emptyset$;
C2. *if* $u \prec v$ *then* $u \notin C(\{u,v\})$;
C3. a. *if* $C(A_2) \subseteq A_2 \setminus A_1$ *and* $A_1 \subseteq A_2 \subseteq A$ *then* $C(A) \subseteq A \setminus A_1$;
b. *if* $C(A_2) \subseteq A_2 \setminus A_1$ *and* $A \subseteq A_1$ *then* $C(A_2 \setminus A) \subseteq A_2 \setminus A_1$;
C4. a. *if* $A_1 \subseteq C(A_2)$ *then* $\lambda A_1 \subseteq C(\lambda A_2)$;
b. *if* $A_1 \subseteq C(A_2)$ *then* $A_1 + \{u\} \subseteq C(A_2 + \{u\})$.
We collect all coherent choice functions on $\mathcal{V}$ in the set $\overline{\mathbf{C}}(\mathcal{V})$, often simply denoted as $\overline{\mathbf{C}}$ when it is clear from the context which vector space we are using.

Definition 7 (Coherent rejection function)**.** *We call a rejection function R on $\mathcal{V}$* coherent *if for all A, A_1 and A_2 in $\mathcal{Q}$, all u and v in $\mathcal{V}$, and all λ in $\mathbb{R}_{>0}$:*
R1. $R(A) \neq A$;
R2. *if* $u \prec v$ *then* $u \in R(\{u,v\})$;
R3. a. *if* $A_1 \subseteq R(A_2)$ *and* $A_2 \subseteq A$ *then* $A_1 \subseteq R(A)$;
b. *if* $A_1 \subseteq R(A_2)$ *and* $A \subseteq A_1$ *then* $A_1 \setminus A \subseteq R(A_2 \setminus A)$;
R4. a. *if* $A_1 \subseteq R(A_2)$ *then* $\lambda A_1 \subseteq R(\lambda A_2)$;
b. *if* $A_1 \subseteq R(A_2)$ *then* $A_1 + \{u\} \subseteq R(A_2 + \{u\})$.
We collect all coherent rejection functions on $\mathcal{V}$ in the set $\overline{\mathbf{R}}(\mathcal{V})$, often simply denoted as $\overline{\mathbf{R}}$ when it is clear from the context which vector space we are using.

Definition 8 (Coherent choice relation)**.** *We call a choice relation $\lhd$ on $\mathcal{V}$* coherent *if for all A, A_1 and A_2 in $\mathcal{Q}$, all u and v in $\mathcal{V}$, and all λ in $\mathbb{R}_{>0}$:*
$\lhd$1. $A \not\lhd A$;
$\lhd$2. *if* $u \prec v$ *then* $\{u\} \lhd \{v\}$;
$\lhd$3. a. *if* $A_1 \lhd A_2$ *and* $A_2 \subseteq A$ *then* $A_1 \lhd A$;
b. *if* $A_1 \lhd A_2$ *and* $A \subseteq A_1$ *then* $A_1 \setminus A \lhd A_2 \setminus A$;
$\lhd$4. a. *if* $A_1 \lhd A_2$ *then* $\lambda A_1 \lhd \lambda A_2$;
b. *if* $A_1 \lhd A_2$ *then* $A_1 + \{u\} \lhd A_2 + \{u\}$.
We collect all coherent choice relations on $\mathcal{V}$ in the set $\overline{\mathbf{S}}(\mathcal{V})$, often simply denoted as $\overline{\mathbf{S}}$ when it is clear from the context which vector space we are using.

Before we show that these definitions of coherence correspond, let us first identify a number of equivalent forms of Axioms C4, R4 and $\lhd$4.

Lemma 12. *Consider any choice function C, any rejection function R and any choice relation* $\lhd$. *Then the following equivalences hold:*

C satisfies Axiom $\mathrm{C4a}_{20}$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in A, \lambda \in \mathbb{R}_{>0})(u \in C(A) \Rightarrow \lambda u \in C(\lambda A)) \tag{C4a.1}$$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in A, \lambda \in \mathbb{R}_{>0})(u \in C(A) \Leftrightarrow \lambda u \in C(\lambda A)) \tag{C4a.2}$$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, \lambda \in \mathbb{R}_{>0})\lambda C(A) = C(\lambda A), \tag{C4a.3}$$

R satisfies Axiom $\mathrm{R4a}_{20}$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in A, \lambda \in \mathbb{R}_{>0})(u \in R(A) \Rightarrow \lambda u \in R(\lambda A)) \tag{R4a.1}$$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in A, \lambda \in \mathbb{R}_{>0})(u \in R(A) \Leftrightarrow \lambda u \in R(\lambda A)) \tag{R4a.2}$$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, \lambda \in \mathbb{R}_{>0})\lambda R(A) = R(\lambda A), \tag{R4a.3}$$

$\lhd$ *satisfies Axiom* $\lhd 4\mathrm{a}_{20}$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in \mathcal{V}, \lambda \in \mathbb{R}_{>0})(\{u\} \lhd A \Rightarrow \lambda\{u\} \lhd \lambda A) \tag{$\lhd$4a.1}$$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in \mathcal{V}, \lambda \in \mathbb{R}_{>0})(\{u\} \lhd A \Leftrightarrow \lambda\{u\} \lhd \lambda A) \tag{$\lhd$4a.2}$$

$$\Leftrightarrow (\forall A_1, A_2 \in \mathcal{Q}, \lambda \in \mathbb{R}_{>0})(A_1 \lhd A_2 \Leftrightarrow \lambda A_1 \lhd \lambda A_2,) \tag{$\lhd$4a.3}$$

C satisfies Axiom $\mathrm{C4b}_{20}$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in A, v \in \mathcal{V})(u \in C(A) \Rightarrow u + v \in C(A + \{v\})) \tag{C4b.1}$$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in A, v \in \mathcal{V})(u \in C(A) \Leftrightarrow u + v \in C(A + \{v\})) \tag{C4b.2}$$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, v \in \mathcal{V})C(A) + \{v\} = C(A + \{v\}), \tag{C4b.3}$$

R satisfies Axiom $\mathrm{R4b}_{20}$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in A, v \in \mathcal{V})(u \in R(A) \Rightarrow u + v \in R(A + \{v\})) \tag{R4b.1}$$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in A, v \in \mathcal{V})(u \in R(A) \Leftrightarrow u + v \in R(A + \{v\})) \tag{R4b.2}$$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, v \in \mathcal{V})R(A) + \{v\} = R(A + \{v\}), \tag{R4b.3}$$

$\lhd$ *satisfies Axiom* $\lhd 4\mathrm{b}_{20}$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in \mathcal{V}, v \in \mathcal{V})(\{u\} \lhd A \Rightarrow \{u + v\} \lhd A + \{v\}) \tag{$\lhd$4b.1}$$

$$\Leftrightarrow (\forall A \in \mathcal{Q}, u \in \mathcal{V}, v \in \mathcal{V})(\{u\} \lhd A \Leftrightarrow \{u + v\} \lhd A + \{v\}) \tag{$\lhd$4b.2}$$

$$\Leftrightarrow (\forall A_1, A_2 \in \mathcal{Q}, v \in \mathcal{V})(A_1 \lhd A_2 \Leftrightarrow A_1 + \{v\} \lhd A_2 + \{v\}). \tag{$\lhd$4b.3}$$

Proof. For the equivalences involving Axioms C4a, R4a and $\lhd$4a, we will establish the following chain of implications:

C satisfies Axiom $\mathrm{C4a}_{20} \Rightarrow$ (C4a.1) $\Rightarrow$ (C4a.2) $\Rightarrow$ (C4a.3) $\Rightarrow C$ satisfies Axiom $\mathrm{C4a}_{20}$, and similarly for R and $\lhd$, all at once. Note that the first implication holds simply by considering $\tilde{A}_1 := \{u\}$ and $\tilde{A}_2 := A$ in the statement of the axiom. The second implication follows by considering $\frac{1}{\lambda} > 0$ instead of λ. The third and fourth implications are then immediate, for $\lhd$ taking into account Definition 3_{15}(ii).

For the equivalences involving Axioms $\mathrm{C4b}_{20}$, $\mathrm{R4b}_{20}$ and $\lhd\mathrm{4b}_{20}$, we will establish the following chain of implications:

$$C \text{ satisfies Axiom } \mathrm{C4b}_{20} \Rightarrow (\mathrm{C4b.1}) \Rightarrow (\mathrm{C4b.2}) \Rightarrow (\mathrm{C4b.3}) \Rightarrow C \text{ satisfies Axiom } \mathrm{C4b}_{20},$$

and similarly, for Axioms $\mathrm{R4b}_{20}$ and $\lhd\mathrm{4b}_{20}$, all in once. Note that the first implication holds simply by considering $\tilde{u} := v$, $\tilde{A}_1 := \{u\}$ and $\tilde{A}_2 := A$ in the statement of the axiom. The second implication follows by considering $-v \in \mathcal{V}$ instead of v. The third and fourth implications are then immediate, for $\lhd$ taking into account Definition 3_{15}(ii). □

Proposition 13. *Consider any corresponding choice function C, rejection function R and choice relation $\lhd$ (satisfy Equation $(2.7)_{18}$). Then for any axiom* $\mathrm{C}*$ *in* $\{\mathrm{C1}_{20}, \mathrm{C2}_{20}, \mathrm{C3a}_{20}, \mathrm{C3b}_{20}, \mathrm{C4a}_{20}, \mathrm{C4b}_{20}\}$*, the following statements are equivalent:*

(i) *C satisfies Axiom* $\mathrm{C}*$*;*
(ii) *R satisfies Axiom* $\mathrm{R}*$*;*
(iii) *$\lhd$ satisfies Axiom $\lhd*$,*

where we denote by $\mathrm{R}*$ *and $\lhd*$ the axiom corresponding to* $\mathrm{C}*$*, for all $*$ in* $\{1, 2, 3a, 3b, 4a, 4b\}$*. As a consequence, the following statements are equivalent:*

(iv) *C is coherent;*
(v) *R is coherent;*
(vi) *$\lhd$ is coherent.*

Proof. The proof has the following structure. For any axiom $\mathrm{C}*$ in the set $\{\mathrm{C1}_{20}, \mathrm{C2}_{20}, \mathrm{C3a}_{20}, \mathrm{C3b}_{20}, \mathrm{C4a}_{20}, \mathrm{C4b}_{20}\}$, we show that (i)⇔(ii), and that (ii)⇔(iii). As a consequence, therefore, trivially (iv)⇔(v)⇔(vi).

Since C, R and $\lhd$ correspond, we may assume that $C(A) = A \setminus R(A)$, $R(A) = A \setminus C(A) = \bigcup\{A' \subseteq A : A' \lhd A\}$, and $A_1 \lhd A_2 \Leftrightarrow A_1 \subseteq R(A_1 \cup A_2)$ for all A, A_1 and A_2 in $\mathcal{Q}$.

We start with Axiom $\mathrm{C1}_{20}$, and its corresponding Axioms $\mathrm{R1}_{20}$ and $\lhd 1_{20}$. To show that (i)⇔(ii), consider any A in $\mathcal{Q}$ and note that, since C and R correspond, indeed $C(A) \neq \emptyset \Leftrightarrow R(A) = A \setminus C(A) \neq A$. To show that (ii)⇔(iii), consider any A in $\mathcal{Q}$ and note that indeed $R(A) = A \Leftrightarrow A \subseteq R(A) \Leftrightarrow A \lhd A$, where the first equivalence is due to the requirement in Definition 2_{14} that $R(A) \subseteq A$.

We now turn to Axiom $\mathrm{C2}_{20}$, and the corresponding Axioms $\mathrm{R2}_{20}$ and $\lhd 2_{20}$. To show that (i)⇔(ii), consider any u and v in $\mathcal{V}$, and infer the following equivalences: $u \notin C(\{u,v\}) \Leftrightarrow u \in \{u,v\} \setminus C(\{u,v\}) \Leftrightarrow u \in R(\{u,v\})$. To show that (ii)⇔(iii), consider any u and v in $\mathcal{V}$, and infer the following equivalence: $u \in R(\{u,v\}) \Leftrightarrow \{u\} \lhd \{v\}$, where we used $A_1 \lhd A_2 \Leftrightarrow A_1 \subseteq R(A_1 \cup A_2)$ with $A_1 = \{u\}$ and $A_2 = \{v\}$.

Next, consider Axiom $\mathrm{C3a}_{20}$, and its corresponding Axioms $\mathrm{R3a}_{20}$ and $\lhd\mathrm{3a}_{20}$. To show that (i)⇔(ii), we will first show that (i)⇒(ii). Assume that C satisfies Axiom $\mathrm{C3a}_{20}$ and consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $A_1 \subseteq R(A_2)$ and $A_2 \subseteq A$. Then $C(A_2) \subseteq A_2 \setminus A_1$, and also $A_1 \subseteq A_2$ by Definition 2_{14}. Axiom $\mathrm{C3a}_{20}$ implies that then

$C(A) \subseteq A \setminus A_1$, or in other words, that $A_1 \subseteq R(A)$, and hence R indeed satisfies Axiom R3a$_{20}$. To show that (ii)⇒(i), assume that R satisfies Axiom R3a$_{20}$ and consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $C(A_2) \subseteq A_2 \setminus A_1$ and $A_1 \subseteq A_2 \subseteq A$. Then $A_1 \subseteq R(A_2)$. Axiom R3a$_{20}$ implies that then $A_1 \subseteq R(A)$, or, in other words, that $C(A) \subseteq A \setminus A_1$, and hence C indeed satisfies Axiom C3a$_{20}$.

To show that (ii)⇔(iii), first we will show that (ii)⇒(iii). Assume that R satisfies Axiom R3a$_{20}$ and consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $A_1 \lhd A_2$ and $A_2 \subseteq A$. The first assumption is equivalent to $A_1 \subseteq R(A_1 \cup A_2)$. Let $A' \coloneqq A \cup A_1$, then $A_1 \cup A_2 \subseteq A' = A' \cup A_1$, and hence Axiom R3a$_{20}$ implies that then $A_1 \subseteq R(A' \cup A_1)$, or, in other words, that $A_1 \lhd A \cup A_1$. By Definition 3$_{15}$(i), this is equivalent to $A_1 \lhd A$, and hence $\lhd$ indeed satisfies Axiom $\lhd$3a$_{20}$. To show that (iii)⇒(ii), assume that $\lhd$ satisfies Axiom $\lhd$3a$_{20}$ and consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $A_1 \subseteq R(A_2)$ and $A_2 \subseteq A$. Then $A_1 \lhd A_2$. Axiom $\lhd$3a$_{20}$ implies that then $A_1 \lhd A$, or, in other words, that $A_1 \subseteq R(A)$, where we used that $A_1 \subseteq A$ so $A = A \cup A_1$, and hence R indeed satisfies Axiom R3a$_{20}$.

Next, consider Axiom C3b$_{20}$, and its corresponding Axioms R3b$_{20}$ and $\lhd$3b$_{20}$. To show that (i)⇔(ii), first we will show that (i)⇒(ii). Assume that C satisfies Axiom C3b$_{20}$ and consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $A_1 \subseteq R(A_2)$ and $A \subseteq A_1$. Then $C(A_2) \subseteq A_2 \setminus A_1$. Axiom C3b implies that then $C(A_2 \setminus A) \subseteq A_2 \setminus A_1$, and since $A \subseteq A_1$—so $A_2 \setminus A_1 \subseteq A_2 \setminus A$—, in other words, that $(A_2 \setminus A) \setminus (A_2 \setminus A_1) \subseteq R(A_2 \setminus A)$. Now use that $(A_2 \setminus A) \setminus (A_2 \setminus A_1) = A_2 \cap A^c \cap (A_2^c \cup A_1) = A_1 \cap A_2 \cap A^c = A_1 \cap A^c = A_1 \setminus A$ to infer that $A_1 \setminus A \subseteq R(A_2 \setminus A)$, and hence R indeed satisfies Axiom R3b$_{20}$. To show that (ii)⇒(i), assume that R satisfies Axiom R3b$_{20}$ and consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $C(A_2) \subseteq A_2 \setminus A_1$ and $A \subseteq A_1$. Then $A_1' \subseteq R(A_2)$ Axiom R3b$_{20}$ implies that then $A_1' \setminus A \subseteq R(A_2 \setminus A)$, or, in other words, that $C(A_2 \setminus A) \subseteq (A_2 \setminus A) \setminus (A_1' \setminus A)$. Infer the following chain of equalities: $(A_2 \setminus A) \setminus (A_1 \setminus A) = A_2 \cap A^c \cap (A_1^c \cup A) = A_2 \cap A^c \cap A_1^c = A_2 \cap A_1^c = A_2 \setminus A_1$, and therefore $C(A_2 \setminus A) \subseteq A_2 \setminus A_1$, whence C indeed satisfies Axiom C3b$_{20}$.

To show that (ii)⇔(iii), first we will show that (ii)⇒(iii). Assume that R satisfies Axiom R3b$_{20}$ and consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $A_1 \lhd A_2$ and $A \subseteq A_1$. Let $A_2' \coloneqq A_1 \cup A_2$, then $A_1 \subseteq R(A_2')$. Axiom R3b$_{20}$ implies that then $A_1 \setminus A \subseteq R(A_2' \setminus A)$, or, in other words, that $A_1 \setminus A \lhd A_2' \setminus A$. Since $A_2' \setminus A = (A_2 \setminus A) \cup (A_1 \setminus A)$, use Definition 3$_{15}$(i) to infer that $A_1 \setminus A \lhd A_2' \setminus A$ is equivalent to $A_1 \setminus A \lhd A_2 \setminus A$, and hence $\lhd$ indeed satisfies Axiom $\lhd$3b$_{20}$. To show that (iii)⇒(ii), assume that $\lhd$ satisfies Axiom $\lhd$3b$_{20}$ and consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $A_1 \subseteq R(A_2)$ and $A \subseteq A_1$. Then $A_1 \lhd A_2$, and, using Definition 2$_{14}$, $A_1 \subseteq A_2$. Axiom $\lhd$3b$_{20}$ implies that then $A_1 \setminus A \lhd A_2 \setminus A$, or, in other words, that $A_1 \setminus A \subseteq R((A_1 \setminus A) \cup (A_2 \setminus A))$. Since $A_1 \subseteq A_2$, therefore $A_1 \setminus A \subseteq A_2 \setminus A$, whence $A_1 \setminus A \subseteq R(A_2 \setminus A)$, and hence R indeed satisfies Axiom R3b$_{20}$.

Subsequently, consider Axiom C4a$_{20}$, and its corresponding Axioms R4a$_{20}$ and $\lhd$4a$_{20}$. To show that (i)⇔(ii)⇔(iii), use Lemma 12$_{21}$ and the fact that for any $A \in \mathcal{Q}$, $u \in A$ and $\lambda \in \mathbb{R}_{>0}$, $u \in C(A) \Leftrightarrow u \notin R(A) \Leftrightarrow \{u\} \ntriangleleft A$ and $\lambda u \in C(\lambda A) \Leftrightarrow \lambda u \notin R(\lambda A) \Leftrightarrow \lambda\{u\} \ntriangleleft \lambda A$.

Finally, consider Axiom C4b$_{20}$, and its corresponding Axioms R4b$_{20}$ and $\lhd$4b$_{20}$. To show that (i)⇔(ii)⇔(iii), use Lemma 12$_{21}$ and the fact that for all $A \in \mathcal{Q}$, $u \in A$ and $v \in \mathcal{V}$, $u \in C(A) \Leftrightarrow u \notin R(A) \Leftrightarrow \{u\} \ntriangleleft A$ and $u+v \in C(A+\{v\}) \Leftrightarrow u+v \notin R(A+\{v\}) \Leftrightarrow \{u+v\} \ntriangleleft A+\{v\}$. □

Taking into account Proposition 11$_{18}$, this means that there are bijections between $\overline{\mathbf{C}}$, $\overline{\mathbf{R}}$ and $\overline{\mathbf{S}}$,[5] by appropriately restricting the bijections between $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{S}$.

Corollary 14. *There are bijections between $\overline{\mathbf{C}}$, $\overline{\mathbf{R}}$ and $\overline{\mathbf{S}}$.*

Proof. Proposition 11_{18} implies that ρ is a bijection between $\mathbf{C}$ and $\mathbf{R}$, σ a bijection between $\mathbf{R}$ and $\mathbf{S}$, and κ a bijection between $\mathbf{S}$ and $\mathbf{C}$. To show that ρ defines a bijection between $\overline{\mathbf{C}}$ and $\overline{\mathbf{R}}$, note that its restriction to $\overline{\mathbf{C}}$ is in particular injective. To show that ρ is surjective, it suffices to show that, for any R in $\overline{\mathbf{R}}$, there is some C in $\overline{\mathbf{C}}$ such that $R = R_C$, or, in other words, that $\rho^{-1}(R) = C_R$ is coherent. Proposition 13_{22} guarantees that this is true. Furthermore, the same proposition guarantees that, for any C in $\overline{\mathbf{C}}$, $\rho(C)$ is an element of $\overline{\mathbf{R}}$. In a similar fashion, we can show that σ is a bijection between $\overline{\mathbf{R}}$ and $\overline{\mathbf{S}}$, and that κ is a bijection between $\overline{\mathbf{S}}$ and $\overline{\mathbf{C}}$. □

For all the properties about coherence that we will see, it therefore suffices to prove the coherence for either choice functions, rejection functions, or choice relations, since by Proposition 13_{22}, if we have coherence for one choice model, we automatically have coherence for all the other corresponding choice models.

2.3.1 *Motivation of the rationality axioms*

Let us give an informal and intuitive motivation for the rationality axioms, and the reasons why we think that rational choice must satisfy them. First of all, it should be noted that, apart from Axiom $\mathrm{C2}_{20}$—or its corresponding Axioms $\mathrm{R2}_{20}$ and $\triangleleft 2_{20}$—, our axioms are, after performing the necessary translation from horse lotteries to linear spaces (which we will do in Section 2.4_{28}), a strict subset[6] of Seidenfeld et al.'s [67]. Any motivation to use their axioms therefore immediately transfers to our setting.

Let us briefly review every axiom. We do so for the axioms of rejection functions; due to Proposition 13_{22}, this translates to other choice models.

It is implicit in every theory of choice that not every option can be rejected, or, in other words, that Axiom $\mathrm{R1}_{20}$ should hold. In many works (see, amongst others, References [45, 67, 69]) it is already implicit in the definition of rejection function—or choice function for that matter—that the rejection cannot be full. We decided to not require this directly in the definition, since, otherwise the theory of coherent rejection functions would not be a belief structure [23], something we consider to be rather important. We discuss this in Section 2.6_{46}. As we will see in Section 2.8_{55}, by instead requiring this as a rationality axiom, our choice functions *are* a belief structure.

[5] As we will see in Section 2.6_{46}, $\overline{\mathbf{C}}$, $\overline{\mathbf{R}}$ and $\overline{\mathbf{S}}$ are even order-isomorphic.

[6] There are two of their axioms that we do not consider as part of the rationality requirements. One is an 'Archimedean' continuity condition, and the other a convexity condition, necessary for their connection with a set of probability-utility pairs; see Section 2.4_{28} for more information.

Axiom R2$_{20}$ and R4$_{20}$ are most clear if we attach the interpretation of gambles to our options.

It is clear that some gamble f that is strictly dominated by another gamble g—in other words, $f < g$—results in a pay-off that is smaller than or equal to g's pay-off, irrespective of the actual outcome. Furthermore, for at least one of the outcomes, f's pay-off is (strictly) smaller than g's, and therefore, f cannot be chosen—and must hence be rejected—from within the option set $\{f, g\}$ without incurring a partial loss: this is what is required by Axiom R2$_{20}$.

Axiom R4a$_{20}$ requires that the utility scale in which the pay-off takes place should not affect our choice. Axiom R4b$_{20}$ requires that the addition of a fixed gamble to every element of the option set, should not affect our choice.

Axioms R1$_{20}$, R2$_{20}$ and R4$_{20}$ where concerned with the linearity of the utility scale, the non-emptiness of the choice, and the (trivial) preference implied by dominance. For us, the two axioms that are really specific to 'choice', are Axioms R3a$_{20}$—also known as Sen's condition α [68, 69] or Chernoff's condition [11]; see [2] for an overview—and R3b$_{20}$—also known as Aizerman's condition [1]. Axiom R3a$_{20}$ requires that a rejected option can never be promoted to a chosen one by adding more options to the option set. Axiom R3b$_{20}$ requires that a rejected option can never be promoted to a chosen one by deleting other rejected options from the option set.

2.3.2 *Other properties imposed on choice models*

As we have already briefly mentioned, and as we will comment on in more detail in Section 2.4$_{28}$, we have decided to not use two of Seidenfeld et al.'s [67] rationality axioms. One of these two is a 'convexity' or 'mixture' axiom, which is hard to reconcile with Walley–Sen maximality as a decision rule. Indeed, as we will see in Example 3$_{64}$, there are coherent and 'non-convex' choice functions that identify the options that are optimal under Walley–Sen maximality. Nevertheless, we feel that this 'convexity' property is interesting enough to merit its own mention here. The other property is related to this 'convexity' property.

Consider any choice function C. The two extra properties are:

C5. if $A \subseteq A_1 \subseteq \mathrm{conv}(A)$ then $C(A) \subseteq C(A_1)$, for all A and A_1 in $\mathcal{Q}$;

C6. if $0 \in C(\{0, u_1, \ldots, u_n\})$ then $0 \in C(\{0, \mu_1 u_1, \ldots, \mu_n u_n\})$, for all n in $\mathbb{N}$, u_1, $\ldots$, u_n in $\mathcal{V}$ and μ_1, $\ldots$, μ_n in $\mathbb{R}_{>0}$.

We will commonly refer to Property C5 as 'mixture' or 'convexity', and to Property C6 as 'separate positive homogeneity'. Its corresponding properties for a rejection function R and a choice relation $\triangleleft$ are:

R5. if $A \subseteq A_1 \subseteq \mathrm{conv}(A)$ then $R(A_1) \cap A \subseteq R(A)$, for all A and A_1 in $\mathcal{Q}$;

R6. if $0 \in R(\{0, u_1, \ldots, u_n\})$ then $0 \in R(\{0, \mu_1 u_1, \ldots, \mu_n u_n\})$, for all n in $\mathbb{N}$, u_1, $\ldots$, u_n in $\mathcal{V}$ and μ_1, $\ldots$, μ_n in $\mathbb{R}_{>0}$;

$\triangleleft$5. if $A \subseteq A_1 \subseteq \mathrm{conv}(A)$ and $u \in A$ then $\{u\} \triangleleft A_1 \Rightarrow \{u\} \triangleleft A$, for all A and A_1 in $\mathcal{Q}$ and u in A;

$\lhd$6. if $\{0\} \lhd \{u_1, \dots, u_n\}$ then $\{0\} \lhd \{\mu_1 u_1, \dots, \mu_n u_n\}$, for all n in $\mathbb{N}$, u_1, ..., u_n in $\mathcal{V}$ and μ_1, ..., μ_n in $\mathbb{R}_{>0}$.

The 0 in Property $\mathrm{C6}_{\curvearrowleft}$ is important. As we will see in Section 2.8[55], it does not imply that

$$u \in C(\{u, u_1, \dots, u_n\}) \Rightarrow u \in C(\{u, \mu_1 u_1, \dots, \mu_n u_n\})$$

for all n in $\mathbb{N}$, u, u_1, ..., u_n in $\mathcal{V}$ and μ_1, ..., μ_n in $\mathbb{R}_{>0}$.

Let us ascertain that Properties $\mathrm{C5}_{\curvearrowleft}$, $\mathrm{R5}_{\curvearrowleft}$ and $\lhd 5_{\curvearrowleft}$ on the one hand, and $\mathrm{C6}_{\curvearrowleft}$, $\mathrm{R6}_{\curvearrowleft}$ and $\lhd$6 on the other hand, indeed correspond:

Proposition 15. *Consider any corresponding choice function C, rejection function R and choice relation $\lhd$. Then the following statements are equivalent:*

(i) *C satisfies Property* $\mathrm{C5}_{\curvearrowleft}$*;*
(ii) *R satisfies Property* $\mathrm{R5}_{\curvearrowleft}$*;*
(iii) *$\lhd$ satisfies Property* $\lhd 5_{\curvearrowleft}$*.*

Similarly, the following statements are equivalent:

(iv) *C satisfies Property* $\mathrm{C6}_{\curvearrowleft}$*;*
(v) *R satisfies Property* $\mathrm{R6}_{\curvearrowleft}$*;*
(vi) *$\lhd$ satisfies Property* $\lhd$6.

Proof. Since C, R and $\lhd$ correspond, we may assume that $C(A) = A \setminus R(A)$, $R(A) = A \setminus C(A) = \bigcup\{A' \subseteq A : A' \lhd A\}$, and $A_1 \lhd A_2 \Leftrightarrow A_1 \subseteq R(A_1 \cup A_2)$ for all A, A_1 and A_2 in $\mathcal{Q}$.

For the first statement, we first show that (i)$\Leftrightarrow$(ii). Consider any A and A_1 in $\mathcal{Q}$ such that $A \subseteq A_1 \subseteq \mathrm{conv}(A)$. It suffices to show that $C(A) \subseteq C(A_1) \Leftrightarrow R(A_1) \cap A \subseteq R(A)$. Consider the following chain of equivalences:

$$\begin{aligned} C(A) \subseteq C(A_1) \Leftrightarrow C(A) \subseteq A \cap C(A_1) &\Leftrightarrow R(A) \supseteq A \setminus (A \cap C(A_1)) = A \cap C(A_1)^c \\ &\Leftrightarrow R(A) \supseteq A \cap R(A_1), \end{aligned}$$

where the first equivalence is due to the requirement of Definition 1[14] that $C(A) \subseteq A$. Therefore indeed (i)$\Leftrightarrow$(ii).

To finish the proof of the first statement, we show that (ii)$\Leftrightarrow$(iii). Consider any A and A_1 in $\mathcal{Q}$ such that $A \subseteq A_1 \subseteq \mathrm{conv}(A)$. It suffices to show that $R(A_1) \cap A \subseteq R(A) \Leftrightarrow (u \in A)(\{u\} \lhd A_1 \Rightarrow \{u\} \lhd A)$. Consider the following chain of equivalences:

$$\begin{aligned} R(A_1) \cap A \subseteq R(A) &\Leftrightarrow (\forall u \in A)(u \in R(A_1) \Rightarrow u \in R(A)) \\ &\Leftrightarrow (\forall u \in A)(\{u\} \lhd A_1 \Rightarrow \{u\} \lhd A). \end{aligned}$$

Therefore indeed (ii)$\Leftrightarrow$(iii).

For the second statement, consider any n in $\mathbb{N}$, u_1, ..., u_n in $\mathcal{V}$, and μ_1, ..., μ_n in $\mathbb{R}_{>0}$. Note that by considering $\frac{1}{\mu_1} > 0$, ..., $\frac{1}{\mu_n} > 0$, Property $\mathrm{C6}_{\curvearrowleft}$ is equivalent to

$$0 \in C(\{0, u_1, \dots, u_n\}) \Leftrightarrow 0 \in C(\{0, \mu_1 u_1, \dots, \mu_n u_n\}).$$

Similarly, Property R6$_{25}$ is equivalent to

$$0 \in R(\{0, u_1, \dots, u_n\}) \Leftrightarrow 0 \in R(\{0, \mu_1 u_1, \dots, \mu_n u_n\}),$$

and Property $\lhd$6 to

$$\{0\} \lhd \{u_1, \dots, u_n\} \Leftrightarrow \{0\} \lhd \{\mu_1 u_1, \dots, \mu_n u_n\}.$$

Since $0 \notin C(\{0, u_1, \dots, u_n\}) \Leftrightarrow 0 \in R(\{0, u_1, \dots, u_n\}) \Leftrightarrow \{0\} \lhd \{0, u_1, \dots, u_n\}$ and $0 \notin C(\{0, \mu_1 u_1, \dots, \mu_n u_n\}) \Leftrightarrow 0 \in R(\{0, \mu_1 u_1, \dots, \mu_n u_n\}) \Leftrightarrow \{0\} \lhd \{0, \mu_1 u_1, \dots, \mu_n u_n\}$, therefore indeed (iv)$\Leftrightarrow$(v)$\Leftrightarrow$(vi). $\square$

Therefore, we can focus on the two additional properties for any one of the choice models, just as for the rationality axioms.

We now show that, under some conditions, Property C6$_{25}$ is implied by Property C5$_{25}$—and therefore, as a consequence, also that Property R6$_{25}$ is implied by Property R5$_{25}$, and that Property $\lhd$6 is implied by Property $\lhd$5$_{25}$.

Proposition 16. *Consider any choice function C that satisfies Axioms* C3a$_{20}$ *and* C4a$_{20}$*. If C satisfies Property* C5$_{25}$*, then it also satisfies Property* C6$_{25}$.

Proof. Consider any n in $\mathbb{N}$, $u_1, \dots, u_n$ in $\mathcal{V}$, and $\mu_1, \dots, \mu_n$ in $\mathbb{R}_{>0}$, let $A \coloneqq \{0, u_1, \dots, u_n\}$ and $\mu^* \coloneqq \max\{\mu_1, \dots, \mu_n\} \in \mathbb{R}_{>0}$, and assume that $0 \in C(A)$. Then Axiom C4a$_{20}$ implies that also $0 \in C(\mu^* A)$. Moreover, for every k in $\{1, \dots, n\}$, we find that $\mu_k u_k \in \operatorname{conv}(\{0, \mu^* u_k\}) \subseteq \operatorname{conv}(\mu^* A)$, so Property C5$_{25}$ guarantees that $0 \in C(\mu^* A \cup \{\mu_1 u_1, \dots, \mu_n u_n\})$. The contraposition of Axiom R3a$_{20}$ [with $\tilde{A} \coloneqq \mu^* A \cup \{\mu_1 u_1, \dots, \mu_n u_n\}$, $\tilde{A}_1 \coloneqq \{0\}$ and $\tilde{A}_2 \coloneqq \{0, \mu_1 u_1, \dots, \mu_n u_n\}$] now yields that indeed $0 \in C(\{0, \mu_1 u_1, \dots, \mu_n u_n\})$. $\square$

As a consequence, under coherence, Property C5$_{25}$ implies Property C6$_{25}$. Furthermore, in turn, under Axiom C4b$_{20}$, Property C6$_{25}$ implies Axiom C4a$_{20}$:

Proposition 17. *Consider any choice function C that satisfies Axiom* C4b$_{20}$*. If C satisfies Property* C6$_{25}$*, then it satisfies Property* C4a$_{20}$.

Proof. We will prove that C satisfies the equivalent version (C4b.1)$_{21}$. So consider any A in $\mathcal{Q}$, any u in A and any λ in $\mathbb{R}_{>0}$ such that $u \in C(A)$. We need to prove that then $\lambda u \in C(\lambda A)$. Since $u \in C(A)$, by Axiom C4b$_{20}$, also $0 \in C(A - \{u\})$, and therefore, by Property C6$_{25}$, also $0 \in C(\lambda(A - \{u\})) = C(\lambda A - \{\lambda u\})$ [with $\mu_i \coloneqq \lambda$ for every i in $\{1, \dots, |A|\}$]. Another application of Axiom C4b$_{20}$ implies than then indeed $\lambda u \in C(\lambda A)$. $\square$

2.4 THE CONNECTION WITH OTHER DEFINITIONS OF CHOICE FUNCTIONS

Before we go on with our exploration of choice functions, let us take some time here to explain why we have chosen to define them in the way we did. Seidenfeld et al. [67] (see also Kadane et al. [45]) define choice functions on *horse lotteries*, instead of options, as this helps them generalise to non-binary preferences the framework established by Anscombe and Aumann [3] for binary preferences.

We consider an arbitrary possibility space $\mathcal{X}$, and a finite set $\mathcal{R}$ of prizes, or rewards.

Definition 9 (Vector-valued gamble). *When the domain is of the type $\mathcal{X}\times\mathcal{R}$, we call elements f of $\mathcal{L}(\mathcal{X}\times\mathcal{R})$* vector-valued gambles *on $\mathcal{X}$. Indeed, for each x in $\mathcal{X}$, the partial map $f(x,\bullet)$ is then an element of the vector space $\mathcal{L}(\mathcal{R})$. We call $\mathcal{X}$ the* state part *of the domain $\mathcal{X}\times\mathcal{R}$ of vector-valued gambles, and $\mathcal{R}$ the* reward part.

Horse lotteries are special vector-valued gambles:

Definition 10 (Horse lottery). *We call a* horse lottery *H any map from $\mathcal{X}\times\mathcal{R}$ to $[0,1]$ such that for all x in $\mathcal{X}$, the partial map $H(x,\bullet)$ is a probability mass function over $\mathcal{R}$:*[7]

$$(\forall x\in\mathcal{X})\Bigg(\sum_{r\in\mathcal{R}}H(x,r)=1\text{ and }(\forall r\in\mathcal{R})H(x,r)\geq 0\Bigg).$$

We collect all the horse lotteries on $\mathcal{X}$ with reward set $\mathcal{R}$ in $\mathcal{H}(\mathcal{X},\mathcal{R})$, which is also denoted more simply by $\mathcal{H}$ when it is clear from the context what the possibility space $\mathcal{X}$ and reward set $\mathcal{R}$ are.

Let us, for the remainder of this section, fix $\mathcal{X}$ and $\mathcal{R}$. It is clear that $\mathcal{H}\subseteq\mathcal{L}(\mathcal{X}\times\mathcal{R})$. Seidenfeld et al. [67] consider choice functions whose domain is $\mathcal{Q}(\mathcal{H})$, the set of all finite subsets of $\mathcal{H}$—choice functions on horse lotteries.[8] We will call them *choice functions on $\mathcal{H}$*. Because of the nature of $\mathcal{H}$, their choice functions are different from ours: they require slightly different rationality axioms. The most significant change is that for Seidenfeld et al. [67], choice functions need not satisfy Axioms $\mathrm{C4a}_{20}$ and $\mathrm{C4b}_{20}$. In fact, choice functions on $\mathcal{H}$ *cannot* even satisfy these axioms, since $\mathcal{H}$ is no linear space:

[7] Note that $H(x,\bullet)$ defines a countably additive probability measure, and that this countable additivity property is necessary for Lemma 19_{31} below to hold.

[8] Actually, Seidenfeld et al. [67] define choice functions on a larger domain: all possibly infinite but *closed* sets of horse lotteries (non-closed sets may not have admissible options). This is an extension we see no need for in our present context.

it is not closed under arbitrary linear combinations, only under *convex combinations*. Instead, on their approach a choice function C^* on $\mathcal{H}$ is required to satisfy

C4*. $A_1^* \lhd^* A_2^* \Leftrightarrow \alpha A_1^* + (1-\alpha)\{H\} \lhd^* \alpha A_2^* + (1-\alpha)\{H\}$, for all α in $(0,1]$, all A_1^* and A_2^* in $\mathcal{Q}(\mathcal{H})$ and all H in $\mathcal{H}$.

The binary relation $\lhd^*$ is the choice relation corresponding to C^*, defined in Definition 4_{16}, and R^* is the corresponding rejection function. Furthermore, for a choice function C^* to be coherent, it needs to additionally satisfy (see Reference [67]):

C1*. $C^*(A^*) \neq \emptyset$ for all A^* in $\mathcal{Q}(\mathcal{H})$;

C2*. for all A^* in $\mathcal{Q}(\mathcal{H})$, all H_1 and H_2 in $\mathcal{H}$ such that $H_1(\bullet,\top) \leq H_2(\bullet,\top)$ and $H_1(\bullet,r) = H_2(\bullet,r) = 0$ for all r in $\mathcal{R} \setminus \{\bot,\top\}$, and all H in $\mathcal{H} \setminus \{H_1,H_2\}$:

a. if $H_2 \in A^*$ and $H \in R^*(\{H_1\} \cup A^*)$ then $H \in R^*(A^*)$;
b. if $H_1 \in A^*$ and $H \in R^*(A^*)$ then $H \in R^*(\{H_2\} \cup A^* \setminus \{H_1\})$;

C3*. for all A^*, A_1^* and A_2^* in $\mathcal{Q}(\mathcal{H})$:

a. if $A_1^* \subseteq R^*(A_2^*)$ and $A_2^* \subseteq A^*$ then $A_1^* \subseteq R^*(A)$;
b. if $A_1^* \subseteq R^*(A_2^*)$ and $A^* \subseteq A_1^*$ then $A_1^* \setminus A^* \subseteq R^*(A_2^* \setminus A)$;

C5*. if $A^* \subseteq A_1^* \subseteq \mathrm{conv}(A)$ then $C^*(A) \subseteq C^*(A_1^*)$, for all A^* and A_1^* in $\mathcal{Q}(\mathcal{H})$;

C6*. for all A^*, $A^{*\prime}$, $A^{*\prime\prime}$, $A_i^{*\prime}$ and $A_i^{*\prime\prime}$ (for i in $\mathbb{N}$) in $\mathcal{Q}(\mathcal{H})$ such that the sequence $A_i^{*\prime}$ converges point-wise to $A^{*\prime}$ and the sequence $A_i^{*\prime\prime}$ converges point-wise to $A^{*\prime\prime}$:

a. if $(\forall i \in \mathbb{N}) A_i^{*\prime\prime} \lhd^* A_i^{*\prime}$ and $A^{*\prime} \lhd^* A^*$ then $A^{*\prime\prime} \lhd^* A^*$;
b. if $(\forall i \in \mathbb{N}) A_i^{*\prime\prime} \lhd^* A_i^{*\prime}$ and $A^* \lhd^* A^{*\prime\prime}$ then $A^* \lhd^* A^{*\prime}$,

where Seidenfeld et al. [67] assume that there is a unique worst reward $\bot$ and a unique best reward $\top$ in $\mathcal{R}$. This is a somewhat stronger assumption than what we will need: further on in this section, we will only need to assume that there is a unique worst reward. Axiom C2* is the counterpart for choice functions on horse lotteries of Proposition 30_{41} further on, which is a consequence of our Axioms $\mathrm{C1}_{20}$–$\mathrm{C4}_{20}$. Seidenfeld et al. [67] need to impose this property as an axiom, essentially because of the absence in their system of a counterpart for our Axiom $\mathrm{C2}_{20}$. Axiom C5* corresponds to the 'convexity' Property $\mathrm{C5}_{25}$. Axioms C6*a and C6*b are Archimedean axioms, hard to reconcile with desirability[9] (see Reference [86, Section 4] for a detailed explanation), which is why will not enforce them here.

We now intend to show that under very weak conditions on the reward set $\mathcal{R}$, choice functions on horse lotteries that satisfy C4* are in a one-to-one correspondence with choice functions on a suitably defined option space that satisfy Axioms $\mathrm{C4a}_{20}$ and $\mathrm{C4b}_{20}$.

[9]Desirability is a very successful and well established imprecise-probabilistic model, and we will link it with choice models in Section 2.8_{55}.

Let us first study the impact of Axiom $\mathrm{C4}^*_{\curvearrowleft}$. We begin by showing that an assessment of $H \in C^*(A^*)$ for some A^* in $\mathcal{Q}(\mathcal{H})$ implies other assessments of this type.

Proposition 18. *Consider any choice function C^* on $\mathcal{Q}(\mathcal{H})$ that satisfies Axiom* $\mathrm{C4}^*_{\curvearrowleft}$*, any option sets A^* and $A^{*\prime}$ in $\mathcal{Q}(\mathcal{H})$, and any H in A^* and H' in $A^{*\prime}$. If there are λ and λ' in $\mathbb{R}_{>0}$ such that $\lambda(A^* - \{H\}) = \lambda'(A^{*\prime} - \{H'\})$, then*

$$H \in C^*(A^*) \Leftrightarrow H' \in C^*(A^{*\prime}).$$

Proof. Consider any A^* and $A^{*\prime}$ in $\mathcal{Q}(\mathcal{H})$, H in A^* and H' in $A^{*\prime}$, and λ and λ' in $\mathbb{R}_{>0}$, and assume that $\lambda(A^* - \{H\}) = \lambda'(A^{*\prime} - \{H'\})$. We will show that $H \in R^*(A^*) \Leftrightarrow H' \in R^*(A^{*\prime})$. We infer from the assumption that

$$\frac{\lambda}{\lambda+\lambda'}A^* + \frac{\lambda'}{\lambda+\lambda'}\{H'\} = \frac{\lambda'}{\lambda+\lambda'}A^{*\prime} + \frac{\lambda}{\lambda+\lambda'}\{H\}.$$

If we let $\alpha := \frac{\lambda}{\lambda+\lambda'}$ to ease the notation along, then $1-\alpha = \frac{\lambda'}{\lambda+\lambda'}$ and $\alpha \in (0,1)$. We now infer from the identity above that $\alpha A^* + (1-\alpha)\{H'\} = (1-\alpha)A^{*\prime} + \alpha\{H\}$. Therefore, infer the following chain of equivalences:

$$\begin{aligned}
H \in R^*(A^*) &\Leftrightarrow \{H\} \lhd^* A^* && \text{by Definition } 3_{15}\\
&\Leftrightarrow \alpha\{H\} + (1-\alpha)\{H'\} \lhd^* \alpha A^* + (1-\alpha)\{H'\} && \text{using Axiom } \mathrm{C4}^*{}_{29}\\
&\Leftrightarrow \alpha\{H\} + (1-\alpha)\{H'\} \lhd^* (1-\alpha)A^{*\prime} + \alpha\{H\} &&\\
&\Leftrightarrow \{H'\} \lhd^* A^{*\prime} && \text{using Axiom } \mathrm{C4}^*{}_{29}\\
&\Leftrightarrow H' \in R^*(A^{*\prime}) && \text{by Definition } 3_{15}. \qquad \square
\end{aligned}$$

For any r in $\mathcal{R}$, we now introduce $\mathcal{R}_r := \mathcal{R} \setminus \{r\}$, the set of all rewards without r. For the connection between choice functions on $\mathcal{H}$ and choice functions on some option space, we need to somehow be able to extend $\mathcal{H}$ to a linear space. The so-called gamblifier φ_r will play a crucial role in this:

Definition 11 (Gamblifier)**.** *Consider any r in $\mathcal{R}$. The* gamblifier *φ_r is the linear map*

$$\varphi_r\colon \mathcal{L}(\mathcal{X}\times\mathcal{R}) \to \mathcal{L}(\mathcal{X}\times\mathcal{R}_r)\colon f \mapsto \varphi_r f,$$

where $\varphi_r f(x,s) := f(x,s)$ for all x in $\mathcal{X}$ and s in $\mathcal{R}_r$.

This map will be important (using its lifted variant $\tilde{\varphi}_r$) for Theorem 23_{35} where we will connect Axioms $\mathrm{C1}_{20}$–$\mathrm{C4}_{20}$ and Property $\mathrm{C5}_{25}$ with Axioms $\mathrm{C1}^*_{\curvearrowleft}$, $\mathrm{C3}^*_{\curvearrowleft}$–$\mathrm{C5}^*_{\curvearrowleft}$ and the notion of having a 'worst reward r'. In particular, the gamblifier φ_r maps any horse lottery H in $\mathcal{H}(\mathcal{X},\mathcal{R})$ to an element $\varphi_r H$ of

$\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ that satisfies the following two conditions:[10]

$$\varphi_r H(\bullet,\bullet)\geq 0 \text{ and } \sum_{s\in\mathcal{R}_r}\varphi_r H(\bullet,s)\leq 1. \tag{2.8}$$

Application of φ_r to sets of the form $\lambda(A^*-\{H\})$ essentially leaves the 'information' they contain unchanged:

Lemma 19. *Consider any r in $\mathcal{R}$. Then the following two properties hold:*

(i) *The gamblifier φ_r is one-to-one on $\mathcal{H}$.*
(ii) *For any A^* and $A^{*\prime}$ in $\mathcal{Q}(\mathcal{H})$, any H in A^* and H' in $A^{*\prime}$ and any λ and λ' in $\mathbb{R}_{>0}$:*

$$\lambda(A^*-\{H\})=\lambda'(A^{*\prime}-\{H'\}) \Leftrightarrow \varphi_r(\lambda(A^*-\{H\}))=\varphi_r(\lambda'(A^{*\prime}-\{H'\})).$$

Proof. We begin with the first statement. Consider any H and H' in $\mathcal{H}$, and assume that $\varphi_r(H)=\varphi_r(H')$. We infer from Definition 11 that

$$H(x,s)=H'(x,s) \text{ for all } x \text{ in } \mathcal{X} \text{ and } s \text{ in } \mathcal{R}_r,$$

and therefore also, since H and H' are horse lotteries,

$$H(x,r)=1-\sum_{s\in\mathcal{R}_r}H(x,s)=1-\sum_{s\in\mathcal{R}_r}H'(x,s)=H'(x,r) \text{ for all } x \text{ in } \mathcal{X}.$$

Hence indeed $H=H'$.

The direct implication in the second statement is trivial; let us prove the converse. Assume that $\varphi_r(\lambda(A^*-\{H\}))=\varphi_r(\lambda'(A^{*\prime}-\{H'\}))$. We may write, without loss of generality, that $A^*=\{H,H_1,\dots,H_n\}$ and $A^{*\prime}=\{H',H'_1,\dots,H'_m\}$ for some n and m in $\mathbb{N}$. Now, consider any element H_i in A^*, then $\varphi_r(\lambda(H_i-H))\in\varphi_r(\lambda(A^*-\{H\}))$. Consider any j in $\{1,\dots,m\}$ such that $\varphi_r(\lambda(H_i-H))=\varphi_r(\lambda'(H'_j-H'))$. It follows from the assumption that there is at least one such j. The proof is complete if we can show that $\lambda(H_i-H)=\lambda'(H'_j-H')$. By Definition 11, we already know that

$$\lambda(H_i(\bullet,s)-H(\bullet,s))=\lambda'(H'_j(\bullet,s)-H'(\bullet,s)) \text{ for all } s \text{ in } \mathcal{R}_r,$$

and therefore, since H, H', H_i and H'_j are horse lotteries, also

$$\begin{aligned}\lambda(H_i(\bullet,r)-H(\bullet,r))&=\lambda\left(\sum_{s\in\mathcal{R}_r}H(\bullet,s)-\sum_{s\in\mathcal{R}_r}H_i(\bullet,s)\right)\\&=\sum_{s\in\mathcal{R}_r}\lambda(H(\bullet,s)-H_i(\bullet,s))=\sum_{s\in\mathcal{R}_r}\lambda'(H'(\bullet,s)-H'_j(\bullet,s))\\&=\lambda'\left(\sum_{s\in\mathcal{R}_r}H'(\bullet,s)-\sum_{s\in\mathcal{R}_r}H'_j(\bullet,s)\right)=\lambda'(H'_j(\bullet,r)-H'(\bullet,r)),\end{aligned}$$

whence indeed $\lambda(H_i-H)=\lambda'(H'_j-H')$. □

[10] $\varphi_r H(\bullet,\bullet)\geq 0$ is a particular instance of the notation introduced in Section 2.1.2$_{12}$: it means $(\forall x\in\mathcal{X},s\in\mathcal{R}_r)(\varphi_r H(x,s)\geq 0)$. A similar remark holds for $\sum_{s\in\mathcal{R}_r}\varphi_r H(\bullet,s)\leq 1$.

We now lift the gamblifier φ_r to a map $\tilde{\varphi}_r$ that turns choice functions on gambles into choice functions on horse lotteries:

$$\tilde{\varphi}_r\colon \mathbf{C}(\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)) \to \mathbf{C}(\mathcal{H}(\mathcal{X},\mathcal{R}))\colon C \mapsto \tilde{\varphi}_r C, \tag{2.9}$$

where $\tilde{\varphi}_r C(A^*) \coloneqq \varphi_r^{-1} C(\varphi_r A^*)$ for every A^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$. This definition makes sense because we have proved in Lemma 19 that φ_r is one-to-one on $\mathcal{H}$, and therefore invertible on $\varphi_r\mathcal{H}$. The result of applying $\tilde{\varphi}_r$ to a choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ is a choice function $\tilde{\varphi}_r C$ on $\mathcal{H}(\mathcal{X},\mathcal{R})$. Observe that we can equally well make $\tilde{\varphi}_r$ apply to rejection functions R, and that for every A^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$:

$$\begin{aligned}\tilde{\varphi}_r R(A^*) \coloneqq \varphi_r^{-1} R(\varphi_r A^*) &= \varphi_r^{-1}(\varphi_r A^* \setminus C(\varphi_r A^*))\\ &= A^* \setminus \varphi_r^{-1} C(\varphi_r A^*) = A^* \setminus \tilde{\varphi}_r C(A^*),\end{aligned} \tag{2.10}$$

so $\tilde{\varphi}_r R$ is the rejection function that corresponds with the choice function $\tilde{\varphi}_r C$, when R is the rejection function for C.

One property of the transformation $\tilde{\varphi}_r$ that will be useful in our subsequent proofs is the following:

Lemma 20. *Consider any r in $\mathcal{R}$ and any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}\times\mathcal{R}_r))$, and define g by $g(x,s) \coloneqq \sum_{f\in A}|f(x,s)|$ for all x in $\mathcal{X}$ and s in $\mathcal{R}_r$. Consider any λ in $\mathbb{R}$ such that*[11]

$$\lambda > \max\Big\{\sup_{x\in\mathcal{X}}\sum_{s\in\mathcal{R}_r} h(x,s) : h \in A+\{g\}\Big\} \geq 0.$$

Then $\frac{1}{\lambda}(A+\{g\}) = \varphi_r A^*$ for some A^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$.

Proof. Consider any h in $A+\{g\}$, and let us show that $\frac{1}{\lambda}h$ is a horse lottery, i.e., satisfies the conditions in Equation (2.8). The first one is satisfied because $\lambda > 0$ and $h = f+g$ for some f in A, so $h = f+g = f+\sum_{f'\in A}|f'| \geq f+|f| \geq 0$ and therefore indeed $\frac{1}{\lambda}h \geq 0$. For the second condition, recall that $\lambda \geq \sum_{s\in\mathcal{R}_r} h(\bullet,s)$ by construction and therefore indeed $\sum_{s\in\mathcal{R}_r}\frac{1}{\lambda}h(\bullet,s) \leq 1$. □

Proposition 21. *Consider any r in $\mathcal{R}$. The operator $\tilde{\varphi}_r$ is a bijection between the choice functions on $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ that satisfy Axioms* C4a$_{20}$ *and* C4b$_{20}$*, and the choice functions on $\mathcal{H}(\mathcal{X},\mathcal{R})$ that satisfy Axiom* C4$^*_{29}$.

Proof. We first show that $\tilde{\varphi}_r$ is injective. Assume *ex absurdo* that it is not, so there are choice functions C and C' on $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ that satisfy Axioms C4a$_{20}$ and C4b$_{20}$, such that $\tilde{\varphi}_r C = \tilde{\varphi}_r C'$ but nevertheless $C \neq C'$. The latter means that there are A in

[11] This is always possible since A consists of a finite number of bounded gambles. Since g is a finite sum of bounded gambles, it is bounded as well, and hence so are the gambles in $A+\{g\}$. Therefore, for any gamble h in $A+\{g\}$, the partial map $\sum_{s\in\mathcal{R}_r} h(\bullet,s)$ is a bounded gamble because of the finiteness of $\mathcal{R}_r$, so its supremum is finite.

$\mathcal{Q}(\mathcal{L}(\mathcal{X}\times\mathcal{R}_r))$ and f in A such that $f\in C(A)$ and $f\notin C'(A)$. Use Lemma 20 to find some λ in $\mathbb{R}_{>0}$ and g in $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ such that $\frac{1}{\lambda}(A+\{g\}) = \varphi_r A^*$ for some A^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$. If we now apply Axioms C4a$_{20}$ and C4b$_{20}$ we find that $\frac{f+g}{\lambda}\in C(\frac{1}{\lambda}(A+\{g\}))$, or equivalently, $\varphi_r^{-1}(\frac{f+g}{\lambda})\in\tilde{\varphi}_r C(A^*)$. Similarly, we find that $\frac{f+g}{\lambda}\notin C'(\frac{1}{\lambda}(A+\{g\}))$, or equivalently, $\varphi_r^{-1}(\frac{f+g}{\lambda})\notin\tilde{\varphi}_r C'(A^*)$. But this contradicts our assumption that $\tilde{\varphi}_r C=\tilde{\varphi}_r C'$.

We now show that application of $\tilde{\varphi}_r$ to any choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ that satisfies Axioms C4a$_{20}$ and C4b$_{20}$, results in a choice function $\tilde{\varphi}_\perp C$ that satisfies Axiom C4$^*{}_{29}$. Consider any A_1^* and A_2^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$, any H in $\mathcal{H}(\mathcal{X},\mathcal{R})$, and any α in $(0,1]$. Infer the following chain of equivalences:

$$\begin{aligned}
&A_1^* \lhd_{\tilde{\varphi}_r C} A_2^*\\
&\quad\Leftrightarrow A_1^*\subseteq\tilde{\varphi}_r R(A_1^*\cup A_2^*) && \text{by Definition } 4_{16}\\
&\quad\Leftrightarrow \varphi_r A_1^*\subseteq R(\varphi_r(A_1^*\cup A_2^*)) && \text{by Equation (2.9)}_{32}\\
&\quad\Leftrightarrow \varphi_r\alpha A_1^*\subseteq R(\varphi_r\alpha(A_1^*\cup A_2^*)) && \text{by Axiom C4a}_{20}\\
&\quad\Leftrightarrow \varphi_r(\alpha A_1^*+(1-\alpha)\{H\})\\
&\qquad\qquad\subseteq R(\varphi_r(\alpha(A_1^*\cup A_2^*)+(1-\alpha)\{H\})) && \text{by Axiom C4b}_{20}\\
&\quad\Leftrightarrow \alpha A_1^*+(1-\alpha)\{H\}\subseteq\tilde{\varphi}_r R(\alpha(A_1^*\cup A_2^*)+(1-\alpha)\{H\}) && \text{by Equation (2.10)}_{32}\\
&\quad\Leftrightarrow (\alpha A_1^*+(1-\alpha)\{H\})\lhd_{\tilde{\varphi}_r C}(\alpha A_2^*+(1-\alpha)\{H\}) && \text{by Definition } 4_{16},
\end{aligned}$$

which tells us that indeed $\tilde{\varphi}_r C$ satisfies Axiom C4$^*{}_{29}$.

The proof is complete if we also show that $\tilde{\varphi}_r$ is surjective—that for every choice function C^* on $\mathcal{H}(\mathcal{X},\mathcal{R})$ that satisfies Axiom C4$^*{}_{29}$, there is a choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ that satisfies Axioms C4a$_{20}$ and C4b$_{20}$ such that $\tilde{\varphi}_r C=C^*$. So consider any choice function C^* on $\mathcal{H}(\mathcal{X},\mathcal{R})$ that satisfies Axiom C4$^*{}_{29}$. We will show that the special choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ based on C^*, defined as

$$\begin{aligned}
f\in C(A)\Leftrightarrow(\exists\lambda\in\mathbb{R}_{>0},A^*\in\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R})),H\in A^*)&\\
\big(\varphi_r(A^*-\{H\})=\lambda(A-\{f\})\text{ and }H\in C^*(A^*)\big)&\qquad(2.11)
\end{aligned}$$

for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}\times\mathcal{R}_r))$ and f in A, satisfies Axioms C4a$_{20}$ and C4b$_{20}$ and $\tilde{\varphi}_r C=C^*$. We first show that C satisfies Axioms C4a$_{20}$ and C4b$_{20}$. For Axiom C4a$_{20}$, we use its equivalent form (C4a.1)$_{21}$. Consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}\times\mathcal{R}_r))$, any f in A, and any μ in $\mathbb{R}_{>0}$, and assume that $f\in C(A)$. To show that then $\mu f\in C(\mu A)$, it suffices to consider $\lambda':=\frac{\lambda}{\mu}$ in Equation (2.11), and note that $\lambda(A-\{f\})=\lambda'(\mu A-\{\mu f\})$; then the desired statement follows at once from Equation (2.11). For Axiom C4b$_{20}$, we use its equivalent form (C4b.1)$_{21}$. Consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}\times\mathcal{R}_r))$, any f in A, and any g in $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$, and assume that $f\in C(A)$. We show that then $f+g\in C(A+\{g\})$. To this end, it suffices to note that $A-\{f\}=(A+\{g\})-\{f+g\}$; then the desired statement follows at once from Equation (2.11). So C as defined in Equation (2.11) does indeed satisfy Axioms C4a$_{20}$ and C4b$_{20}$, and is therefore a suitable candidate for showing that $\tilde{\varphi}_r C=C^*$.

We now finish the proof by showing that $\tilde{\varphi}_r C=C^*$. To do so, we consider any A^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$, and show that $\tilde{\varphi}_r C(A^*)\subseteq C^*(A^*)$ and $C^*(A^*)\subseteq\tilde{\varphi}_r C(A^*)$. To show that $\tilde{\varphi}_r C(A^*)\subseteq C^*(A^*)$, consider any H in $\tilde{\varphi}_r C(A^*)$. By the definition of $\tilde{\varphi}_r$

(Equation (2.9)$_{32}$) then $H \in \varphi_r^{-1}(C(\varphi_r A^*))$, and therefore $\varphi_r H \in C(\varphi_r A^*)$. Using Equation (2.11)↜ [with $A = \varphi_r A^*$ and $f = \varphi_r H$], we find that then

$$\varphi_r(A^{*\prime} - \{H'\}) = \lambda'(\varphi_r A^* - \{\varphi_r H\}) \text{ and } H' \in C^*(A^{*\prime})$$

for some λ' in $\mathbb{R}_{>0}$, $A^{*\prime}$ in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$ and $H' \in A^{*\prime}$. By Lemma 19$_{31}$ and since $\lambda'(\varphi_r A^* - \{\varphi_r H\}) = \varphi_r(\lambda'(A^* - \{H\}))$, infer that then

$$A^{*\prime} - \{H'\} = \lambda'(A^* - \{H\}) \text{ and } H' \in C^*(A^{*\prime}),$$

and because C^* satisfies Axiom C4*$_{29}$, and using Proposition 18$_{30}$ this means that indeed $H \in C^*(A^*)$. So we have shown that $H \in C^*(A^*)$, and since the choice of H was arbitrary in $\tilde{\varphi}_r C(A^*)$, therefore indeed $\tilde{\varphi}_r C(A^*) \subseteq C^*(A^*)$.

To show that $C^*(A^*) \subseteq \tilde{\varphi}_r C(A^*)$, consider any H in $C^*(A^*)$. Let $A := \varphi_r A^*$, $f := \varphi_r H$ and $\lambda := 1$, then

$$\varphi_r(A^* - \{H\}) = \lambda(A - \{f\}),$$

whence by Equation (2.11)↜ $\varphi_r H = f \in C(A) = C(\varphi_r A^*)$. Since φ_r is one-to-one on $\mathcal{H}(\mathcal{X},\mathcal{R})$ (see Lemma 19$_{31}$), therefore indeed $H \in \tilde{\varphi}_r C(A^*)$. So we have shown that $\tilde{\varphi}_r C = C^*$, which completes the proof. □

Specifying a choice function C^* on $\mathcal{H}$ induces a strict preference relation on the reward set, as follows. With any reward r in $\mathcal{R}$ we can associate the constant and degenerate lottery H_r by letting

$$H_r(x,s) := \begin{cases} 1 & \text{if } s = r \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } x \text{ in } \mathcal{X} \text{ and } s \text{ in } \mathcal{R}.$$

This is the lottery that associates the certain reward r with all states. Then a reward r is strictly preferred to a reward s when $H_s \in R^*(\{H_r, H_s\})$.

Definition 12 (C^* has worst reward r)**.** *Consider any reward r in $\mathcal{R}$, and any choice function C^* on $\mathcal{H}(\mathcal{X},\mathcal{R})$. We say that C^** has worst reward *r if $H_r \in R^*(\{H, H_r\})$ for all H in $\mathcal{H}(\mathcal{X},\mathcal{R}) \setminus \{H_r\}$.*

The worst reward is unique when C^* satisfies Axiom C1*$_{29}$: indeed, if there were two different worst rewards r and s, by Definition 12 then $\{H_r, H_s\} = R^*(\{H_r, H_s\})$, contradicting Axiom C1*$_{29}$.

The notion of *having a worst reward* is closely related with what would be the natural translation of Axiom C2$_{20}$ to choice functions C^* on $\mathcal{H}(\mathcal{X},\mathcal{R})$: if C^* satisfies

$$(\forall H_1, H_2 \in \mathcal{H})\Big(\big(H_1 \neq H_2 \text{ and } (\forall s \in \mathcal{R}_r)(H_1(\bullet,s) \leq H_2(\bullet,s))\big) \Rightarrow H_1 \in R^*(\{H_1, H_2\})\Big) \quad (2.12)$$

for some r in $\mathcal{R}$, then we say that C^* *satisfies the dominance relation for worst reward r*.

Proposition 22. *Consider any r in $\mathcal{R}$ and any choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ that satisfies Axiom* C4b$_{20}$. *Then $\tilde{\varphi}_r C$ satisfies the dominance relation for worst reward r (Equation* (2.12)*) if and only if $\tilde{\varphi}_r C$ has worst reward r.*

Proof. For the direct implication, consider any H in $\mathcal{H}(\mathcal{X},\mathcal{R})\setminus\{H_r\}$. Then $H_r(\bullet,s) = 0 \leq H(\bullet,s)$ for all s in $\mathcal{R}_r$, and also $H \neq H_r$, whence indeed $H_r \in \tilde{\varphi}_r R(\{H,H_r\})$, because by assumption $\tilde{\varphi}_r C$ satisfies Equation (2.12) for r.

For the converse implication, consider any H_1 and H_2 in $\mathcal{H}(\mathcal{X},\mathcal{R})$ such that $H_1 \neq H_2$ and $H_1(\bullet,s) \leq H_2(\bullet,s)$ for all s in $\mathcal{R}_r$. Then $\varphi_r H_1 < \varphi_r H_2$, whence $0 < \varphi_r(H_2 - H_1)$. Observe that for the horse lottery H' in $\mathcal{H}(\mathcal{X},\mathcal{R})$ defined by

$$H'(\bullet,s) := \begin{cases} H_2(\bullet,s) - H_1(\bullet,s) & \text{if } s \in \mathcal{R}_r \\ 1 - \sum_{s\in\mathcal{R}_r}\big(H_2(\bullet,s) - H_1(\bullet,s)\big) & \text{if } s = r, \end{cases}$$

we have that $\varphi_r H' = \varphi_r(H_2 - H_1)$. Because $\tilde{\varphi}_r C$ is assumed to have worst reward r, we know that in particular $H_r \in \tilde{\varphi}_r R(\{H',H_r\})$, so we infer from Equation (2.9)$_{32}$ that $0 = \varphi_r H_r \in R(\{\varphi_r H_r, \varphi_r H'\}) = R(\{0, \varphi_r H_2 - \varphi_r H_1\})$. Now use Axiom C4b$_{20}$ to infer that $\varphi_r H_1 \in R(\{\varphi_r H_1, \varphi_r H_2\})$, whence indeed $H_1 \in \tilde{\varphi}_r R(\{H_1,H_2\})$, by Equation (2.9)$_{32}$. □

Applying the lifting $\tilde{\varphi}_r$ furthermore preserves coherence:

Theorem 23. *Consider any reward r in $\mathcal{R}$, and any choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ that satisfies Axioms* C4a$_{20}$ *and* C4b$_{20}$. *Then the following statements hold:*

(i) *C satisfies Axiom* C1$_{20}$ *if and only if $\tilde{\varphi}_r C$ satisfies Axiom* C1$^*{}_{29}$*;*
(ii) *C satisfies Axiom* C2$_{20}$ *if and only if $\tilde{\varphi}_r C$ has worst reward r;*
(iii) *C satisfies Axiom* C3a$_{20}$ *if and only if $\tilde{\varphi}_r C$ satisfies Axiom* C3*a$_{29}$*;*
(iv) *C satisfies Axiom* C3b$_{20}$ *if and only if $\tilde{\varphi}_r C$ satisfies Axiom* C3*b$_{29}$*;*
(v) *$\tilde{\varphi}_r C$ satisfies Axiom* C4$^*{}_{29}$*;*
(vi) *C satisfies Property* C5$_{25}$ *if and only if $\tilde{\varphi}_r C$ satisfies Axiom* C5$^*{}_{29}$.

Proof. For the direct implication of (i), assume that C satisfies Axiom C1$_{20}$. Consider any A^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$. Then $\tilde{\varphi}_r C(A^*) = \varphi_r^{-1} C(\varphi_r A) \neq \emptyset$.

For the converse implication, assume that $\tilde{\varphi}_r C$ satisfies Axiom C1$^*{}_{29}$. Consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}\times\mathcal{R}_r))$. By Lemma 20$_{32}$, there are λ in $\mathbb{R}_{>0}$ and g in $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ such that $\frac{1}{\lambda}(A+\{g\}) = \varphi_r A^*$ for some A^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$. Applying Axioms C4a$_{20}$ and C4b$_{20}$ and the definition of $\tilde{\varphi}_r$ [Equation (2.9)$_{32}$], we infer that indeed

$$C(A) = \lambda C\Big(\frac{1}{\lambda}(A+\{g\})\Big) - \{g\} = \lambda C(\varphi_r A^*) - \{g\} = \lambda\varphi_r\tilde{\varphi}_r C(A^*) - \{g\} \neq \emptyset.$$

For the direct implication of (ii), assume that C satisfies Axiom C2$_{20}$. Consider any H_1 and H_2 in $\mathcal{H}(\mathcal{X},\mathcal{R})$ such that $H_1 \neq H_2$ and $H_1(\bullet,s) \leq H_2(\bullet,s)$ for all s in $\mathcal{R}_r$. Then $\varphi_r H_1 < \varphi_r H_2$, so Axiom C2$_{20}$ guarantees that $\varphi_r H_1 \in R(\{\varphi_r H_1, \varphi_r H_2\})$. Equation (2.9)$_{32}$ turns this into $H_1 \in \tilde{\varphi}_r R(\{H_1,H_2\})$. Proposition 22 now tells us that then $\tilde{\varphi}_r C$ has worst reward r.

For the converse implication, assume that $\tilde{\varphi}_r C$ has worst reward r. Consider any f_1 and f_2 in $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ such that $f_1 < f_2$. Let

$$\lambda := \sup_{x\in\mathcal{X}} \sum_{s\in\mathcal{R}_r} (f_2(x,s) - f_1(x,s)) > 0,$$

being a real number because $f_2 - f_1$ is a bounded gamble, and therefore so is the partial map $\sum_{s\in\mathcal{R}_r}(f_2(\bullet,s) - f_1(\bullet,s))$ because of the finiteness of $\mathcal{R}_r$, so its supremum is finite. Then clearly $\frac{1}{\lambda}(f_2 - f_1) = \varphi_r H$ for some H in $\mathcal{H}(\mathcal{X},\mathcal{R})$. Also, $H \neq H_r$ because $f_1 \neq f_2$. Using the assumption that $\tilde{\varphi}_r C$ has worst reward r, we find that then $H_r \in \tilde{\varphi}_r R(\{H_r,H\})$. As a consequence, by Equation (2.9)$_{32}$, we find that $0 = \varphi_r H_r \in R(\{0,\varphi_r H\}) = R(\{0,\frac{1}{\lambda}(f_2 - f_1)\})$. Using Axiom C4a$_{20}$ we infer that $0 \in R(\{0, f_2 - f_1\})$, and using Axiom C4b$_{20}$ that indeed $f_1 \in R(\{f_1,f_2\})$.

For the direct implication of (iii)$_{\curvearrowleft}$, assume that C satisfies Axiom C3a$_{20}$. Consider any A^*, A_1^* and A_2^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$ and assume that $A_1^* \subseteq \tilde{\varphi}_r R(A_2^*)$ and $A_2^* \subseteq A^*$. Then $\varphi_r A_1^* \subseteq R(\varphi_r A_2^*)$ by Equation (2.9)$_{32}$, and $\varphi_r A_1^* \subseteq \varphi_r A^*$. Use Axiom R3a$_{20}$ to infer that then $\varphi_r A_1^* \subseteq R(\varphi_r A^*)$, whence indeed $A_1^* \subseteq \tilde{\varphi}_r R(A^*)$ by Equation (2.9)$_{32}$.

For the converse implication, assume that $\tilde{\varphi}_r C$ satisfies Axiom C3*a$_{29}$. Consider any A, A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(\mathcal{X}\times\mathcal{R}_r))$ and assume that $A_1 \subseteq R(A_2)$ and $A_2 \subseteq A$. Use Lemma 20$_{32}$ to find λ in $\mathbb{R}_{>0}$ and g in $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ such that $\frac{1}{\lambda}(A + \{g\}) = \varphi_r A^*$ for some A^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$. Analogously, we find that $\frac{1}{\lambda}(A_2 + \{g\}) = \varphi_r(A_2^*)$ for some $A_2^* \subseteq A^*$. $A_1 \subseteq R(A_2)$ implies $A_1 \subseteq A_2$, so also $\frac{1}{\lambda}(A_1 + \{g\}) = \varphi_r(A_1^*)$ for some $A_1^* \subseteq A_2^*$. Using Axioms C4a$_{20}$ and C4b$_{20}$, we infer from the assumptions that $\frac{1}{\lambda}(A_1 + \{g\}) \subseteq R\big(\frac{1}{\lambda}(A_2 + \{g\})\big)$, or in other words, $\varphi_r A_1^* \subseteq R(\varphi_r A_2^*)$. Equation (2.9)$_{32}$ then yields that $A_1^* \subseteq \tilde{\varphi}_r R(A_2^*)$. As a result, using Axiom C3*a$_{29}$, $A_1^* \subseteq \tilde{\varphi}_r R(A^*)$, which, again applying Equation (2.9)$_{32}$, results in $\frac{1}{\lambda}(A_1 + \{g\}) = \varphi_r A_1^* \subseteq R(\varphi_r A^*) = R\big(\frac{1}{\lambda}(A + \{g\})\big)$, and as a consequence, by Axioms C4a$_{20}$ and C4b$_{20}$, we find eventually that indeed $A_1 \subseteq R(A)$.

For the direct implication of (iv)$_{\curvearrowleft}$, assume that C satisfies Axiom C3b$_{20}$. Consider any A^*, A_1^* and A_2^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$ and assume that $A_1^* \subseteq \tilde{\varphi}_r R(A_2^*)$ and $A^* \subseteq A_1^*$. Then $\varphi_r A_1^* \subseteq R(\varphi_r A_2^*)$ by Equation (2.9)$_{32}$, and $\varphi_r A^* \subseteq \varphi_r A_1^*$. Use Axiom R3b$_{20}$ to infer that then $\varphi_r(A_1^* \setminus A^*) = (\varphi_r A_1^*) \setminus (\varphi_r A) \subseteq R((\varphi_r A_2^*) \setminus (\varphi_r A^*)) = R(\varphi_r(A_2^* \setminus A^*))$, whence indeed $A_1^* \setminus A^* \subseteq \tilde{\varphi}_r R(A_2^* \setminus A^*)$.

For the converse implication, assume that $\tilde{\varphi}_r C$ satisfies Axiom C3*b$_{29}$. Consider any A, A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(\mathcal{X}\times\mathcal{R}_r))$ and assume that $A_1 \subseteq R(A_2)$ and $A \subseteq A_1$. Use Lemma 20$_{32}$ to find λ in $\mathbb{R}_{>0}$ and g in $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ such that $\frac{1}{\lambda}(A_2 + \{g\}) = \varphi_r A_2^*$ for some A_2^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$. $A_1 \subseteq R(A_2)$ implies $A_1 \subseteq A_2$, whence $\frac{1}{\lambda}(A_1 + \{g\}) = \varphi_r(A_1^*)$ for some $A_1^* \subseteq A_2^*$, and analogously, $\frac{1}{\lambda}(A + \{g\}) = \varphi_r(A^*)$ for some $A^* \subseteq A_1^*$. Using Axioms C4a$_{20}$ and C4b$_{20}$ we find that $\frac{1}{\lambda}(A_1 + \{g\}) \subseteq R\big(\frac{1}{\lambda}(A_2 + \{g\})\big)$, or in other words, $\varphi_r A_1^* \subseteq R(\varphi_r A_2^*)$. Equation (2.9)$_{32}$ then tells us that $A_1^* \subseteq \tilde{\varphi}_r R(A_2^*)$, which, using Axiom C3*b$_{29}$, results in $A_1^* \setminus A^* \subseteq \tilde{\varphi}_r R(A_2^* \setminus A^*)$. Again applying Equation (2.9)$_{32}$ results in

$$\begin{aligned}
\frac{1}{\lambda}((A_1 \setminus A) + \{g\}) = \frac{1}{\lambda}(A_1 + \{g\}) \setminus \frac{1}{\lambda}(A + \{g\}) &= (\varphi_r A_1^*) \setminus (\varphi_r A^*) = \varphi_r(A_1^* \setminus A^*) \\
&\subseteq R(\varphi_r(A_2^* \setminus A^*)) = R((\varphi_r A_2^*) \setminus (\varphi_r A^*)) \\
&= R\Big(\frac{1}{\lambda}(A_2 + \{g\}) \setminus \frac{1}{\lambda}(A + \{g\})\Big) = R\Big(\frac{1}{\lambda}((A_2 \setminus A) + \{g\})\Big),
\end{aligned}$$

and as a consequence, by Axioms C4a$_{20}$ and C4b$_{20}$, we find eventually that indeed $A_1 \setminus A \subseteq R(A_2 \setminus A)$.

For (v)$_{35}$, since by Proposition 21$_{32}$, $\tilde{\varphi}_r$ is a bijection between the choice functions on $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ that satisfy Axioms C4a$_{20}$ and C4b$_{20}$, and the choice functions on $\mathcal{H}(\mathcal{X},\mathcal{R})$ that satisfy Axiom C4*$_{29}$, therefore indeed $\tilde{\varphi}_r C$ satisfies Axiom C4*$_{29}$.

For the direct implication of (vi)$_{35}$, assume that C satisfies Property C5$_{25}$. Consider any A^* and A_1^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$ and assume that $A^* \subseteq A_1^* \subseteq \operatorname{conv}(A^*)$. Then $\varphi_r A^* \subseteq \varphi_r A_1^* \subseteq \operatorname{conv}(\varphi_r A^*)$, whence $C(\varphi_r A^*) \subseteq C(\varphi_r A_1^*)$ by Property C5$_{25}$. Use Equation (2.9)$_{32}$ to infer that then indeed $\tilde{\varphi}_r C(A^*) \subseteq \tilde{\varphi}_r C(A_1^*)$.

For the converse implication, assume that $\tilde{\varphi}_r C$ satisfies Axiom C5*$_{29}$. Consider any A and A_1 in $\mathcal{Q}(\mathcal{L}(\mathcal{X}\times\mathcal{R}_r))$ and assume that $A \subseteq A_1 \subseteq \operatorname{conv}(A)$. Use Lemma 20$_{32}$ to find λ in $\mathbb{R}_{>0}$ and g in $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ such that $\frac{1}{\lambda}(A_1 + \{g\}) = \varphi_r A_1^*$ for some A_1^* in $\mathcal{Q}(\mathcal{H}(\mathcal{X},\mathcal{R}))$, and analogously, $\frac{1}{\lambda}(A + \{g\}) = \varphi_r(A^*)$ for some $A^* \subseteq A_1^*$. From $A_1 \subseteq \operatorname{conv}(A)$ infer that $\frac{1}{\lambda}(A_1 + \{g\}) \subseteq \operatorname{conv}(\frac{1}{\lambda}(A + \{g\}))$, or in other words, $\varphi_r A_1^* \subseteq \operatorname{conv}(\varphi_r A^*)$. Then we claim that $A_1^* \subseteq \operatorname{conv}(A^*)$. To prove this, consider any H in A_1^*. Then there are n in $\mathbb{N}$, H_i in A, and $\alpha_i \geq 0$ such that $\sum_{i=1}^n \alpha_i = 1$ and $H(\bullet,s) = \sum_{i=1}^n \alpha_i H_i(\bullet,s)$ for all s in $\mathcal{R}_r$. Moreover,

$$\begin{aligned} H(\bullet,r) &= 1 - \sum_{s\in\mathcal{R}_r} H(\bullet,s) = 1 - \sum_{s\in\mathcal{R}_r}\sum_{i=1}^n \alpha_i H_i(\bullet,s) \\ &= \sum_{i=1}^n \alpha_i - \sum_{i=1}^n \alpha_i \sum_{s\in\mathcal{R}_r} H_i(\bullet,s) = \sum_{i=1}^n \alpha_i \Big(1 - \sum_{s\in\mathcal{R}_r} H_i(\bullet,s)\Big) = \sum_{i=1}^n \alpha_i H_i(\bullet,r), \end{aligned}$$

so indeed $H \in \operatorname{conv}(A^*)$. Use Axiom C5*$_{29}$ to infer that then $\tilde{\varphi}_r C(A^*) \subseteq \tilde{\varphi}_r C(A_1^*)$. Equation (2.9)$_{32}$ turns this into $C\big(\frac{1}{\lambda}(A + \{g\})\big) = C(\varphi_r A^*) \subseteq C(\varphi_r A_1^*) = C\big(\frac{1}{\lambda}(A_1 + \{g\})\big)$, which by Axioms C4a$_{20}$ and C4b$_{20}$, results in $C(A) \subseteq C(A_1)$. □

We conclude that our discussion of choice functions on linear spaces subsumes the treatment of choice functions on horse lotteries satisfying Axiom C4*$_{29}$. By combining Proposition 21$_{32}$ and Theorem 23$_{35}$, we have the following important result: the coherent choice functions on vector-valued gambles $\mathcal{L}(\mathcal{X}\times\mathcal{R}_r)$ are isomorphic to the choice functions on horse lotteries that satisfy Axioms C1*$_{29}$, C2*$_{29}$, C3*a$_{29}$, C3*b$_{29}$ and C4*$_{29}$, under the assumption that there is a unique worst reward in r in $\mathcal{R}$. Therefore we have embedded the choice functions considered in Reference [67] in our account of choice functions: it suffices to consider as option set a particular set of vector-valued gambles. Using this connection, all the results that we will prove later on for coherent choice functions on vector-valued gambles, are also applicable to choice functions on horse lotteries that satisfy the corresponding rationality axioms. However, as mentioned, Seidenfeld et al. [67] allow for infinite but closed option sets, something we don't allow.

2.5 CONSEQUENCES OF COHERENCE

In this section, we will investigate some of the consequences of coherence. We divide the results into three parts.

Note that, due to Proposition 13_{22}, the coherence axioms for the different types of choice models correspond *one by one* with each other: it was our deliberate choice to state them in such an order and form that this should be the case. However, another choice could be to state more elegant forms of them, such that the coherence axioms *as a whole* correspond for different the choice models. For instance, the requirement in Axiom $\mathrm{C2}_{20}$ that $u \notin C(\{u,v\})$, is under Axiom $\mathrm{C1}_{20}$ equivalent to $\{v\} = C(\{u,v\})$, which can be perceived as more elegant, but cannot be derived from Axiom $\mathrm{C2}_{20}$—or Axiom $\mathrm{R2}_{20}$ for its corresponding rejection function—alone. In the first part of this section, Section 2.5.1, we will collect some more elegant forms of the axioms, which are equivalent under coherence.

In the second part, Section $2.5.2_{40}$, we will derive some consequences of coherence for choice functions—or rejection functions or choice relations for that matter—that will be useful later on in this thesis.

In the third part, Section $2.5.3_{43}$, we consider an important preorder on the option sets, and investigate its connections with coherence for choice functions.

2.5.1 *Equivalent forms of the axioms*

We have learned from dire experience that in verifying whether a rejection function is coherent, Axiom $\mathrm{R3}_{20}$ is often hardest to check. But under various additional conditions, it has a number of equivalent formulations that may simplify this task, which we consider in the next two propositions.

Proposition 24. *Let R be any rejection function on $\mathcal{Q}$, and consider the following statements:*

(i) *R satisfies Axiom* $\mathrm{R3a}_{20}$*;*
(ii) $(\forall A \in \mathcal{Q})(\forall u \in R(A))(\forall v \in \mathcal{V})u \in R(A \cup \{v\})$*;*
(iii) $(\forall A \in \mathcal{Q})(\forall v \in \mathcal{V})(0 \in R(A) \Rightarrow 0 \in R(A \cup \{v\}))$.

Then (i) *and* (ii) *are equivalent, and imply* (iii)*. Moreover, if R satisfies Axiom* $\mathrm{R4b}_{20}$*, then* (i)*,* (ii)*, and* (iii) *are equivalent.*

Proof. That (i) implies (ii), follows immediately from Axiom $\mathrm{R3a}_{20}$ [with $\tilde{A} \coloneqq A \cup \{v\}$, $\tilde{A}_1 \coloneqq \{u\}$ and $\tilde{A}_2 \coloneqq A$].

To prove that (ii) implies (i), we assume that (ii) holds, and we prove that R satisfies (i). Let $A_1 \coloneqq \{u_1,\dots,u_n\}$, $A_2 \coloneqq A_1 \cup \{v_1,\dots,v_m\}$ and $A \coloneqq A_2 \cup \{w_1,\dots,w_r\}$, where n belongs to $\mathbb{N}$, and m and r to $\mathbb{Z}_{\geq 0}$, and assume that $A_1 \subseteq R(A_2)$. Consider any j in $\{1,\dots,n\}$, then we have to prove that $u_j \in R(A) = R(A_2 \cup \{w_1,\dots,w_r\})$. Since $u_j \in R(A_2)$, it follows from (ii) that $u_j \in R(A_2 \cup \{w_1\})$, and therefore, again using (ii), also that $u_j \in R(A_2 \cup \{w_1, w_2\})$. We can go on in this way until we reach the desired statement, that $u_j \in R(A_2 \cup \{w_1, \dots, w_r\})$, after a finite number of steps.

The last statement is now immediate, once we realise that Axiom R4b$_{20}$ implies that $u \in R(A)$ is equivalent to $0 \in R(A - \{u\})$, for any A in $\mathcal{Q}$ and u in A. □

Proposition 25. *Let R be any rejection function on $\mathcal{Q}$, and consider the following statements:*

(i) *R satisfies Axiom* R3b$_{20}$*;*

(ii) $(\forall A \in \mathcal{Q})(\forall u \in R(A))(\forall v \in R(A) \setminus \{u\}) u \in R(A \setminus \{v\})$*;*

(iii) $(\forall A \in \mathcal{Q})(\forall u \in R(A)) u \in R(\{u\} \cup A \setminus R(A))$*;*

(iv) $(\forall A \in \mathcal{Q})(\forall v \in R(A) \setminus \{0\})(0 \in R(A) \Rightarrow 0 \in R(A \setminus \{v\}))$*;*

(v) $(\forall A \in \mathcal{Q})(0 \in R(A) \Rightarrow 0 \in R(\{0\} \cup A \setminus R(A)))$.

Generally, (i) *and* (ii) *are equivalent, and imply* (iii)*. Moreover,* (i)*,* (ii) *and* (iii) *are equivalent if R satisfies Axiom* R3a$_{20}$*. Finally, if R satisfies Axiom* R4b$_{20}$*, then* (ii) *and* (iv) *are equivalent, and so are* (iii) *and* (v).

Proof. That (i) implies (ii), follows immediately from Axiom R3b$_{20}$ [with $\tilde{A} := \{v\}$, $\tilde{A}_1 := \{u, v\}$ and $\tilde{A}_2 := A$].

To prove that (ii) implies (i), we assume that (ii) holds, and we prove that R satisfies Axiom R3b$_{20}$. Let $A := \{u_1, \dots, u_n\}$, $A_1 := A \cup \{v_1, \dots, v_m\}$ and $A_2 := A_1 \cup \{w_1, \dots, w_r\}$, where $n \in \mathbb{N}$ and $m, r \in \mathbb{Z}_{\geq 0}$, and assume that $A_1 \subseteq R(A_2)$. Consider any j in $\{1, \dots, m\}$, then we have to prove that $v_j \in R(\{v_1, \dots, v_m, w_1, \dots, w_r\}) = R(A_2 \setminus \{u_1, \dots, u_n\})$. Since $\{u_1, u_2\} \subseteq R(A_2)$ and $\{v_j, u_1\} \subseteq R(A_2)$, it follows from (ii) that $\{u_2, v_j\} \subseteq R(A_2 \setminus \{u_1\})$, whence, again using (ii), $v_j \in R(A_2 \setminus \{u_1, u_2\})$. Also, $\{u_1, u_3\} \subseteq R(A_2)$, whence $u_3 \in R(A_2 \setminus \{u_1\})$ using (ii). Since we already know that also $u_2 \in R(A_2 \setminus \{u_1\})$, we infer that $u_3 \in R(A_2 \setminus \{u_1, u_2\})$, again using (ii) . In turn, this implies that $v_j \in R(A_2 \setminus \{u_1, u_2, u_3\})$. We can go on in this way until we reach the desired statement, that $v_j \in R(A_2 \setminus \{u_1, \dots, u_n\})$, after a finite number of steps.

That (i) implies (iii), follows immediately from Axiom R3b$_{20}$ [with $\tilde{A} := R(A) \setminus \{u\}$, $\tilde{A}_1 := R(A)$ and $\tilde{A}_2 := A$].

To prove that (iii) implies (i) under Axiom R3a$_{20}$, consider any A, A_1 and A_2 in $\mathcal{Q}$ and assume that $A \subseteq A_1 \subseteq R(A_2)$. Then in particular $u \in R(A_2)$, and therefore, using (iii), $u \in R(\{u\} \cup A_2 \setminus R(A_2))$, for every u in $A_1 \setminus A$. Applying Axiom R3a$_{20}$, we infer that $u \in R(A_2 \setminus A)$ for every u in $A_1 \setminus A$, whence indeed $A_1 \setminus A \subseteq R(A_2 \setminus A)$.

The last statement is now immediate, once we realise that Axiom R4b$_{20}$ implies that $u \in R(A)$ is equivalent to $0 \in R(A - \{u\})$, for any A in $\mathcal{Q}$ and u in A. □

Using Proposition 25, we can find an easy characterisation of Axiom R1$_{20}$ under Axioms R3b$_{20}$ and R4b$_{20}$:

Corollary 26. *Consider any rejection function R that satisfies Axioms* R3b$_{20}$ *and* R4b$_{20}$*. Then the following two statements are equivalent:*

(i) *R satisfies Axiom* R1$_{20}$*;*

(ii) $0 \notin R(\{0\})$.

Proof. That (i) implies (ii) follows immediately by considering $A := \{0\}$ in Axiom R1$_{20}$. It therefore suffices to show that (ii) implies (i); we will prove the contraposition. Assume that R does not satisfy Axiom R1$_{20}$. Therefore, we have that

$A = R(A)$ for some A in $\mathcal{Q}$. Consider any u in A, then by Proposition $25_{\curvearrowleft}$(iii) we find that $u \in R(\{u\} \cup A \setminus R(A)) = R(\{u\} \cup A \setminus A) = R(\{u\})$. By Axiom $\mathrm{R4b}_{20}$ therefore indeed $0 \in R(\{0\})$. □

As we will see later on, this characterisation will be very helpful in establishing that a rejection function satisfies Axiom $\mathrm{R1}_{20}$.

For Axiom $\mathrm{R4}_{20}$, we have the following useful characterisation:

Proposition 27. *Consider any choice function R. Then the following statements are equivalent:*

(i) *R satisfies Axiom* $\mathrm{R4}_{20}$*;*

(ii) *$R(\lambda A + \{u\}) = \lambda R(A) + \{u\}$ for all A in $\mathcal{Q}$, λ in $\mathbb{R}_{>0}$ and u in $\mathcal{V}$.*

Proof. To show that (i) implies (ii), consider any rejection function R that satisfies Axiom $\mathrm{R4a}_{20}$, and any A in $\mathcal{Q}$, λ in $\mathbb{R}_{>0}$ and u in $\mathcal{V}$. Consider the following equalities:

$$\begin{aligned} R(\lambda A + \{u\}) = R\big(\lambda(A + 1/\lambda\{u\})\big) &= \lambda R\big(A + 1/\lambda\{u\}\big) && \text{by Equation (R4a.3)}_{21} \\ &= \lambda(R(A) + 1/\lambda\{u\}) && \text{by Equation (R4b.3)}_{21} \\ &= \lambda R(A) + \{u\}. \end{aligned}$$

To show that (ii) implies (i), infer that, by letting $u = 0$, $R(\lambda A) = \lambda R(A)$ for all A in $\mathcal{Q}$ and λ in $\mathbb{R}_{>0}$, whence by Lemma 12_{21} R satisfies Axiom $\mathrm{R4a}_{20}$. Furthermore, by letting $\lambda = 1$, $R(A + \{u\}) = R(A) + \{u\}$, so R satisfies Axiom $\mathrm{R4b}_{20}$ by Lemma 12_{21}. □

2.5.2 *Consequences*

In this section, we collect useful properties of coherent choice models. First, let us show that a coherent choice function is an idempotent operator on $\mathcal{Q}$:

Proposition 28. *Any coherent choice function C is* insensitive to the omission of non-chosen options *(see Reference [33, Definition 11]): $C(A') = C(A)$ for all A and A' in $\mathcal{Q}$ such that $C(A) \subseteq A' \subseteq A$. As an immediate consequence, C is idempotent: $C \circ C = C$, or, in other words, $C(C(A)) = C(A)$ for all A in $\mathcal{Q}$.*

Proof. Consider any A and A' in $\mathcal{Q}$ such that $C(A) \subseteq A' \subseteq A$. Let R be the rejection function corresponding to C. That $C(A) = C(A')$ is, by the requirement in Definition 1_{14} that $C(A') \subseteq A'$, equivalent to $A' \cap C(A) = C(A')$, and hence also to $R(A') = A' \setminus (A' \cap C(A)) = A' \cap C(A)^c = A' \cap R(A)$, and therefore it suffices to prove that $R(A') = A' \cap R(A)$. Since $A' \subseteq A$, by Axiom $\mathrm{R3a}_{20}$ we have that $R(A') \subseteq R(A)$, and by the requirement in Definition 2_{14} that $R(A') \subseteq A'$, therefore also $R(A') \subseteq A' \cap R(A)$, so it suffices to show that $A' \cap R(A) \subseteq R(A')$. To establish this, consider any u in $A' \cap R(A)$. Since $C(A) \subseteq A' \subseteq A$, infer that $A = A' \cup A''$ for some $A'' \subseteq R(A)$, and we may assume without loss of generality that A'' and A' are disjoint, and therefore $A' = A \setminus A''$ and $u \notin A''$. Since $u \in R(A)$, by Proposition $25_{\curvearrowleft}$(iii) therefore $u \in R(\{u\} \cup A \setminus R(A))$, and hence, since $A'' \subseteq R(A)$, by Axiom $\mathrm{R3a}_{20}$ therefore $u \in R(\{u\} \cup A \setminus A'')$. Using the fact that $u \notin A''$, we find that $u \in A \setminus A''$ and hence indeed $u \in R(A \setminus A'') = R(A')$. □

The following result implies that any coherent choice relation is a strict partial order:

Proposition 29. *Consider any choice relation $\lhd$. Then the following two statements hold:*
(i) *If $\lhd$ satisfies Axiom $\lhd 1_{20}$, then is $\lhd$ irreflexive.*
(ii) *If $\lhd$ satisfies Axioms $\lhd 3a_{20}$ and $\lhd 3b_{20}$, then is $\lhd$ transitive.*
As a consequence, if $\lhd$ satisfies Axioms $\lhd 1_{20}$, $\lhd 3a_{20}$ and $\lhd 3b_{20}$, then $\lhd$ is a strict partial order on $\mathcal{Q}$.

Proof. For (i), irreflexivity is precisely the rationality Axiom $\lhd 1_{20}$.

For (ii), assume that $\lhd$ satisfies Axioms $\lhd 3a_{20}$ and $\lhd 3b_{20}$, and consider any A_1, A_2 and A_3 in $\mathcal{Q}$ such that $A_1 \lhd A_2$ and $A_2 \lhd A_3$. By Definition 3_{15}(i), this implies that $A_1 \lhd A_1 \cup A_2$ and $A_2 \lhd A_2 \cup A_3$. Use Axiom $\lhd 3a_{20}$ to infer that then $A_1 \lhd A_1 \cup A_2 \cup A_3$ and $A_2 \lhd A_1 \cup A_2 \cup A_3$, and hence, by Definition 3_{15}(ii) therefore $A_1 \cup A_2 \lhd A_1 \cup A_2 \cup A_3$. Now, use Axiom $\lhd 3b_{20}$ with $A \coloneqq A_2 \setminus (A_1 \cup A_3) \subseteq A_1 \cup A_2$ to infer that $(A_1 \cup A_2) \setminus A \lhd (A_1 \cup A_2 \cup A_3) \setminus A$. By repeated application of De Morgan's laws, infer the following equalities:

$$\begin{aligned}(A_1 \cup A_2) \setminus A &= (A_1 \cup A_2) \setminus (A_2 \setminus (A_1 \cup A_3))\\ &= (A_1 \cup A_2) \cap (A_2^c \cup A_1 \cup A_3)\\ &= ((A_1 \cup A_2) \cap (A_2^c \cup A_1)) \cup ((A_1 \cup A_2) \cap A_3)\\ &= A_1 \cup ((A_1 \cup A_2) \cap A_3) = A_1 \cup (A_1 \cap A_3) \cup (A_2 \cap A_3) = A_1 \cup (A_2 \cap A_3),\end{aligned}$$

and

$$\begin{aligned}(A_1 \cup A_2 \cup A_3) \setminus A &= (A_1 \cup A_2 \cup A_3) \setminus (A_2 \setminus (A_1 \cup A_3))\\ &= (A_1 \cup A_2 \cup A_3) \cap (A_2^c \cup A_1 \cup A_3) = A_1 \cup A_3,\end{aligned}$$

so $A_1 \cup (A_2 \cap A_3) \lhd A_1 \cup A_3$. Using Definition 3_{15}(ii) again, therefore $A_1 \lhd A_1 \cup A_3$, and by Definition 3_{15}(i) therefore indeed $A_1 \lhd A_3$.

For the consequence, if $\lhd$ satisfies Axioms $\lhd 1_{20}$, $\lhd 3a_{20}$ and $\lhd 3b_{20}$, then we have just shown that it is irreflexive and transitive, being indeed the two characteristic properties of strict partial orders. □

The following basic property—on which many results, such as Proposition 34_{44}, build—is useful in finding rejected options from other option sets.

Proposition 30. *Consider any coherent rejection function R. Then for all u_1 and u_2 in $\mathcal{V}$ such that $u_1 \preceq u_2$, all A in $\mathcal{Q}$ and all v in $A \setminus \{u_1, u_2\}$:*
(i) *if $u_2 \in A$ and $v \in R(A \cup \{u_1\})$ then $v \in R(A)$;*
(ii) *if $u_1 \in A$ and $v \in R(A)$ then $v \in R(\{u_2\} \cup A \setminus \{u_1\})$.*

Proof. Consider any u_1 and u_2 in $\mathcal{V}$ such that $u_1 \preceq u_2$, any A in $\mathcal{Q}$ and any v in $A \setminus \{u_1, u_2\}$. The proof is trivial if $u_1 = u_2$, so assume that $u_1 \neq u_2$ and therefore $u_1 \prec u_2$, whence $u_1 \in R(\{u_1, u_2\})$ by Axiom $R2_{20}$.

To prove (i), assume that $u_2 \in A$ and $v \in R(A \cup \{u_1\})$, then using Axiom $R3a_{20}$ [with $\tilde{A} \coloneqq A \cup \{u_1\}$, $\tilde{A}_1 \coloneqq \{u_1\}$ and $\tilde{A}_2 \coloneqq \{u_1, u_2\}$; then $\tilde{A}_2 \subseteq \tilde{A}$ since $u_2 \in A$] we infer from $u_1 \in$

$R(\{u_1,u_2\})$ that $u_1 \in R(A \cup \{u_1\})$ and therefore $\{u_1,v\} \subseteq R(A \cup \{u_1\})$. Axiom R3b$_{20}$ [with $\tilde{A} \coloneqq \{u_1\}$, $\tilde{A}_1 \coloneqq \{u_1,v\}$ and $\tilde{A}_2 \coloneqq A \cup \{u_1\}$] then implies that $v \in R(A \setminus \{u_1\})$ and Axiom R3a$_{20}$ [with $\tilde{A}_1 \coloneqq \{v\}$, $\tilde{A}_2 \coloneqq A \setminus \{u_1\}$ and $\tilde{A} \coloneqq A$] then implies that indeed $v \in R(A)$.

For (ii)$_{\curvearrowleft}$, assume that $u_1 \in A$ and $v \in R(A)$, then by Axiom R3a$_{20}$ [with $\tilde{A} \coloneqq \{u_2\} \cup A$, $\tilde{A}_1 \coloneqq \{u_1\}$ and $\tilde{A}_2 \coloneqq \{u_1,u_2\}$; then $\tilde{A}_2 \subseteq \tilde{A}$ since $u_1 \in A$] we infer from $u_1 \in R(\{u_1,u_2\})$ that $u_1 \in R(\{u_2\} \cup A)$. Similarly, using Axiom R3a$_{20}$ [with $\tilde{A} \coloneqq \{u_2\} \cup A$, $\tilde{A}_1 \coloneqq \{v\}$ and $\tilde{A}_2 \coloneqq A$] we infer from $v \in R(A)$ that $v \in R(\{u_2\} \cup A)$ and therefore $\{u_1,v\} \subseteq R(\{u_2\} \cup A)$. By Axiom R3b$_{20}$ [with $\tilde{A} \coloneqq \{u_1\}$, $\tilde{A}_1 \coloneqq \{u_1,v\}$ and $\tilde{A}_2 \coloneqq \{u_2\} \cup A$], this implies that indeed $v \in R(\{u_2\} \cup A \setminus \{u_1\})$. ☐

As mentioned in Section 2.4$_{28}$, the two statements in Proposition 30$_{\curvearrowleft}$ are imposed as Axiom C2*$_{29}$ in Reference [67]. Using our rationality axioms, we are able to prove them as consequences of coherence.

Given any option set, we pay special attention to a particular subset consisting of its undominated options, and derive a useful property of it, necessary for Proposition 69$_{79}$ and Lemma 80$_{96}$, amongst others:

Proposition 31. *Consider any option set A, and its subset of maximal, or undominated, options:*

$$\max A \coloneqq \{u \in A : (\forall v \in A) u \nprec v\} \subseteq A.$$

We have that $\max A \neq \emptyset$. *Consider any choice function C that satisfies Axioms* C2$_{20}$ *and* C3$_{20}$*, and its corresponding rejection function R. Then $A \setminus \max A \subseteq R(A)$, and $(\forall u \in \max A)(u \in C(A) \Leftrightarrow u \in C(\max A))$. As a consequence:*

$$C(A) = C(\max A) \text{ and } R(A) = (A \setminus \max A) \cup R(\max A). \tag{2.13}$$

Proof. First, we prove that $\max A \neq \emptyset$. Assume *ex absurdo* that $\max A = \emptyset$ for some A in $\mathcal{Q}$. Without loss of generality, we may assume that $A = \{u_1,\dots,u_m\}$ for some m in $\mathbb{N}$. Then $\max A = \emptyset$ implies that $(\forall i \in \{1,\dots,m\})(\exists j \in \{1,\dots,m\}) u_i \prec u_j$. So in particular $u_1 \prec u_{j_1}$ for some j_1 in $\{1,\dots,m\}$, and because $\prec$ is irreflexive, we have that $j_1 \neq 1$ and therefore $j_1 \in \{2,\dots,m\}$. Without loss of generality, we let $j_1 = 2$ be the second index, so $u_1 \prec u_2$. Also, $u_2 \prec u_{j_2}$ for some j_2 in $\{1,\dots,m\}$, and because $\prec$ is irreflexive, we find similarly that $j_2 \neq 2$. But $\prec$ is also transitive, and therefore $u_1 \prec u_{j_2}$, whence $j_2 \neq 1$, so $j_2 \in \{3,\dots,m\}$. Without loss of generality, we let $j_2 = 3$ be the third index, so $u_2 \prec u_3$. We can go on the same vein until after $m-1$ steps we find that $u_1 \prec u_2 \prec \dots \prec u_{m-1} \prec u_m$. Since there is an element that dominates u_m, we have that $u_m \prec u_j$ for some j in $\{1,\dots,m\}$. But, since $\prec$ is irreflexive, we have that $j \neq m$. Also, since $\prec$ is transitive, we have that $j \notin \{1,\dots,m-1\}$, a contradiction. Therefore indeed $\max A \neq \emptyset$.

Next, we prove that $A \setminus \max A \subseteq R(A)$. Consider any u in $A \setminus \max A$, so we know that there is some $v \in A$ such that $u \prec v$, whence, by Axiom R2$_{20}$, $u \in R(\{u,v\})$. Axiom R3a$_{20}$ then guarantees that indeed $u \in R(A)$.

To prove the equivalence, consider any u in $\max A$. First, assume that $u \in C(A)$. Then, by the contraposition of Axiom R3a$_{20}$ [with $\tilde{A} \coloneqq A$, $\tilde{A}_1 \coloneqq \{u\}$ and $\tilde{A}_2 \coloneqq \max A$], we have that $u \in C(\max A)$. Conversely, assume that $u \in C(\max A)$, and assume *ex absurdo* that $u \in R(A)$. Then also $\{u\} \cup A \setminus \max A \subseteq R(A)$, using the first statement we proved above. Now, use Axiom R3b$_{20}$ [with $\tilde{A}_1 \coloneqq \{u\} \cup A \setminus \max A$, $\tilde{A} \coloneqq A \setminus \max A$ and $\tilde{A}_2 \coloneqq A$, and observe that $\tilde{A}_2 \setminus \tilde{A} = \max A$ and $\tilde{A}_1 \setminus \tilde{A} = \{u\}$] to infer that $u \in R(\max A)$, a contradiction. Hence indeed $u \in C(A)$.

The equalities in Equation (2.13) are now immediate. □

The following property will be useful in connecting choice models with desirability.

Proposition 32. *Consider any coherent rejection function R and $n \in \mathbb{N}$ options $u_1, \dots, u_n$ in $\mathcal{V}$. If $0 \in R(\{0, u_k\})$ for every k in $\{1, \dots, n\}$, then $0 \in R(\{0, \sum_{k=1}^{n} u_k\})$.*

Proof. We will use induction on n. For the base case $n = 1$, the result follows trivially from the assumption. Consider now the case $n > 1$. Then by the induction hypothesis, we may assume that $0 \in R(\{0, \sum_{k=1}^{n-1} u_k\})$. Using Axiom R4b$_{20}$, infer that $u_n \in R(\{u_n, \sum_{k=1}^{n} u_k\})$, and therefore, by Axiom R3a$_{20}$ [with $\tilde{A} \coloneqq \{0, u_n, \sum_{k=1}^{n} u_k\}$, $\tilde{A}_1 \coloneqq \{u_n\}$ and $\tilde{A}_2 \coloneqq \{u_n, \sum_{k=1}^{n} u_k\}$], also $u_n \in R(\{0, u_n, \sum_{k=1}^{n} u_k\})$. Since by assumption $0 \in R(\{0, u_n\})$, by Axiom R3a$_{20}$ [with $\tilde{A} \coloneqq \{0, u_n, \sum_{k=1}^{n} u_k\}$, $\tilde{A}_1 \coloneqq \{0\}$ and $\tilde{A}_2 \coloneqq \{0, u_n\}$] also $0 \in R(\{0, u_n, \sum_{k=1}^{n} u_k\})$. Therefore, by Axiom R3b$_{20}$ [use Proposition 25$_{39}$(ii)] indeed $0 \in R(\{0, \sum_{k=1}^{n} u_k\})$. □

2.5.3 *An important ordering of the option sets*

There is an ordering of the option sets—a set-wise generalisation of $\leq$—that is closely related with coherent choice relations, as we will show in Proposition 35$_{\curvearrowright}$. This order will turn out to be of crucial importance, mainly because it allows to write the natural extension in a natural way, as we will see in Chapter 3$_{89}$.

Definition 13. *We define the ordering $\lessdot$ on $\mathcal{Q}$ as:*

$$A_1 \lessdot A_2 \Leftrightarrow (\forall u \in A_1)(\exists v \in A_2) u \leq v$$

for all A_1 and A_2 in $\mathcal{Q}$.

Proposition 33. *The ordering $\lessdot$ satisfies the following properties:*

(i) *if $A_1 \subseteq A_2$ then $A_1 \lessdot A_2$;*
(ii) *$\lessdot$ is transitive;*
(iii) *$(A_1 \lessdot A_3$ and $A_2 \lessdot A_3) \Leftrightarrow A_1 \cup A_2 \lessdot A_3$;*
(iv) *$A \lessdot \max A$;*
(v) *if $A_1 \lessdot A_2$ then $A_1 + \{u\} \lessdot A_2 + \{u\}$;*
(vi) *if $A_1 \lessdot A_2$ then $\lambda A_1 \lessdot \lambda A_2$;*
(vii) *if $A_1 \lessdot A_2$ then $A_1 \cup A_3 \lessdot A_2 \cup A_3$,*

for all A_1, A_2 and A_3 in $\mathcal{Q}$, u in $\mathcal{V}$ and λ in $\mathbb{R}_{>0}$.

Proof. For Property (i)$_{\curvearrowleft}$, consider any u_1 in A_1, so also $u_1 \in A_2$. Then by the reflexivity of $\leq$, $u_1 \leq u_1$, so trivially $A_1 \preccurlyeq A_2$.

For Property (ii)$_{\curvearrowleft}$, consider the arbitrary option sets A_1, A_2 and A_3 and assume that $A_1 \preccurlyeq A_2$ and $A_2 \preccurlyeq A_3$, so $(\forall u_1 \in A_1)(\exists u_2 \in A_2) u_1 \leq u_2$ and $(\forall u_2 \in A_2)(\exists u_3 \in A_3) u_2 \leq u_3$, whence indeed $(\forall u_1 \in A_1)(\exists u_3 \in A_3) u_1 \leq u_3$.

Property (iii)$_{\curvearrowleft}$ is a direct consequence of Definition 13$_{\curvearrowleft}$.

For Property (iv)$_{\curvearrowleft}$, we know by Property (i)$_{\curvearrowleft}$ that $\max A \preccurlyeq \max A$. Consider any element u of $A \setminus \max A$, then by the definition of $\max A$, there is some v in $\max A$ for which $u < v$, and therefore in particular $A \setminus \max A \preccurlyeq \max A$. Then, using Property (iii)$_{\curvearrowleft}$, indeed $A \preccurlyeq \max A$.

For Properties (v)$_{\curvearrowleft}$ and (vi)$_{\curvearrowleft}$, since $A_1 \preccurlyeq A_2$, we have that $(\forall u_1 \in A_1)(\exists u_2 \in A_2) u_1 \leq u_2$, and because $\leq$ is a vector ordering, therefore also $(\forall u_1 \in A_1)(\exists u_2 \in A_2) u_1 + u \leq u_2 + u$ and $(\forall u_1 \in A_1)(\exists u_2 \in A_2) \lambda u_1 \leq \lambda u_2$, whence indeed $A_1 + \{u\} \preccurlyeq A_2 + \{u\}$ and $\lambda A_1 \preccurlyeq \lambda A_2$.

For Property (vii)$_{\curvearrowleft}$, assume that $A_1 \preccurlyeq A_2$. By Definition 13$_{\curvearrowleft}$, it suffices for any u in $A_1 \cup A_3$ to find some v in $A_2 \cup A_3$ such that $u \leq v$. So consider any u in $A_1 \cup A_3$. If $u \in A_1$ then there is some v in A_2 such that $u \leq v$ by the assumption, and if $u \in A_3$, it suffices to consider $v := u$ in A_3 and to note that in particular $u \leq v$. □

In particular, Property (i)$_{\curvearrowleft}$ implies that $\preccurlyeq$ is reflexive. So, due to Property (ii)$_{\curvearrowleft}$, $\preccurlyeq$ is a preorder—a reflexive and transitive binary relation—, but it is not antisymmetric: due to Properties (i)$_{\curvearrowleft}$ and (iv)$_{\curvearrowleft}$, $\max A \preccurlyeq A$ and $A \preccurlyeq \max A$ for all A in $\mathcal{Q}$, but $\max A$ might be different from A. So $\preccurlyeq$ is *not necessarily a partial order*, but it is useful to infer rejected options from other option sets.

Proposition 34. *Consider any coherent rejection function, any two option sets A and A', and any option u in $A \cap A'$. If $u \in R(A)$ and $A \preccurlyeq A'$, then $u \in R(A')$.*

Proof. Consider any A and A' in $\mathcal{Q}$, any u in $A \cap A'$, and assume that $u \in R(A)$ and $A \preccurlyeq A'$. Let $A := \{u, u_1, \dots, u_m\}$ and $A' := \{u, v_1, \dots, v_n\}$ for some m and n in $\mathbb{Z}_{\geq 0}$, where some of the u_i's and some of the v_j's are possible equal to each other. Since $A \preccurlyeq A'$, by Definition 13$_{\curvearrowleft}$ we have that $(\forall i \in \{1, \dots, m\})(\exists j_i \in \{1, \dots, n\}) u_i \leq v_{j_i}$. Use $u_1 \leq v_{j_1}$ and Proposition 30$_{41}$(ii) to infer from $u \in R(\{u, u_1, \dots, u_m\}) = R(A)$ that $u \in R(\{u, v_{j_1}, u_2, \dots, u_m\}) = R(\{v_{j_1}\} \cup A \setminus \{u_1\})$. Similarly, use $u_2 \leq v_{j_2}$ to infer that $u \in R(\{u, v_{j_1}, v_{j_2}, u_3, \dots, u_m\}) = R(\{v_{j_1}, v_{j_2}\} \cup A \setminus \{u_1, u_2\})$. Repeating this procedure m times, we infer that $u \in R(\{u, v_{j_1}, \dots, v_{j_m}\})$. Since $\{u, v_{j_1}, \dots, v_{j_m}\} \subseteq A'$, Axiom R3a$_{20}$ [with $\tilde{A} := A'$, $\tilde{A}_1 := \{u\}$ and $\tilde{A}_2 := \{u, v_{j_1}, \dots, v_{j_m}\}$] finally tells us that indeed $u \in R(A')$. □

As a consequence, if $u \in R(\{u\} \cup A)$ and $A \preccurlyeq A'$, then $u \in R(\{u\} \cup A')$ for all A and A' in $\mathcal{Q}$ and u in $\mathcal{V}$.

The connection of this order $\preccurlyeq$ with choice models is more elegant using choice relations: as we will see in the next proposition, there is a *mixed transitivity* [58] property.

Proposition 35. *Consider any coherent choice relation $\lhd$. Then $\lhd$ and $\preccurlyeq$ are mixed transitive, in the sense that:*

(i) *if* $A_1 \lhd A_2$ *and* $A_2 \preccurlyeq A_3$ *then* $A_1 \lhd A_3$*;*
(ii) *if* $A_1 \preccurlyeq A_2$ *and* $A_2 \lhd A_3$ *then* $A_1 \lhd A_3$*,*
for all A_1*,* A_2 *and* A_3 *in* $\mathcal{Q}$*.*

Proof. Consider the rejection function R corresponding to $\lhd$. For (i) consider any A_1, A_2 and A_3 in $\mathcal{Q}$ such that $A_1 \lhd A_2$ and $A_2 \preccurlyeq A_3$. Since R and $\lhd$ correspond, this implies that $A_1 \subseteq R(A_1 \cup A_2)$, or, in other words, $u \in R(A_1 \cup A_2)$ for every u in A_1. Since $A_2 \preccurlyeq A_3$, by Proposition 33_{43}(vii) therefore $A_1 \cup A_2 \preccurlyeq A_1 \cup A_3$, whence by Proposition 34 $u \in R(A_1 \cup A_3)$ for every u in A_1. Therefore $A_1 \subseteq R(A_1 \cup A_3)$, or, in other words, indeed $A_1 \lhd A_3$.

For (ii), consider any A_1, A_2 and A_3 in $\mathcal{Q}$ such that $A_1 \preccurlyeq A_2$ and $A_2 \lhd A_3$. Since R and $\lhd$ correspond, this implies that $A_2 \subseteq R(A_2 \cup A_3)$, or, in other words, $v \in R(A_2 \cup A_3)$ for every v in A_2. Use Proposition 25_{39}(iii) to infer that then $v \in R(\{v\} \cup (A_2 \cup A_3) \setminus R(A_2 \cup A_3))$, and since $A_2 \subseteq R(A_2 \cup A_3)$, by Axiom R3a$_{20}$ therefore $v \in R(\{v\} \cup A_3)$ for all v in A_2. We will show that this implies that $u \in R(\{u\} \cup A_3)$ for all u in A_1. Consider any u in A_1. Since $A_1 \preccurlyeq A_2$, therefore $u \leq v$ for some v in A_2. There are two possibilities: either (i) $u = v$, or (ii) $u < v$. If (i) $u = v$, then $u \in R(\{u\} \cup A_3)$. If (ii) $u < v$, by Axiom R2$_{20}$ then $u \in R(\{u, v\})$. Use Axiom R3a$_{20}$ to infer that then $u \in R(\{u, v\} \cup A_3)$. Since we already know that $v \in R(\{v\} \cup A_3)$, by Axiom R3a$_{20}$ therefore $v \in R(\{u, v\} \cup A_3)$, so $\{u, v\} \subseteq R(\{u, v\} \cup A_3)$. Use Proposition 25_{39}(ii) to infer that then $u \in R(\{u\} \cup A_3)$. So in both cases we conclude that $u \in R(\{u\} \cup A_3)$. Use Axiom R3a$_{20}$ once again to infer that now $u \in R(A_1 \cup A_3)$, and, since the choice of u in A_1 was arbitrary, therefore $A_1 \subseteq R(A_1 \cup A_3)$. Since R and $\lhd$ correspond, therefore indeed $A_1 \lhd A_3$. □

There is no *weakening*, in the sense that, for any A_1 and A_2 in $\mathcal{Q}$, $A_1 \preccurlyeq A_2$ need not imply $A_1 \lhd A_2$, nor *vice versa*.[12] For the first statement, it suffices to consider that $\preccurlyeq$ is reflexive and that $\lhd$ is irreflexive. For the second statement, it suffices to note, as we will see in Example 4_{65}, that undominated options can be rejected—that $R(A) \cap \max A$ need not be empty, for any A in $\mathcal{Q}$. Therefore, implications like

(iii) if $A_1 \lhd A_2$ and $A_2 \preccurlyeq A_3$ then $A_1 \preccurlyeq A_3$;
(iv) if $A_1 \preccurlyeq A_2$ and $A_2 \lhd A_3$ then $A_1 \preccurlyeq A_3$,
for all A_1, A_2 and A_3 in $\mathcal{Q}$, do *not* generally hold.

The following result will be useful, mainly for Chapter 4_{125}:

Corollary 36. *Consider any coherent rejection function* R*, and any* A *in* $\mathcal{Q}$*. Then*

$$0 \in R(A) \Leftrightarrow 0 \in R(A \cap \mathcal{V}_{<0}^c).$$

Proof. Note first that both $0 \in R(A)$ and $0 \in R(A \cap \mathcal{V}_{<0}^c)$ imply that $0 \in A$. For the converse implication, observe that $A \cap \mathcal{V}_{<0}^c \subseteq A$ and use Axiom R3a$_{20}$.

[12] As we will see in Corollary 42_{50}, there is weakening with a strict variant of $\preccurlyeq$, based on the vector order $<$.

For the direct implication, note that $A \cap \mathcal{V}_{<0} \lessdot \{0\}$, and therefore $\max A \subseteq A \cap \mathcal{V}_{<0}^c$. Use Proposition 33_{43}(i) to infer that then $\max A \lessdot A \cap \mathcal{V}_{<0}^c$, and, since by Proposition 33_{43}(iv) $A \lessdot \max A$, by Proposition 33_{43}(ii) therefore $A \lessdot A \cap \mathcal{V}_{<0}^c$. Now use Proposition 34_{44} to infer that indeed $0 \in R(A) \Rightarrow 0 \in R(A \cap \mathcal{V}_{<0}^c)$. □

2.6 ORDER-THEORETIC PROPERTIES

We will be concerned with *conservative reasoning* using choice models: we will look for the implications of a given assessment that are as uninformative as possible. In order to do this, we need some binary relation on the choice models, having the specific interpretation of being 'at most as informative as', 'at most as committal as', or 'at least as conservative as'.

2.6.1 *Basic relations*

Definition 14 ('At most as informative as' relation)**.** *Given two choice functions C_1 and C_2 in* $\mathbf{C}$*, we call C_1* at most as informative as *C_2—and we write $C_1 \sqsubseteq_{\mathbf{C}} C_2$—if*

$$C_1 \sqsubseteq_{\mathbf{C}} C_2 \Leftrightarrow (\forall A \in \mathcal{Q}) C_1(A) \supseteq C_2(A).$$

Given two rejection functions R_1 and R_2 in $\mathbf{R}$*, we call R_1* at most as informative as *R_2—and we write $R_1 \sqsubseteq_{\mathbf{R}} R_2$—if*

$$R_1 \sqsubseteq_{\mathbf{R}} R_2 \Leftrightarrow (\forall A \in \mathcal{Q}) R_1(A) \subseteq R_2(A).$$

Given two choice relations $\lhd_1$ and $\lhd_2$ in $\mathbf{S}$*, we call $\lhd_1$* at most as informative as *$\lhd_2$—and we write $\lhd_1 \sqsubseteq_{\mathbf{S}} \lhd_2$—if*

$$\lhd_1 \sqsubseteq_{\mathbf{S}} \lhd_2 \Leftrightarrow \lhd_1 \subseteq \lhd_2,$$

or, in other words, if $A_1 \lhd_1 A_2$ implies $A_1 \lhd_2 A_2$, for all A_1 and A_2 in $\mathcal{Q}$.

The idea underlying this natural ordering of choice models in Definition 14 is that rejection function R_1 is at most as informative as rejection function R_2 whenever any option rejected by R_1 is rejected by R_2 as well, so R_2 rejects at least as many options as R_1. For choice functions, the idea is similar. For choice relations, a choice relation $\lhd_1$ is at most as informative as choice relation $\lhd_2$ when every comparison between option sets that belongs to $\lhd_1$, also belongs to $\lhd_2$.

Since by Definition 14, the ordering $\sqsubseteq_{\mathbf{S}}$ is simply a set inclusion, the following result is immediate [17, Example 2.6].

Proposition 37. *The structure $(\mathbf{S}; \sqsubseteq_{\mathbf{S}})$ of all choice relations, provided with the order $\sqsubseteq_{\mathbf{S}}$, is a complete lattice:*

(i) *it is a* partially ordered set*, or* poset*, meaning that the binary relation $\sqsubseteq_{\mathbf{S}}$ on* $\mathbf{S}$ *is* reflexive*,* antisymmetric *and* transitive*;*

(ii) *for any subset* $\mathcal{S}$ *of* $\mathbf{S}$*, its infimum* $\inf\mathcal{S}$ *and its supremum* $\sup\mathcal{S}$ *with respect to the ordering* $\sqsubseteq_{\mathbf{S}}$ *exist in* $\mathbf{S}$*, and are given by* $\inf\mathcal{S} = \bigcap\mathcal{S}$ *and* $\sup\mathcal{S} = \bigcup\mathcal{S}$.

Similarly, since by Definition 14, $\sqsubseteq_{\mathbf{C}}$ and $\sqsubseteq_{\mathbf{R}}$ are product orderings of set inclusions, the following result is also immediate [17, Section 2.15].

Proposition 38. *The structures* $(\mathbf{C};\sqsubseteq_{\mathbf{C}})$ *and* $(\mathbf{R};\sqsubseteq_{\mathbf{R}})$ *of all choice functions provided with the order* $\sqsubseteq_{\mathbf{C}}$*, and all rejection functions provided with the order* $\sqsubseteq_{\mathbf{R}}$*, are complete lattices:*

(i) *they are partially ordered sets;*

(ii) *for any subset* $\mathcal{C}$ *of* $\mathbf{C}$*, its infimum* $\inf\mathcal{C}$ *and its supremum* $\sup\mathcal{C}$ *with respect to the ordering* $\sqsubseteq_{\mathbf{C}}$ *exist in* $\mathbf{C}$*, and are given by* $(\inf\mathcal{C})(A) = \bigcup_{C\in\mathcal{C}} C(A)$ *and* $\sup\mathcal{C}(A) = \bigcap_{C\in\mathcal{C}} C(A)$ *for all* A *in* $\mathcal{Q}$.
Similarly, for any subset $\mathcal{R}$ *of* $\mathbf{R}$*, its infimum* $\inf\mathcal{R}$ *and its supremum* $\sup\mathcal{R}$ *with respect to the ordering* $\sqsubseteq_{\mathbf{R}}$ *exist in* $\mathbf{R}$*, and are given by* $\inf\mathcal{R}(A) = \bigcap_{R\in\mathcal{R}} R(A)$ *and* $\sup\mathcal{R}(A) = \bigcup_{R\in\mathcal{R}} R(A)$ *for all* A *in* $\mathcal{Q}$.

The importance of Propositions 37 and 38 lies in the fact that for any $\mathcal{C} \subseteq \mathbf{C}$, $\inf\mathcal{C}$ is the most informative model that is at least as informative as any of the models in $\mathcal{C}$, and $\sup\mathcal{C}$ the least informative model that is not less informative than any of the models in $\mathcal{C}$, and similar for other choice models.

We will also consider the poset $(\overline{\mathbf{C}};\sqsubseteq_{\mathbf{C}})$ of all coherent choice functions, where $\overline{\mathbf{C}} \subseteq \mathbf{C}$ inherits the partial order $\sqsubseteq_{\mathbf{C}}$ from $\mathbf{C}$. Similarly, in the poset $(\overline{\mathbf{R}};\sqsubseteq_{\mathbf{R}})$ of all coherent rejection functions, $\overline{\mathbf{R}}$ is assumed to inherit the partial order $\sqsubseteq_{\mathbf{R}}$ from $\mathbf{R}$, and in the poset $(\overline{\mathbf{S}};\sqsubseteq_{\mathbf{S}})$ of all coherent choice relations, $\overline{\mathbf{S}}$ is assumed to inherit the partial order $\sqsubseteq_{\mathbf{S}}$ from $\mathbf{S}$.

If there is a bijection between two posets that preserves the order, the two posets are in some way equivalent to each other. The following definition formalises this idea.

Definition 15 (Order isomorphism)**.** *Two posets* $(P_1;\leq_1)$ *and* $(P_2;\leq_2)$ *are called* order-isomorphic *if there is a bijective function* f *from* P_1 *to* P_2 *with the property that* $x \leq_1 y \Leftrightarrow f(x) \leq_2 f(y)$ *for all* x *and* y *in* P_1.

Also from an order-theoretic point of view, our different types of choice models are equivalent:

Proposition 39. *The posets* $(\mathbf{C};\sqsubseteq_{\mathbf{C}})$*,* $(\mathbf{R};\sqsubseteq_{\mathbf{R}})$ *and* $(\mathbf{S};\sqsubseteq_{\mathbf{S}})$ *are (pairwise) order-isomorphic. Moreover, the posets* $(\overline{\mathbf{C}};\sqsubseteq_{\mathbf{C}})$*,* $(\overline{\mathbf{R}};\sqsubseteq_{\mathbf{R}})$ *and* $(\overline{\mathbf{S}};\sqsubseteq_{\mathbf{S}})$ *are (pairwise) order-isomorphic.*

Proof. For the first statement, Proposition 11_{18} already shows that there are bijections ρ, σ and κ between $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{S}$. We prove that those bijections preserve the order. The proof has the following structure: we first prove that (i) $C_1 \sqsubseteq_{\mathbf{C}} C_2 \Rightarrow \rho(C_1) = R_{C_1} \sqsubseteq_{\mathbf{R}} R_{C_2} = \rho(C_2)$ for all choice functions C_1 and C_2, then that (ii) $R_1 \sqsubseteq_{\mathbf{R}} R_2 \Rightarrow \sigma(R_1) =$

$\lhd_{R_1} \sqsubseteq_{\mathbf{S}} \lhd_{R_2} = \sigma(R_2)$ for all rejection functions R_1 and R_2, and finally that (iii) $\lhd_1 \sqsubseteq_{\mathbf{S}} \lhd_2 \Rightarrow \kappa(\lhd_1) = C_{\lhd_1} \sqsubseteq_{\mathbf{C}} C_{\lhd_2} = \kappa(\lhd_2)$ for all choice relations $\lhd_1$ and $\lhd_2$.

For (i), consider any C_1 and C_2 in $\mathbf{C}$ such that $C_1 \sqsubseteq_{\mathbf{C}} C_2$, or, in other words, that $C_1(A) \supseteq C_2(A)$—and hence $R_{C_1}(A) \subseteq R_{C_2}(A)$—for all A in $\mathcal{Q}$. Therefore indeed $R_{C_1} \sqsubseteq_{\mathbf{R}} R_{C_2}$. For (ii), consider any R_1 and R_2 in $\mathbf{R}$ such that $R_1 \sqsubseteq_{\mathbf{R}} R_2$, or, in other words, that $R_1(A) \subseteq R_2(A)$ for all A in $\mathcal{Q}$. Therefore $A_1 \subseteq R_1(A_1 \cup A_2) \Rightarrow A_1 \subseteq R_2(A_1 \cup A_2)$—using Definition 4_{16}, equivalently $A_1 \lhd_{R_1} A_2 \Rightarrow A_1 \lhd_{R_2} A_2$—for all A_1 and A_2 in $\mathcal{Q}$, whence indeed $\lhd_{R_1} \sqsubseteq_{\mathbf{S}} \lhd_{R_2}$. For (iii), consider any $\lhd_1$ and $\lhd_2$ in $\mathbf{S}$ such that $\lhd_1 \sqsubseteq_{\mathbf{S}} \lhd_2$, or, in other words, that $\lhd_1 \subseteq \lhd_2$. By Definition 5_{17} therefore $C_{\lhd_1}(A) \supseteq C_{\lhd_2}(A)$ for all A in $\mathcal{Q}$, whence indeed $C_{\lhd_1} \sqsubseteq_{\mathbf{C}} C_{\lhd_2}$.

For the second statement, it suffices to note that, by Corollary 14_{24}, ρ, σ and κ define bijections between $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{S}$, and that we just have shown that they preserve the order. □

Proposition $39_{\curvearrowleft}$ implies that we can regard $\sqsubseteq_{\mathbf{C}}$, $\sqsubseteq_{\mathbf{R}}$ and $\sqsubseteq_{\mathbf{S}}$ as essentially the same partial orders, each defined on their proper domain, in the sense that one of the partial orders can be obtained from another one just by renaming. From now on, we will denote each of the three 'at most as informative' relations simply by $\sqsubseteq$ when it is clear from the context which specific order is meant. Hence, it is of no importance which of the posets $(\mathbf{C};\sqsubseteq_{\mathbf{C}})$, $(\mathbf{R};\sqsubseteq_{\mathbf{R}})$ and $(\mathbf{S};\sqsubseteq_{\mathbf{S}})$ we use to prove order-theoretic properties: any given property in one of the posets, immediately transfer to the other posets. The same remark holds for the posets $(\overline{\mathbf{C}};\sqsubseteq_{\mathbf{C}})$, $(\overline{\mathbf{R}};\sqsubseteq_{\mathbf{R}})$ and $(\overline{\mathbf{S}};\sqsubseteq_{\mathbf{S}})$.

2.6.2 *Intersection structures*

In the subsequent sections, we focus on the poset $(\overline{\mathbf{C}};\sqsubseteq)$ of *coherent* choice functions—or equivalently, on the poset $(\overline{\mathbf{R}};\sqsubseteq)$ of *coherent* rejection functions or the poset $(\overline{\mathbf{S}};\sqsubseteq)$ of *coherent* choice relations, when it suits our purpose better. An important property is that of being an *intersection structure*, or *complete infimum-semilattice* [23].

Proposition 40 (Intersection structure). $(\overline{\mathbf{C}};\sqsubseteq)$ *is an intersection structure (a complete infimum-semilattice):* $\overline{\mathbf{C}}$ *is closed under arbitrary non-empty infima, so* $\inf\mathcal{C} \in \overline{\mathbf{C}}$ *for any non-empty subset* $\mathcal{C}$ *of* $\overline{\mathbf{C}}$.

Proof. Consider any collection $\mathcal{C}$ of coherent choice functions. We will show that $\inf\mathcal{C}$ satisfies the rationality axioms of Definition 6_{20}:

C1$_{20}$. Consider any C in $\mathcal{C}$ [always possible since $\mathcal{C} \neq \emptyset$] and any A in $\mathcal{Q}$, then $\emptyset \subset C(A) \subseteq (\inf\mathcal{C})(A)$.

C2$_{20}$. Note that, for all C in $\mathcal{C}$, $u \notin C(\{u,v\})$ whenever $u \prec v$, implying that $u \notin \bigcup_{C\in\mathcal{C}} C(\{u,v\}) = (\inf\mathcal{C})(\{u,v\})$.

C3a$_{20}$. Consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $(\inf\mathcal{C})(A_2) = \bigcup_{C\in\mathcal{C}} C(A_2) \subseteq A_2 \setminus A_1$ and $A_1 \subseteq A_2 \subseteq A$. This implies that for all C in $\mathcal{C}$, $C(A_2) \subseteq A_2 \setminus A_1$ and by their coherence [Axiom C3a$_{20}$] therefore also $C(A) \subseteq A \setminus A_1$. So indeed $(\inf\mathcal{C})(A) = \bigcup_{C\in\mathcal{C}} C(A) \subseteq A \setminus A_1$.

C3b$_{20}$. Consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $(\inf\mathcal{C})(A_2) = \bigcup_{C\in\mathcal{C}} C(A_2) \subseteq A_2 \setminus A_1$ and $A \subseteq A_1$. This implies that for all C in $\mathcal{C}$, $C(A_2) \subseteq A_2 \setminus A_1$, and by their coherence [Axiom C3b$_{20}$] therefore also $C(A_2 \setminus A) \subseteq A_2 \setminus A_1$. So indeed $(\inf\mathcal{C})(A_2 \setminus A) = \bigcup_{C\in\mathcal{C}} C(A_2 \setminus A) \subseteq A_2 \setminus A_1$.

C4$_{20}$. Consider any C in $\mathcal{C}$, λ in $\mathbb{R}_{>0}$, u in $\mathcal{V}$ and A in $\mathcal{Q}$, and infer from Proposition 27$_{40}$ that $\lambda C(A) + \{u\} = C(\lambda A + \{u\})$. Hence $\lambda(\inf\mathcal{C})(A) + \{u\} = \lambda \bigcup_{C\in\mathcal{C}} C(A) + \{u\} = \bigcup_{C\in\mathcal{C}}(\lambda C(A) + \{u\}) = \bigcup_{C\in\mathcal{C}} C(\lambda A + \{u\}) = (\inf\mathcal{C})(\lambda A + \{u\})$, which implies using Proposition 27$_{40}$ that $\inf\mathcal{C}$ indeed satisfies axioms C4a$_{20}$ and C4b$_{20}$. □

Since by Proposition 39$_{47}$, $(\overline{\mathbf{C}};\sqsubseteq)$, $(\overline{\mathbf{R}};\sqsubseteq)$ and $(\overline{\mathbf{S}};\sqsubseteq)$ are order-isomorphic, therefore $(\overline{\mathbf{R}};\sqsubseteq)$ and $(\overline{\mathbf{S}};\sqsubseteq)$ are intersection structures as well. This, for instance, allows us to do conservative inference with choice functions: if we consider that there is a coherent choice function that represents a subject's beliefs (coherent choices) and we can only tell that it belongs to a family $\{C_i : i \in I\}$, the conservative option is to consider its infimum $\inf_{i\in I} C_i$. This choice function is still guaranteed to be coherent, and as a consequence, it satisfies all the rationality requirements discussed above.

Proposition 40 also guarantees that there is a unique smallest—least informative—coherent rejection function. We will call it the *vacuous rejection function*, and denote it by R_{v}.

Proposition 41 (Vacuous rejection function)**.** *The vacuous rejection function R_{v} is given by $R_{\mathrm{v}}(A) = A \setminus \max A = \{u \in A : (\exists v \in A) u \prec v\}$ for all A in $\mathcal{Q}$. It selects from any set of options the ones that are dominated under the strict vector ordering $\prec$.*

Proof. See Reference [9, Theorem 3] for an alternative proof by Bradley. First, we will show that any coherent rejection function must dominate R_{v}: $R_{\mathrm{v}} \sqsubseteq R$ for all R in $\overline{\mathbf{R}}$. So consider any R in $\overline{\mathbf{R}}$, and any option set A in $\mathcal{Q}$, and any option u in $R_{\mathrm{v}}(A)$. Then $u \in A \setminus \max A$, so $u \prec v$ for some v in A, and therefore, by Axiom R2$_{20}$, $u \in R(\{u,v\})$. Use Axiom R3a$_{20}$ to infer that then $u \in R(A)$. Since the choice of u in $R_{\mathrm{v}}(A)$ was arbitrary, therefore $R_{\mathrm{v}}(A) \subseteq R(A)$, for every A in $\mathcal{Q}$. This means that indeed $R_{\mathrm{v}} \sqsubseteq R$.

The proof is complete if we also show that R_{v} is coherent. We will show that R satisfies the rationality axioms of Definition 7$_{20}$:

R1$_{20}$. Since by Proposition 31$_{42}$, $\max A \neq \emptyset$, therefore indeed $R_{\mathrm{v}}(A) = A \setminus \max A \neq A$ for every A in $\mathcal{Q}$.

R2$_{20}$. Consider any u and v in $\mathcal{V}$ such that $u \prec v$. Let $A \coloneqq \{u,v\}$. Then $u \in A \setminus \max A$, whence indeed $u \in R_{\mathrm{v}}(A) = R_{\mathrm{v}}(\{u,v\})$.

R3a$_{20}$. We will prove the equivalent version of Proposition 24$_{38}$(ii). Consider any A in $\mathcal{Q}$, any u in $R_{\mathrm{v}}(A)$ and any v in $\mathcal{V}$. Since $u \in R_{\mathrm{v}}(A) = A \setminus \max A$, therefore $u \prec w$ for some w in A, whence trivially $u \prec w$ for some w in $A \cup \{v\}$. Therefore indeed $u \in R_{\mathrm{v}}(A \cup \{v\})$.

R3b$_{20}$. We will prove the equivalent version of Proposition 25$_{39}$(ii). Consider any A in $\mathcal{Q}$, any u in $R_{\mathrm{v}}(A)$ and any v in $R_{\mathrm{v}}(A) \setminus \{u\}$. Then $u \prec w_1$ for some w_1 in A and $v \prec w_2$ for some w_2 in A. If $w_1 \neq v$, then $u \in R_{\mathrm{v}}(A \setminus \{v\})$ and the proof is done, so assume that $w_1 = v$. Then, since $v \prec w_2$ therefore $u \prec w_2$, and, trivially, $w_2 \neq v$. Therefore indeed $u \in R_{\mathrm{v}}(A \setminus \{v\})$.

R4$_{20}$. This follows using Proposition 27$_{40}$, after observing that $u \in \max A \Leftrightarrow \lambda u + \{v\} \in \max(\lambda A + \{v\})$ for all A in $\mathcal{Q}$, u in A, λ in $\mathbb{R}_{>0}$ and v in $\mathcal{V}$. □

As a result, the vacuous choice function C_{v} is given by

$$C_{\mathrm{v}}(A) = \max A = \{u \in A : (\forall v \in A) u \nprec v\} \text{ for all } A \text{ in } \mathcal{Q}, \tag{2.14}$$

while the vacuous choice relation $\lhd_{\mathrm{v}}$ is determined in our next result:

Corollary 42. *The vacuous choice relation* $\lhd_{\mathrm{v}}$ *is given by*

$$A_1 \lhd_{\mathrm{v}} A_2 \Leftrightarrow (\forall u \in A_1)(\exists v \in A_2) u \prec v, \textit{ for all } A_1 \textit{ and } A_2 \textit{ in } \mathcal{Q}.$$

Proof. Because by Proposition 39$_{47}$, $(\overline{\mathbf{R}};\sqsubseteq)$ and $(\overline{\mathbf{S}};\sqsubseteq)$ are order-isomorphic, their minimal elements correspond, and therefore $\lhd_{\mathrm{v}}$ is the choice relation corresponding to R_{v}. So $A_1 \lhd_{\mathrm{v}} A_2 \Leftrightarrow A_1 \subseteq R_{\mathrm{v}}(A_1 \cup A_2)$, or, in other words,

$$(\forall u \in A_1)(\exists v \in A_1 \cup A_2) u \prec v \tag{2.15}$$

for any A_1 and A_2 in $\mathcal{Q}$. We show that this is equivalent to

$$(\forall u \in A_1)(\exists v \in A_2) u \prec v \tag{2.16}$$

for any A_1 and A_2 in $\mathcal{Q}$. That Statement (2.16) implies Statement (2.15) is immediate, so it suffices to show that Statement (2.15) implies Statement (2.16). Consider any A_1 and A_2 in $\mathcal{Q}$. Without loss of generality, let $A_1 := \{u_1, \dots, u_n\}$. Consider any u in A_1; without loss of generality let $u = u_1$. By Statement (2.15), $u_1 \prec v$ for some v in $A_1 \cup A_2$. If v belongs to A_2, then the proof is done, so assume that $v \in A_1$. By the irreflexivity of $\prec$, we infer that $v \neq u_1$ and therefore, without loss of generality, let $v = u_2$. By Statement (2.15), $u_2 \prec v'$ for some v' in $A_1 \cup A_2$, by the transitivity of $\prec$, therefore $u_1 \prec v'$. If v' belongs to A_2, then the proof is done, so assume that $v' \in A_1$. By the irreflexivity of $\prec$, we infer that $v' \notin \{u_1, u_2\}$ and therefore, without loss of generality, let $v' = u_3$. We can go on in the same vein until we find that $u_1 \prec u_2 \prec \dots \prec u_k \prec w$ for some k in $\{1, \dots, n\}$ and w in A_2, and therefore, by the transitivity of $\prec$, indeed $u_1 \prec w$. □

Compare this with the set-wise generalisation $\prec'$ of $\prec$, defined as

$$A_1 \prec' A_2 \Leftrightarrow (\forall u \in A_1)(\exists v \in A_2) u \prec v, \text{ for all } A_1 \text{ and } A_2 \text{ in } \mathcal{Q}.$$

Corollary 42 states that $\lhd_{\mathrm{v}} = \prec'$, and therefore $\prec' \subseteq \lhd$ for every coherent choice relation $\lhd$.

2.6.3 *Maximal coherent choice models*

Recall that an element of a poset is maximal if it is not dominated by any other element of the poset.[13]

[13]The definition in Proposition 31$_{42}$ of $\max A$ is an instance of this: any element of $\max A$ is maximal in A, under the partial order $\preceq$.

Definition 16 (Maximal elements)**.** *Consider any poset $(P;\leq)$. We denote the set of all maximal elements in P by $\hat{P}$:*[14]

$$\hat{P} \coloneqq \{p \in P : (\forall q \in P)(p \leq q \Rightarrow p = q)\} = \{p \in P : (\forall q \in P)p \nless q\} \subseteq P.$$

Applying this definition to the poset $(\overline{\mathbf{C}};\sqsubseteq)$, we obtain

$$\hat{\mathbf{C}} \coloneqq \{C \in \overline{\mathbf{C}} : (\forall C' \in \overline{\mathbf{C}})(C \sqsubseteq C' \Rightarrow C = C')\} = \{C \in \overline{\mathbf{C}} : (\forall C' \in \overline{\mathbf{C}})C \not\sqsubset C'\} \subseteq \overline{\mathbf{C}}$$

as the collection of all maximal coherent choice functions.[15] For rejection functions, we introduce the notation $\hat{\mathbf{R}} \coloneqq \{R \in \overline{\mathbf{R}} : (\forall R' \in \overline{\mathbf{R}})R \sqsubseteq R' \Rightarrow R = R'\} \subseteq \overline{\mathbf{R}}$ for the collection of maximal elements of the poset $(\overline{\mathbf{R}};\sqsubseteq)$. By Proposition 39$_{47}$, there is a connection between sets $\hat{\mathbf{C}}$ and $\hat{\mathbf{R}}$: any choice function belongs to $\hat{\mathbf{C}}$ if and only if its corresponding rejection function belongs to $\hat{\mathbf{R}}$. A similar remark holds for choice relations.

As we will see in Section 2.8$_{55}$, there is an easy characterisation of the maximal coherent *sets of desirable options*, which are, essentially, coherent choice functions representing binary choice only. This characterisation will allow us to prove the important representation result[16] in Proposition 52$_{59}$ that every coherent set of desirable options is an infimum of such maximal elements. However, for the more general coherent choice models, no such representation has been found yet. As a result, it is unknown whether choice models can be expressed as infima of their dominating *maximal* models. For more information about this, see Chapter 4$_{125}$.

We cannot take for granted that every coherent choice function is dominated by a maximal one, nor that there are even maximal choice functions. But, even though we are not yet able to characterise them, we will prove in Proposition 46$_{\curvearrowright}$ that there are maximal choice functions, and, moreover, that every coherent choice function is dominated by some maximal one. Its proof relies heavily on Zorn's Lemma (a version of the Axiom of Choice; see References [38, Section 16] and [17, Section 10.2] for more information), and the following two concepts of *upper bound* and *chain*.

Definition 17 (Upper bound)**.** *Consider any poset $(P_1;\leq)$ and any subset P_2 of P_1. An* upper bound *of P_2 is an element $p \in P_1$ such that $q \leq p$ for every q in P_2.*

Definition 18 (Chain)**.** *Consider any poset $(P;\leq)$ and any subset $\mathcal{K}$ of P. We say that $\mathcal{K}$ is a* chain *when it is totally ordered by $\leq$:*

$$p \leq q \text{ or } q \leq p \text{ for every } p \text{ and } q \text{ in } \mathcal{K}.$$

[14] The right-most equality holds since $p \nless q \Leftrightarrow \neg(p \leq q \text{ and } p \neq q) \Leftrightarrow (p \leq q \Rightarrow p = q)$.

[15] Actually, we should call this set $\hat{\overline{\mathbf{C}}}$, but since we will only be concerned with maximal *coherent* choice functions, we can use $\hat{\mathbf{C}}$ to denote the maximal elements of $(\overline{\mathbf{C}};\sqsubseteq)$ without confusion.

[16] See Reference [13, Theorem 21] for a constructive proof for finite possibility spaces, and Reference [31, Corollary 4] for a proof that relies on *Zorn's Lemma* for arbitrary possibility spaces.

Lemma 43. *Consider any chain $\mathcal{K} \subseteq \overline{\mathbf{C}}$. Then*

$$(\forall A \in \mathcal{Q})(\exists C \in \mathcal{K})(\sup\mathcal{K})(A) = C(A).$$

Proof. Consider any A in $\mathcal{Q}$. Since A is finite, therefore so is the set $\{C(A) : C \in \mathcal{K}\} \Subset \mathcal{P}_{\varnothing}(A)$, and we let $\ell \in \mathbb{N}$ be its cardinality. Denote $\{C(A) : C \in \mathcal{K}\} = \{A_1, \dots, A_\ell\}$, where $A_j \subseteq A$ for every j in $\{1, \dots, \ell\}$. This partitions $\mathcal{K}$ into ℓ classes $\mathcal{K}_1$, ..., $\mathcal{K}_\ell$ such that for every j in $\{1, \dots, \ell\}$, $C(A) = A_j$ for all C in $\mathcal{K}_j$. For every choice of C_1 in $\mathcal{K}_1$, ..., C_ℓ in $\mathcal{K}_\ell$, therefore $\{A_1, \dots, A_\ell\} = \{C_1(A), \dots, C_\ell(A)\}$, and since $\mathcal{K}$ is a chain, the set $\{C_1(A), \dots, C_\ell(A)\}$ is also a chain: it is totally ordered by $\subseteq$. Without loss of generality, let $C_1(A) \subseteq C_2(A) \subseteq \dots \subseteq C_\ell(A)$. This implies that $\{C(A) : C \in \mathcal{K}\}$ is a finite chain, so its infimum exists and is given by $\bigcap\{C(A) : C \in \mathcal{K}\} = \bigcap_{j=1}^{\ell} C_j(A) = C_1(A)$. Since $(\sup\mathcal{K})(A) = \bigcap\{C(A) : C \in \mathcal{K}\}$, therefore indeed $(\sup\mathcal{K})(A) = C_1(A)$, for some C_1 in $\mathcal{K}_1 \subseteq \mathcal{K}$. □

Lemma 44 (Zorn's Lemma)**.** *Consider any poset $(P; \leq)$. If every non-empty chain $\mathcal{K} \subseteq P$ has an upper bound (in P), then $(P; \leq)$ has at least one maximal element.*

Lemma 45. *Consider any chain $\mathcal{K} \subseteq \overline{\mathbf{C}}$. Then* $\sup\mathcal{K}$ *is coherent.*

Proof. We will show that $\sup\mathcal{C}$ satisfies the rationality axioms of Definition 6_{20}:

- C1_{20}. Consider any A in $\mathcal{Q}$. By Lemma 43, then $(\sup\mathcal{K})(A) = C'(A)$ for some C' in $\mathcal{K}$, and since C' is coherent, therefore in particular $C'(A) \neq \varnothing$. Hence indeed $(\sup\mathcal{K})(A) \neq \varnothing$.
- C2_{20}. Note that, for all C in $\mathcal{K}$, $u \notin C(\{u, v\})$ whenever $u \prec v$, implying that $u \notin \bigcap_{C \in \mathcal{K}} C(\{u, v\}) = (\sup\mathcal{K})(\{u, v\})$.
- C3a_{20}. Consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $(\sup\mathcal{K})(A_2) = \bigcap_{C \in \mathcal{K}} C(A_2) \subseteq A_2 \setminus A_1$ and $A_1 \subseteq A_2 \subseteq A$. By Lemma 43 then there is some C in $\mathcal{K}$ such that $(\sup\mathcal{K})(A_2) = C(A_2)$, so $C(A_2) \subseteq A_2 \setminus A_1$, and by its coherence [Axiom C3a_{20}] therefore also $C(A) \subseteq A \setminus A_1$. So indeed $(\sup\mathcal{K})(A) = \bigcap_{C \in \mathcal{K}} C(A) \subseteq A \setminus A_1$.
- C3b_{20}. Consider any A, A_1 and A_2 in $\mathcal{Q}$ such that $(\sup\mathcal{K})(A_2) = \bigcap_{C \in \mathcal{K}} C(A_2) \subseteq A_2 \setminus A_1$ and $A \subseteq A_1$. By Lemma 43 then there is some C in $\mathcal{K}$ such that $(\sup\mathcal{K})(A_2) = C(A_2)$, so $C(A_2) \subseteq A_2 \setminus A_1$, and by its coherence [Axiom C3b_{20}] therefore also $C(A_2 \setminus A) \subseteq A_2 \setminus A_1$. So indeed $(\sup\mathcal{K})(A_2 \setminus A) = \bigcap_{C \in \mathcal{K}} C(A_2 \setminus A) \subseteq A_2 \setminus A_1$.
- C4_{20}. Consider any C in $\mathcal{K}$, λ in $\mathbb{R}_{>0}$, u in $\mathcal{V}$ and A in $\mathcal{Q}$, and infer from Proposition 27_{40} that $\lambda C(A) + \{u\} = C(\lambda A + \{u\})$. Hence $\lambda(\sup\mathcal{K})(A) + \{u\} = \lambda \bigcap_{C \in \mathcal{K}} C(A) + \{u\} = \bigcap_{C \in \mathcal{K}} (\lambda C(A) + \{u\}) = \bigcap_{C \in \mathcal{K}} C(\lambda A + \{u\}) = (\sup\mathcal{K})(\lambda A + \{u\})$, which implies using Proposition 27_{40} that $\sup\mathcal{K}$ indeed satisfies axioms C4a_{20} and C4b_{20}. □

Now we are ready to show the following important result.

Proposition 46. *For any choice function C in $\overline{\mathbf{C}}$, its set of dominating maximal coherent choice functions $\hat{\mathbf{C}}_C := \{\hat{C} \in \hat{\mathbf{C}} : C \sqsubseteq \hat{C}\}$ is non-empty. As a consequence, $\hat{\mathbf{C}} \neq \varnothing$.*

Proof. We will apply Zorn's Lemma 44 to the poset $(\uparrow C;\sqsubseteq)$, where $\uparrow C \coloneqq \{C' \in \overline{\mathbf{C}} : C \sqsubseteq C'\}$. This set is non-empty because $C \in \uparrow C$. Consider any non-empty chain $\mathcal{K} \subseteq \uparrow C$. By Lemma 45, the upper bound $\sup\mathcal{K}$ of $\mathcal{K}$ is a coherent choice function (that dominates C) and hence an element of $\uparrow C$. Therefore, by Zorn's Lemma 44, $(\uparrow C;\sqsubseteq)$ has a maximal element, that—since every element of $\uparrow C$ dominates C—therefore indeed dominates C.

For the second statement, that $\hat{\mathbf{C}} \neq \emptyset$, it suffices to note that the maximal element $\hat{C}$ of the poset $(\uparrow C;\sqsubseteq)$ that we have found in the first part of this proof, is also a maximal element of the poset $(\overline{\mathbf{C}};\sqsubseteq)$: indeed, if this was not the case, then by Definition 16_{51} there is some C' in $\overline{\mathbf{C}}$ such that $\hat{C} \sqsubset C'$. But then $C \sqsubseteq C'$, whence $C' \in \uparrow C$, a contradiction with the fact that $\hat{C}$ is a maximal element of the poset $(\uparrow C;\sqsubseteq)$. □

Later, in Proposition 62_{70}, we will find explicit expressions for some special types of elements of $\hat{\mathbf{C}}$, but at this point, we content ourselves with the result that $\hat{\mathbf{C}}$ is non-empty. Since $\overline{\mathbf{C}}$, $\overline{\mathbf{R}}$ and $\overline{\mathbf{S}}$ are order-isomorphic, note that therefore every coherent rejection function is dominated by some maximal rejection function, and similarly for choice relations.

2.6.4 *What about the other properties imposed on choice models?*

In the previous sections, we have shown that $(\overline{\mathbf{C}};\sqsubseteq)$ is an infimum-semilattice—guaranteeing that there is a unique smallest (vacuous) choice function—and that every element of $\overline{\mathbf{C}}$ is dominated by some maximal one. Central in those sections, was the set $\overline{\mathbf{C}}$ of *coherent* choice functions: we disregarded the additional Properties $\mathrm{C5}_{25}$ and $\mathrm{C6}_{25}$.

In this section we investigate which results of Sections $2.6.2_{48}$ and $2.6.3_{50}$ remain valid if we additionally assume Properties $\mathrm{C5}_{25}$ and $\mathrm{C6}_{25}$.

Let us first focus on infima. It turns out that both Properties $\mathrm{C5}_{25}$ and $\mathrm{C6}_{25}$ are closed under arbitrary infima:

Proposition 47. *Consider any set $\mathcal{C}$ of choice functions that satisfy Property* $\mathrm{C5}_{25}$. *Then* $\inf\mathcal{C}$ *satisfies Property* $\mathrm{C5}_{25}$. *Moreover, consider any set $\mathcal{C}$ of choice functions that satisfy Property* $\mathrm{C6}_{25}$. *Then* $\inf\mathcal{C}$ *satisfies Property* $\mathrm{C6}_{25}$.

Proof. For the first statement, consider any A and A_1 in $\mathcal{Q}$ and assume that $A \subseteq A_1 \subseteq \operatorname{conv}(A)$. Then $C(A) \subseteq C(A_1)$ for all C in $\mathcal{C}$, whence indeed $(\inf\mathcal{C})(A) = \bigcup_{C\in\mathcal{C}} C(A) \subseteq \bigcup_{C\in\mathcal{C}} C(A_1) = (\inf\mathcal{C})(A_1)$.

For the second statement, consider any n in $\mathbb{N}$, $u_1, \dots, u_n$ in $\mathcal{V}$ and $\mu_1, \dots, \mu_n$ in $\mathbb{R}_{>0}$ and assume that $0 \in (\inf\mathcal{C})(\{0,u_1,\dots,u_n\})$. Then $0 \in C(\{0,u_1,\dots,u_n\})$, and hence $0 \in C(\{0,\mu_1u_1,\dots,\mu_nu_n\})$ for some C in $\mathcal{C}$, whence indeed $0 \in \bigcup_{C\in\mathcal{C}} C(\{0,\mu_1u_1,\dots,\mu_nu_n\}) = (\inf\mathcal{C})(\{0,\mu_1u_1,\dots,\mu_nu_n\})$. □

Due to Proposition 47, there is a unique least informative coherent choice function that is coherent and satisfies Properties $\mathrm{C5}_{25}$ and/or $\mathrm{C6}_{25}$. Since the following lemma guarantees that the vacuous choice function C_{v} satisfies Property $\mathrm{C6}_{25}$, therefore the least informative coherent choice function that satisfies Property $\mathrm{C6}_{25}$ is exactly C_{v}.

Lemma 48. *The vacuous choice function* C_{v} *satisfies Property* $\mathrm{C6}_{25}$.

Proof. Since by Equation $(2.14)_{50}$ $C_{\mathrm{v}}(A)$ is given by $\max A$ for all A in $\mathcal{Q}$, it suffices to show that $0 \in \max\{u_1,\dots,u_n\} \Leftrightarrow 0 \in \max\{\mu_1 u_1,\dots,\mu_n u_n\}$ for all n in $\mathbb{N}$, $u_1, \dots, u_n$ in $\mathcal{V}$ and $\mu_1, \dots, \mu_n$ in $\mathbb{R}_{>0}$. To this end, assume the following chain of equivalences:

$$0 \in \max\{u_1,\dots,u_n\} \Leftrightarrow (\forall i \in \{1,\dots,n\}) 0 \nprec u_i \Leftrightarrow (\forall i \in \{1,\dots,n\}) 0 \nprec \mu_i u_i \Leftrightarrow 0 \in \max\{\mu_1 u_1,\dots,\mu_n u_n\}. \qquad \square$$

However, as we will see in Example 3_{64}, the vacuous choice function C_{v} does not generally[17] satisfy Property $\mathrm{C5}_{25}$; we will find the least informative coherent choice function that also satisfies Property $\mathrm{C5}_{25}$ in Corollary 109_{146}.

Let us establish the counterparts for arbitrary suprema:

Proposition 49. *Consider any set* $\mathcal{C}$ *of choice functions that satisfy Property* $\mathrm{C5}_{25}$. *Then* $\sup\mathcal{C}$ *satisfies Property* $\mathrm{C5}_{25}$. *Moreover, consider any set* $\mathcal{C}$ *of choice functions that satisfy Property* $\mathrm{C6}_{25}$. *Then* $\sup\mathcal{C}$ *satisfies Property* $\mathrm{C6}_{25}$.

Proof. This follows readily from the proof of Proposition $47_{\curvearrowleft}$, by changing 'some' to 'all', 'inf' to 'sup', and '$\bigcup$' to '$\bigcap$'. $\square$

Taking Proposition 46_{52} into account, this implies that every coherent choice function that satisfies Property $\mathrm{C5}_{25}$ and/or $\mathrm{C6}_{25}$ is dominated by some maximal element of the set of all coherent choice functions that satisfy Property $\mathrm{C5}_{25}$ and/or $\mathrm{C6}_{25}$: given any coherent choice function C that satisfies Property $\mathrm{C5}_{25}$ and/or $\mathrm{C6}_{25}$, its set $\{\hat{C} \in \hat{\mathbf{C}} : C \sqsubseteq \hat{C}$ and $\hat{C}$ satisfies Property $\mathrm{C5}_{25}$ and/or $\mathrm{C6}_{25}\}$ of dominating maximal coherent choice functions that satisfy Property $\mathrm{C5}_{25}$ and/or $\mathrm{C6}_{25}$, is non-empty.

2.7 CHANGE OF OPTION SPACE

Sometimes, for instance in Chapter 8_{247}, it will be useful to consider another option space $\mathcal{W}$ instead of $\mathcal{V}$.

Consider two isomorphic ordered vector spaces $\mathcal{V}$ and $\mathcal{W}$, a linear order isomorphism ϕ—a bijective map that is linear and preserves the order—between $\mathcal{V}$ and $\mathcal{W}$. By letting ϕ work on sets—by letting, as usual, $\phi A \coloneqq \{\phi u : u \in A\}$ for all A in $\mathcal{Q}(\mathcal{V})$—, this induces an isomorphism between $\mathcal{Q}(\mathcal{V})$ and $\mathcal{Q}(\mathcal{W})$. Furthermore, by lifting it to $\mathbf{C}(\mathcal{V})$, ϕ defines an isomorphism $\tilde{\phi}$ between $\mathbf{C}(\mathcal{V})$ and $\mathbf{C}(\mathcal{W})$:

$$\tilde{\phi} : \mathbf{C}(\mathcal{V}) \to \mathbf{C}(\mathcal{W}) : C \mapsto \tilde{\phi}C \quad \text{where } (\tilde{\phi}C)(B) \coloneqq \phi(C(\phi^{-1}(B))) \text{ for all } B \text{ in } \mathcal{Q}(\mathcal{W}).$$

[17] C_{v} only satisfies Property $\mathrm{C5}_{25}$ if the vector ordering $\prec$ is *lexicographic*. This happens precisely if $\mathcal{K}^c$ is a convex cone (i.e., if $\mathrm{posi}(\mathcal{K}^c) = \mathcal{K}^c$). For more information about lexicographic orderings, see Section 4.2_{128}.

Then

$$u \in C(A) \Leftrightarrow \phi(u) \in \phi(C(A)) = \phi(C(\phi^{-1}(\phi(A)))) = (\tilde{\phi}C)(\phi(A)),$$

for all A in $\mathcal{Q}(\mathcal{V})$ and u in A. The rejection function $\tilde{\phi}R$ corresponding to $\tilde{\phi}C$ is determined by

$$\begin{aligned}(\tilde{\phi}R)(B) = B \setminus (\tilde{\phi}C)(B) = B \setminus \phi(C(\phi^{-1}(B))) &= \phi(\phi^{-1}(B) \setminus C(\phi^{-1}(B))) \\ &= \phi(R(\phi^{-1}(B)))\end{aligned}$$

for all B in $\mathcal{Q}(\mathcal{W})$, and the choice relation $\tilde{\phi}\lhd$ corresponding to $\tilde{\phi}C$ by

$$\begin{aligned}B_1(\tilde{\phi}\lhd)B_2 &\Leftrightarrow B_1 \subseteq (\tilde{\phi}R)(B_1 \cup B_2) \\ &\Leftrightarrow B_1 \subseteq \phi(R(\phi^{-1}(B_1 \cup B_2))) \\ &\Leftrightarrow \phi^{-1}(B_1) \subseteq R(\phi^{-1}(B_1 \cup B_2)) = R(\phi^{-1}(B_1) \cup \phi^{-1}(B_2)) \\ &\Leftrightarrow \phi^{-1}(B_1) \lhd \phi^{-1}(B_2)\end{aligned}$$

for all B_1 and B_2 in $\mathcal{Q}(\mathcal{W})$.

Since ϕ preserves the order, so does the isomorphism $\tilde{\phi}$. This implies that $(\mathbf{C}(\mathcal{V});\sqsubseteq)$ and $(\mathbf{C}(\mathcal{W});\sqsubseteq)$ are order-isomorphic: C and $\tilde{\phi}C$ have the same order-theoretic properties.

Essentially, C and $\tilde{\phi}C$ are the same choice functions—they represent the same choices. Note that C is coherent if and only if $\tilde{\phi}C$ is. Indeed, because ϕ is a bijection, C satisfies Axioms C1$_{20}$ and C3$_{20}$ if and only if $\tilde{\phi}C$ does; furthermore, because ϕ is order preserving, C satisfies Axiom C2$_{20}$ if and only if $\tilde{\phi}C$ does; and finally, because ϕ is linear, C satisfies Axiom C4$_{20}$ if and only if $\tilde{\phi}C$ does: such isomorphisms preserve coherence. Moreover, C satisfies Property C5$_{25}$ if and only if $\tilde{\phi}C$ does, and C satisfies Property C6$_{25}$ if and only if $\tilde{\phi}C$ does.

This observation implies moreover that $(\overline{\mathbf{C}}(\mathcal{V});\sqsubseteq)$ and $(\overline{\mathbf{C}}(\mathcal{W});\sqsubseteq)$ are order-isomorphic. Similar remarks can be made for rejection functions and choice relations as well.

2.8 PURELY BINARY CHOICE FUNCTIONS

In general, a rejection function—or a choice function or choice relation for that matter—cannot be characterised using only pairwise comparisons of options, meaning that a binary relation on options will not in general uniquely determine a choice function. Indeed, in principle none of the rationality Axioms R1$_{20}$–R4$_{20}$ excludes the following behaviour for a coherent rejection function R:

$$u \notin R(\{u,v\}) \qquad u \notin R(\{u,w\}) \qquad u \in R(\{u,v,w\})$$

where u, v and w are options. This is an instance of a *non-binary* rejection function: in every binary (or pairwise) comparison of u with v and w, u is in the choice set, but u is rejected from $\{u,v,w\}$—it is rejected using v and w *together*. As shown by Schervish et al. [61], E-admissibility is an example of an important non-binary decision rule. In this section, we study a special class of choice functions that *are* determined by their restrictions to binary option sets—option sets of cardinality two. Remark already that the corresponding choice relation of a binary rejection function will be completely determined by its restriction to $\{\{u\} : u \in \mathcal{V}\} \subseteq \mathcal{Q}$.

2.8.1 *Motivation*

Imprecise probabilities is an umbrella term for mathematical models that are meant to be used in situations of imprecise or incomplete information, where it may not be possible (or advisable) to use (precise) probabilities. In particular, it covers sets of probability measures and various types of non-additive measures and functionals, such as coherent lower previsions [51], belief functions [35, 70] and possibility measures [12, 22]. All of these models can be expressed in terms of coherent sets of desirable gambles [57, 64, 82, 83],[18] which encode the gambles that a subject, whose beliefs we want to model, strictly prefers to the status quo. One of their advantages is that they avoid problems with conditioning on events of probability zero. They can be—and have been—used to replace probabilities in Bayesian networks, for predictive inference, and so on [13, 19, 24, 31, 55].

Sets of desirable gambles are typically a binary concept: they are characterised by pairwise comparisons between the available options, whereas in practice choice may be more complex. One of the aims of this section is to study how the more general—not necessarily binary—choice functions relate to the sets of desirable gambles that are now more commonly used in imprecise probabilities papers.

2.8.2 *Sets of desirable options*

Sets of desirable options are a(n obvious and immediate) generalisation of *sets of desirable gambles*: instead of working with the linear space $\mathcal{L}$ of gambles, we will work with our general (abstract) vector space $\mathcal{V}$ of (abstract) options. We will see that sets of desirable options amount to a pairwise comparison of options and therefore correspond to a special kind of choice functions.

Definition 19 (Set of desirable options)**.** *A set of desirable options D is simply a subset of the vector space of options $\mathcal{V}$. We collect all possible such sets of*

[18] In their article from 1990, Reference [64], Seidenfeld et al. use the term 'favorable gambles'.

desirable options in the set $\mathbf{D}(\mathcal{V})$*, often simply denoted as* $\mathbf{D}$ *when it is clear from the contact what the option space* $\mathcal{V}$ *is.*

Its interpretation will be that D contains those options that some subject strictly prefers to the status quo 0. As we did for choice functions, we pay special attention to *coherent* sets of desirable options. The following is an immediate generalisation of existing coherence definitions [19, 29, 31, 57, 64, 82] from gambles to abstract options.

Definition 20 (Coherent set of desirable options)**.** *We call a set of desirable options* $D \subseteq \mathcal{V}$ coherent *if for all* u *and* v *in* $\mathcal{V}$ *and* λ *in* $\mathbb{R}_{>0}$*:*
D1. $0 \notin D$;
D2. *if* $0 < u$ *then* $u \in D$;
D3. *if* $u \in D$ *then* $\lambda u \in D$;
D4. *if* $u, v \in D$ *then* $u + v \in D$.
We collect all coherent sets of desirable options in the set $\overline{\mathbf{D}}(\mathcal{V})$*, often simply denoted as* $\overline{\mathbf{D}}$ *when it is clear from the context which vector space we are using.*

Axioms D3 and D4 turn coherent sets of desirable options D into convex cones—meaning that $\mathrm{posi}(D) = D$. They include the positive options due to Axiom D2, but not the zero option due to Axiom D1. As an immediate consequence, their intersection with $\mathcal{V}_{<0}$ is empty.

As usual, we may associate with the convex cone D a strict—irreflexive—vector order $\prec_D$—called *preference relation*—on $\mathcal{V}$, by letting $u \prec_D v \Leftrightarrow 0 \prec_D v - u \Leftrightarrow v - u \in D$ for all u and v in $\mathcal{V}$ [31, 57]. If a preference relation $\prec$ is equal to $\prec_D$, then we say that D and $\prec$ *correspond*, in the sense that D can be retrieved from $\prec_D$ as $D = \{u \in \mathcal{V} : 0 \prec_D u\}$.

Definition 21 (Coherent preference relation)**.** *We call a preference relation* $\prec$ *on* $\mathcal{V}$ coherent *if for all* u*,* v *and* w *in* $\mathcal{V}$ *and* α *in* $(0, 1]$*:*
$\prec$1. $u \nprec u$;
$\prec$2. $< \subseteq \prec$;
$\prec$3. $u \prec v \Leftrightarrow \alpha u + (1 - \alpha) w \prec \alpha v + (1 - \alpha) w$;
$\prec$4. $\prec$ *is transitive: if* $u \prec v$ *and* $v \prec w$ *then* $u \prec w$.
We collect all coherent preference relations on the linear space $\mathcal{V}$ *in the set* $\overline{\mathbf{P}}(\mathcal{V})$*, often simply denoted as* $\overline{\mathbf{P}}$ *when it is clear from the context which vector space we are using.*

Axioms $\prec$1 and $\prec$4 turn coherent choice relations into strict partial orders that include the vector order $<$ (Axiom $\prec$2) and are mixture independent (Axiom $\prec$3). A straightforward verification of the axioms shows that any set of desirable options is coherent if and only if its corresponding preference relation is. We therefore focus on either one of them, and use ‘desirability’ as an umbrella term for sets of desirable options and preference relations.

More details about coherent sets of desirable gambles (options) and coherent preference relations can be found in a number of papers and books [13, 19, 29, 31, 54, 55, 57, 58, 64, 82, 83].

2.8.3 *Order-theoretic properties of desirability*

Sets of desirable options can be ordered according to an 'at most as informative as' relation, analogously to the ordering introduced for choice functions. None of the material in this section is new, except for the straight-forward generalisation to arbitrary option spaces; see References [13, 31, 31, 57, 82] for more information.

Definition 22. *Given two sets of desirable options D_1 and D_2, we call D_1* at most as informative as *D_2 when $D_1 \subseteq D_2$. Similarly, given two preference relations $\prec_1$ and $\prec_2$, we call $\prec_1$* at most as informative as *$\prec_2$ when $\prec_1 \subseteq \prec_2$.*

Clearly, for any sets of desirable options D_1 and D_2, and their corresponding choice relations $\prec_1$ and $\prec_2$, we have that $D_1 \subseteq D_2 \Leftrightarrow \prec_1 \subseteq \prec_2$, so we can focus on sets of desirable options for the remainder of this section.

Because the ordering of sets of desirable options ($\subseteq$) is just set inclusion, it is a partial ordering on $\mathbf{D}$, and the poset $(\mathbf{D};\subseteq)$ is a complete lattice, with supremum operator $\bigcup$, and infimum operator $\bigcap$. Next we investigate the structure of the set of all coherent sets of desirable options:

Proposition 50. *The poset $(\overline{\mathbf{D}};\subseteq)$ is a complete infimum-semilattice, or alternatively, $\overline{\mathbf{D}}$ is an intersection structure—closed under arbitrary non-empty intersections. The unique least informative (smallest) set of desirable options D_{v} is given by $D_{\mathrm{v}} \coloneqq \mathcal{V}_{>0}$.*

Proof. Consider any set of coherent sets of desirable options $\mathcal{D} \subseteq \overline{\mathbf{D}}$, and its infimum $\inf\mathcal{D} = \bigcap\mathcal{D}$, which is of course also a set of desirable options. We show that $\inf\mathcal{D}$ is coherent, meaning that it satisfies the rationality axioms of Definition 20:

- $\mathrm{D}1_{\curvearrowleft}$. Since $0 \notin D$ for all D in $\mathcal{D}$, also $0 \notin \inf\mathcal{D}$.
- $\mathrm{D}2_{\curvearrowleft}$. Consider any u in $\mathcal{V}_{>0}$, so $u \in D$ for all D in $\mathcal{D}$, implying that indeed $u \in \inf\mathcal{D}$.
- $\mathrm{D}3_{\curvearrowleft}$. Consider any λ in $\mathbb{R}_{>0}$ and any $u \in \inf\mathcal{D}$, meaning that $u \in D$ for all D in $\mathcal{D}$. Then also $\lambda u \in D$ for all D in $\mathcal{D}$, implying that indeed $\lambda u \in \inf\mathcal{D}$.
- $\mathrm{D}4_{\curvearrowleft}$. Consider any u and v in $\inf\mathcal{D}$, meaning that u and v belong to all D in $\mathcal{D}$. Hence $u + v \in D$ for all D in $\mathcal{D}$, implying that indeed $u + v \in \inf\mathcal{D}$.

Now, as a convex cone, $\mathcal{V}_{>0}$ satisfies Axioms $\mathrm{D}3_{\curvearrowleft}$ and $\mathrm{D}4_{\curvearrowleft}$, and by definition it satisfies $\mathrm{D}1_{\curvearrowleft}$ and $\mathrm{D}2_{\curvearrowleft}$. So $\mathcal{V}_{>0}$ is coherent, and by Axiom $\mathrm{D}2_{\curvearrowleft}$ it is included in any other coherent set of desirable options. □

We will refer to D_{v} as the *vacuous set of desirable options*.

It will be useful to also consider the maximally informative, or *maximal*, coherent sets of desirable options.[19] They are the undominated elements of the complete infimum-semilattice $(\overline{\mathbf{D}};\subseteq)$; we collect them into a set $\hat{\mathbf{D}}$:

$$\hat{\mathbf{D}} \coloneqq \{D \in \overline{\mathbf{D}} : (\forall D' \in \overline{\mathbf{D}})(D \subseteq D' \Rightarrow D = D')\}.$$

First, the following useful result allows us to characterise these maximal elements very elegantly.

Proposition 51. *Given any coherent set of desirable options D and any non-zero option $u \notin D$,* $\operatorname{posi}(D \cup \{-u\})$ *is a coherent set of desirable options. As a consequence, a coherent set of desirable options D is maximal if and only if*

$$(\forall u \in \mathcal{V} \setminus \{0\})(u \in D \text{ or } -u \in D). \tag{2.17}$$

Proof. Let, to ease the notation, $D' \coloneqq \operatorname{posi}(D \cup \{-u\})$. It is clear that D' satisfies Axioms D2$_{57}$–D4$_{57}$, so we only need to prove that $0 \notin D'$. Assume *ex absurdo* that $0 \in D'$. Since $0 \notin D$ and $0 \neq u$, there must be v in D and λ in $\mathbb{R}_{>0}$ such that $v + \lambda(-u) = 0$, implying that $u = \frac{1}{\lambda} v \in D$ [Axiom D3$_{57}$], a contradiction.

Next, consider any maximal coherent set of desirable options D and any option u in $\mathcal{V} \setminus \{0\}$ such that $-u \notin D$. Assume *ex absurdo* that also $u \notin D$, then $D' \coloneqq \operatorname{posi}(D \cup \{-u\}) \supset D$ is a coherent set of desirable options by the first part, contradicting the maximality of D.

Conversely, consider any coherent set of desirable options D that satisfies Equation (2.17), and any coherent set of desirable options $D' \supseteq D$. Consider any u in D' then clearly $u \neq 0$ [Axiom D1$_{57}$], so we infer from (2.17) that $u \in D$ or $-u \in D$. Assume *ex absurdo* that $-u \in D$, then also $-u \in D'$, which together with $u \in D'$ implies that $0 = u + (-u) \in D'$ [Axiom D4$_{57}$], which contradicts the coherence of D' [Axiom D1$_{57}$]. Hence $u \in D$, which implies that $D' = D$, so D is indeed maximal. ☐

This immediately shows that a coherent preference relation is maximal if and only if it is a total order.

Next, note that the set of all coherent sets of desirable options is *dually atomic*, meaning that any coherent set of desirable options is the infimum of its non-empty set of dominating maximal coherent sets of desirable options:

Proposition 52 (Sets of desirable options are dually atomic)**.** *For any coherent set of desirable options D, its set of dominating maximal coherent sets of desirable options $\hat{\mathbf{D}}_D \coloneqq \{\hat{D} \in \hat{\mathbf{D}} : D \subseteq \hat{D}\}$ is non-empty. Moreover, $D = \inf \hat{\mathbf{D}}_D$.*

Proof. We have to prove that the set $\{\hat{D} \in \overline{\mathbf{D}} : D \subseteq \hat{D}\}$ has a maximal element. This will follow directly from Zorn's Lemma 44$_{52}$ if we can show that any chain $\mathcal{K}$ in this

[19] The discussion in the rest of this section is based on similar discussions about sets of desirable gambles [13, 31, 58]. I repeat the details here *mutatis mutandis* in order to make this dissertation more self-contained.

poset has a greatest element. It is a matter of straightforward verification of the axioms to see that $\bigcup\mathcal{K}$ is a coherent set of desirable options, so $\bigcup\mathcal{K}$ is this greatest element.

Let us now establish the dual atomicity. By definition, $D \subseteq \bigcap \hat{\mathbf{D}}_D$, so we concentrate on proving that $\bigcap \hat{\mathbf{D}}_D \subseteq D$. Consider any u in $\bigcap \hat{\mathbf{D}}_D$, meaning that $u \in \hat{D}$ for all $\hat{D} \in \hat{\mathbf{D}}_D$. Assume *ex absurdo* that $u \notin D$, then $-u \in \operatorname{posi}(D \cup \{-u\})$, and $\operatorname{posi}(D \cup \{-u\})$ is a coherent set of desirable options by Proposition 51$_{\curvearrowleft}$. Consider any maximal coherent set of desirable options $\hat{D}$ that dominates this set [there is such a coherent maximal set by Proposition 52$_{\curvearrowleft}$], then also $-u \in \hat{D}$ and therefore $u \notin \hat{D}$ [use Axioms D4$_{57}$ and D1$_{57}$]. But since the maximal $\hat{D}$ also dominates D, this is a contradiction. □

As we have seen, the counterpart of this result for choice models is Proposition 46$_{52}$, except for the dual atomicity, which I was not able to establish for choice models.

2.8.4 *Connection between choice functions and sets of desirable options*

We now set out to establish a connection between choice functions and sets of desirable options.

Definition 23 (Compatibility between choice models and desirability)**.** *Given a choice function C, and its corresponding rejection function R and choice relation $\lhd$, we say that an option v is* chosen over *some option u whenever $u \notin C(\{u,v\})$, or equivalently, whenever $u \in R(\{u,v\})$, or $\{u\} \lhd \{v\}$. Similarly, given a set of desirable options D, we say that an option v is* preferred to *some option u whenever $v - u \in D$, or equivalently, $u \prec v$. We call a choice function C and a set of desirable options D* compatible *when v is chosen over u if and only if v is preferred to u for all the options u and v.*

Compatibility means that the behaviour of the choice function *restricted to pairs of options* reflects the behaviour of the set of desirable options.[20] So, a choice function C will have at most one compatible set of desirable options, whereas conversely, a set of desirable options D may have many compatible choice functions: compatibility only directly influences the behaviour of a choice function on doubletons.

Definition 23 means that choice function C and a set of desirable options D are compatible if

$$u \notin C(\{u,v\}) \Leftrightarrow v - u \in D, \text{ for all } u \text{ and } v \text{ in } \mathcal{V}.$$

This is even clearer in terms of the relations: a choice relation $\lhd$ is compatible with some preference relation $\prec$ if $\{u\} \lhd \{v\} \Leftrightarrow u \prec v$, for all u and v in $\mathcal{V}$.

[20] See Reference [65] for an axiomatisation of imprecise preferences in the context of binary comparisons of horse lotteries, rather than gambles.

Loosely speaking, $\lhd$ and $\prec$ are compatible if $\lhd$ extends $\prec$ from options to sets of options. This observation also sheds light on the necessity of our Axiom C2$_{20}$, which Seidenfeld et al. [67] uses a weakened version of. Indeed, if a choice relation does not satisfy Axiom C2$_{20}$ then necessarily $0 \not\prec u$ for some $0 \prec u$, contradicting Axiom D2$_{57}$. So on their account, the relation between choice functions and desirability is more tenuous.

From choice functions to desirability

We begin by studying the properties of the set of desirable options compatible with a given coherent choice function. Since compatibility is related to the restriction of the choice function to pairwise comparisons, it is not surprising that each choice function has a unique compatible set of desirable options:

Proposition 53. *For any coherent choice function C in $\overline{\mathbf{C}}$ (and its corresponding rejection function R and choice relation $\lhd$), the unique compatible coherent set of desirable options D_C is given by*

$$D_C \coloneqq \{u \in \mathcal{V} : 0 \notin C(\{0,u\})\} = \{u \in \mathcal{V} : 0 \in R(\{0,u\})\} = \{u \in \mathcal{V} : \{0\} \lhd \{u\}\}. \tag{2.18}$$

Proof. We first show that D_C is coherent:

D1$_{57}$. Since $0 \notin C(\{0\})$, $0 \in D_C$.

D2$_{57}$. Consider any option u in $\mathcal{V}_{\succ 0}$, so $0 \prec u$. The coherence of C [Axiom C2$_{20}$] then implies that $\{u\} = C(\{0,u\})$, whence $0 \notin C(\{0,u\})$, implying that indeed $u \in D_C$.

D3$_{57}$. Consider any λ in $\mathbb{R}_{>0}$, and any u in D_C, meaning that $0 \notin C(\{0,u\})$. Lemma 12$_{21}$ then implies that $0 \notin C(\{0,\lambda u\})$, whence indeed $\lambda u \in D_C$.

D4$_{57}$. Consider any u and v in D, meaning that $0 \in R(\{0,u\})$ and $0 \in R(\{0,v\})$. By Proposition 32$_{43}$ then $0 \in R(\{0,u+v\})$, whence indeed $u+v \in D_C$.

We complete the proof by showing that C and D_C are compatible: $v \in R(\{u,v\}) \Leftrightarrow u - v \in D_C$ for all u and v in $\mathcal{V}$. For the direct implication, consider any u and v in $\mathcal{V}$ such that $v \in R(\{u,v\})$. Lemma 12$_{21}$ then guarantees that $0 \in R(\{0,u-v\})$, implying that indeed $u - v \in D_C$. For the converse implication, consider any u and v in $\mathcal{V}$ such that $u - v \in D_C$. Then $0 \in R(\{0,u-v\})$, implying that indeed $v \notin C(\{u,v\})$, by Lemma 12$_{21}$. □

Proposition 53 implies that, quite elegantly, the unique preference relation $\prec_\lhd$ compatible with a coherent choice relation $\lhd$ is coherent and given by

$$u \prec_\lhd v \Leftrightarrow \{u\} \lhd \{v\}, \text{ for all } u \text{ and } u \text{ in } \mathcal{V}. \tag{2.19}$$

Example 1. Consider the two-dimensional option space $\mathcal{V} = \mathcal{L}$ of gambles on the binary possibility space $\mathcal{X} = \{\mathrm{H},\mathrm{T}\}$, ordered by the standard point-wise ordering $\leq$. Let C be the choice function given by

$$C(A) \coloneqq \{f \in A : (\forall g \in A) g(\mathrm{H}) + g(\mathrm{T}) \leq f(\mathrm{H}) + f(\mathrm{T})\} \text{ for all } A \text{ in } \mathcal{Q}. \tag{2.20}$$

It is easy to check that C is a coherent choice function—it is an example of what we will call an E-admissible choice function later. Use Proposition 53 to infer that

$$D_C = \{f \in \mathcal{L} : f(\mathrm{H}) + f(\mathrm{T}) > 0\} \tag{2.21}$$

is its unique compatible coherent set of desirable options. ◊

From desirability to choice functions

We collect in $\overline{\mathbf{C}}_D$ all the coherent choice functions that are compatible with a given coherent set of desirable options D:

$$\overline{\mathbf{C}}_D := \{C \in \overline{\mathbf{C}} : (\forall u, v \in \mathcal{V})(v \notin C(\{u,v\}) \Leftrightarrow u - v \in D)\} = \{C \in \overline{\mathbf{C}} : D_C = D\}.$$

The correspondence between choice functions and sets of desirable options can be many-to-one, in the sense that several different coherent choice functions may be compatible with the same coherent set of desirable options: $\overline{\mathbf{C}}_D$ is in general not a singleton. The least informative of them plays an important role:

Proposition 54. *Given a coherent set of desirable options D, the infimum—least informative element—$\inf \overline{\mathbf{C}}_D$ of its set of compatible coherent choice functions $\overline{\mathbf{C}}_D$ is the coherent choice function C_D, given by*

$$\begin{aligned} C_D(A) &:= \{u \in A : (\forall v \in A) v - u \notin D\} \\ &= \{u \in A : (\forall v \in A) u \nprec v\} \text{ for all } A \text{ in } \mathcal{Q}. \end{aligned} \tag{2.22}$$

Proof. The proof is structured as follows: we show (a) that C_D is compatible with D; (b) that C_D is coherent; and (c) that $C_D \sqsubseteq C$ for all $C \in \overline{\mathbf{C}}_D$.

(a). First, we show that C_D is compatible with D: Consider any u and v in $\mathcal{V}$ then it follows from the definition of C_D that indeed

$$v \notin C_D(\{u,v\}) \Leftrightarrow (\exists w \in \{u,v\}) w - v \in D \Leftrightarrow (u - v \in D \text{ or } v - v \in D) \Leftrightarrow u - v \in D,$$

where the last equivalence follows from $0 \notin D$, because D is coherent [Axiom D1$_{57}$].

(b). Next, we show that C_D is coherent. Taking Proposition 13$_{22}$ into account, we will incidentally, for Axioms R3a$_{20}$ and R3b$_{20}$, use R_D, given by

$$R_D(A) = \{u \in A : (\exists v \in A) v - u \in D\} = A \setminus C_D(A) \text{ for all } A \text{ in } \mathcal{Q}. \tag{2.23}$$

C1$_{20}$. Consider any A in $\mathcal{Q}$. Since A is finite and $\prec$ is a strict partial order, we know that there is at least one maximal element u_m for $\prec$, meaning that $(\forall v \in A) u_m \nprec v$, or equivalently $(\forall v \in A) v - u_m \notin D$. Hence $u_m \in C_D(A)$.

C2$_{20}$. Consider any u and v in $\mathcal{V}$ such that $u \prec v$. Then $0 \prec v - u$, whence $v - u \in D$ by Axiom D2$_{57}$. So indeed $u \notin C_D(\{u,v\})$ by compatibility.

R3a$_{20}$. Consider any option sets A, A_1 and A_2 in $\mathcal{Q}$ such that $A_1 \subseteq R_D(A_2)$ and $A_2 \subseteq A$. Then $u \in R_D(A_2)$ for all u in A_1, or, equivalently, $(\forall u \in A_1)(\exists v \in A_2) v - u \in D$. Since $A_2 \subseteq A$, this implies (trivially) that $(\forall u \in A_1)(\exists v \in A) v - u \in D$, whence indeed $A_1 \subseteq R_D(A)$.

R3b$_{20}$. Consider any option sets A, A_1 and A_2 in $\mathcal{Q}$ such that $A_1 \subseteq R_D(A_2)$—whence $(\forall u \in A_1)(\exists v \in A_2) v - u \in D$—and $A \subseteq A_1$. It suffices to prove that then $(\forall u \in A_1 \setminus A)(\exists v \in A_2 \setminus A) v - u \in D$, since this implies that $A_1 \setminus A \subseteq R_D(A_2 \setminus A)$. We may assume without loss of generality that $A = \{v_1, \dots, v_n\}$ with n in $\mathbb{Z}_{\geq 0}$. Consider any u in $A_1 \setminus A$, then we know that $u_1 - u \in D$ for some u_1 in A_2. Then either $u_1 \in A_2 \setminus A$—in which case the proof is finished—or $u_1 \in A$, so we may assume that $u_1 = v_1$ without loss of generality. But then, since $A \subseteq A_1$, we know that $u_2 - v_1 \in D$ for some u_2 in A_2. By coherence of D [Axiom D1$_{57}$], we know that $u_2 \neq v_1$, and therefore also that u_2 in $A_2 \setminus \{v_1\}$. Then either $u_2 \in A_2 \setminus A$—in which case the proof is finished—or $u_2 \in A \setminus \{v_1\}$, so we may assume that $u_2 = v_2$ without loss of generality. This tells us that $v_2 - v_1 \in D$. But then, again, since $A \subseteq A_1$, we know that $u_3 - v_2 \in D$ for some u_3 in A_2. The coherence of D [Axiom D4$_{57}$] then implies that also $u_3 - v_1 = u_3 - v_2 + v_2 - v_1 \in D$, and hence that [use Axiom D1$_{57}$ twice] $u_3 \notin \{v_1, v_2\}$, so we know that u_3 in $A_2 \setminus \{v_1, v_2\}$. When we proceed in this way, we are guaranteed to find indeed, after $k \leq n+1$ steps, some u_k in $A_2 \setminus A$ such that $u_k - u \in D$.

C4$_{20}$. Consider any A_1 and A_2 in $\mathcal{Q}$ such that $A_1 \subseteq C_D(A_2)$, meaning that $(\forall u \in A_1)(\forall v \in A_2) v - u \notin D$.

C4a$_{20}$. It follows that $(\forall u \in A_1)(\forall v \in A_2) \lambda v - \lambda u \notin D$ for any λ in $\mathbb{R}_{>0}$, whence indeed $\lambda A_1 \subseteq C_D(\lambda A_2)$.

C4b$_{20}$. Also $(\forall u \in A_1)(\forall v \in A_2)(v + w) - (u + w) \notin D$ for any w in $\mathcal{V}$, whence indeed $A_1 + \{w\} \subseteq C_D(A_2 + \{w\})$.

(c). Finally, we show that $C_D \sqsubseteq C$ for all $C \in \overline{\mathbf{C}}_D$: Consider any C in $\overline{\mathbf{C}}_D$ and A in $\mathcal{Q}$, then we have to prove that $C(A) \subseteq C_D(A)$. Consider any u in A and assume that $u \notin C_D(A)$. Then $v - u \in D$ for some v in A, implying that $u \notin C(\{u, v\})$ because C is compatible with D. The coherence of C [Axiom C3a$_{20}$] then implies that also $u \notin C(A)$. □

The coherent choice function C_D is the least informative choice function that is compatible with a coherent set of desirable options D: it is based on the binary ordering represented by D *and nothing else*. As we will see in Proposition 70$_{82}$, there typically are other coherent choice functions C compatible with D, but they encode more information than just the binary ordering represented by D, and coherence.

Proposition 54 is especially interesting because it shows that the most conservative choice function based on a strict partial order of options, is the choice function based on *maximality*[21]—the one that selects the *undominated* options under the strict partial order $\prec_D$ corresponding to a coherent set of desirable options D. Any choice function that is based on maximality under such a strict partial order is coherent.

Proposition 54 can also be interpreted in terms of rejection functions and choice relations: given a coherent set of desirable gambles D—or a coherent preference relation $\prec$—, the least informative compatible rejection function is

[21]Note that maximility here refers to the optimality decision criterion [71, 82], and it does not mean that the set of desirable options is a maximal one, in the sense considered in Section 2.8.3$_{58}$.

already given in Equation (2.23)$_{62}$, and the least informative compatible choice relation $\lhd_D$, is by Proposition 39$_{47}$ equal to $\lhd_{C_D}$, and determined by

$$A_1 \lhd_D A_2 \Leftrightarrow (\forall u \in A_1)(\exists v \in A_2) u \prec v, \text{ for all } A_1 \text{ and } A_2 \text{ in } \mathcal{Q}.$$

This shows that the partial order $\lhd_D$ is simply the result of lifting $\prec$ from elements to sets.

Example 2. Let D be the coherent set of desirable options given by Equation (2.21)$_{62}$. Its associated coherent choice function C_D is given by

$$C_D(A) := \{f \in A : (\forall g \in A) g(\mathrm{H}) + g(\mathrm{T}) \leq f(\mathrm{H}) + f(\mathrm{T})\} \text{ for all } A \text{ in } \mathcal{Q}.$$

so it coincides with the choice function given by Equation (2.20)$_{61}$. ◊

Although in this particular example the two procedures coincide, there is in general more than one coherent choice function that is compatible with a coherent set of desirable options. In other words, a coherent choice function is not uniquely determined by its restriction to binary comparisons; we will give an example in Example 11$_{86}$ further on.

There is an easy characterisation of C_D and R_D, given any coherent set of desirable options D:

Proposition 55. *Given any coherent set of desirable options D, then*

$$0 \in C_D(\{0\} \cup A) \Leftrightarrow D \cap A = \emptyset, \textit{ for all } A \textit{ in } \mathcal{Q},$$

and, as a consequence

$$0 \in R_D(\{0\} \cup A) \Leftrightarrow D \cap A \neq \emptyset, \textit{ for all } A \textit{ in } \mathcal{Q}.$$

Proof. By Equation (2.22)$_{62}$, $0 \in C_D(\{0\} \cup A) \Leftrightarrow (\forall v \in \{0\} \cup A) v \notin D \Leftrightarrow (\{0\} \cup A) \cap D = \emptyset$, which is equivalent to $A \cap D = \emptyset$, because $0 \notin D$ for any coherent D. □

Although C_D is coherent when D is, it does not necessarily satisfy the additional Property C5$_{25}$, as the following counterexample shows.

Example 3. Consider the two-dimensional option space $\mathcal{V} = \mathcal{L}$ of gambles on the binary possibility space $\mathcal{X} = \{\mathrm{H}, \mathrm{T}\}$, ordered by the standard point-wise ordering $\leq$, and consider the vacuous set of desirable options $D_{\mathrm{v}} = \{\mathcal{L}(\{\mathrm{H}, \mathrm{T}\}) \in \mathcal{V} : f > 0\} = \mathcal{L}(\{\mathrm{H}, \mathrm{T}\})_{>0}$, which is coherent. By Proposition 55, $0 \in C_{D_{\mathrm{v}}}(\{0\} \cup A) \Leftrightarrow A \cap \mathcal{L}(\{\mathrm{H}, \mathrm{T}\})_{>0} = \emptyset$, for all A in $\mathcal{Q}$. To show that $C_{D_{\mathrm{v}}}$ does not satisfy Property C5$_{25}$, consider $A = \{0, f, g\}$, where $f = (f(\mathrm{H}), f(\mathrm{T})) := (-1, 2)$ and $g := (2, -1)$. We find that $0 \in C_{D_{\mathrm{v}}}(A)$ because $\{f, g\} \cap \mathcal{L}(\{\mathrm{H}, \mathrm{T}\})_{>0} = \emptyset$, since $f \not> 0$ and $g \not> 0$.

However, for the option set $A_1 = A \cup \{\frac{f+g}{2}\} \subseteq \mathrm{conv}(A)$, we find that $\frac{f+g}{2} = (1/2, 1/2) > 0$ and therefore $0 \notin C_{D_{\mathrm{v}}}(A_1)$, meaning that the coherent choice function $C_{D_{\mathrm{v}}}$ does not satisfy Property C5$_{25}$. ◊

For the specific coherent set of desirable options D considered in Example 3, the corresponding choice function C_D fails to satisfy C5$_{25}$. However, as we will see in Chapter 4$_{125}$, there are other sets of desirable options D—which we will call the *lexicographic sets of desirable options*—for which C_D does satisfy Property C5$_{25}$.

On the other hand, for any coherent set of desirable options D, its least informative compatible coherent choice function C_D is guaranteed to satisfy the other additional Property C6$_{25}$, as we will show in Proposition 56. Let us first define a special class of choice functions, the 'infimum of purely binary choice functions'. We will come back to this in Section 2.10$_{81}$, but it will turn out useful to introduce them now.

Definition 24. *For any set of coherent sets of desirable options $\mathcal{D} \subseteq \overline{\mathbf{D}}$, we define the* 'infimum of purely binary' *choice function as $C_{\mathcal{D}} \coloneqq \inf\{C_D : D \in \mathcal{D}\}$.*

Proposition 56. *Any arbitrary infimum of purely binary rejection functions satisfies Property* C6$_{25}$*: consider any $\mathcal{D} \subseteq \overline{\mathbf{D}}$, then $R_{\mathcal{D}}$ satisfies Property* R6$_{25}$.

Proof. Consider any n in $\mathbb{N}$, $u_1, \ldots, u_n$ in $\mathcal{V}$ and $\mu_1, \ldots, \mu_n$ in $\mathbb{R}_{>0}$, and assume that $0 \in R_{\mathcal{D}}(\{0, u_1, \ldots, u_n\})$. Consider any D in $\mathcal{D}$, then $0 \in R_D(\{0, u_1, \ldots, u_n\})$, so, by Proposition 55, there is some i_D such that $u_{i_D} \in D$. Due to Axiom D3$_{57}$, therefore also $\mu_{i_D} u_{i_D} \in D$. Then, using Equation (2.23)$_{62}$, $0 \in R_D(\{0, \mu_{i_D} u_{i_D}\})$, and using Axiom R3a$_{20}$, $0 \in R_D(\{0, \mu_1 u_1, \ldots, \mu_n u_n\})$. Hence indeed $R_{\mathcal{D}}(\{0, \mu_1 u_1, \ldots, \mu_n u_n\})$. □

Proposition 56 implies that there are coherent choice functions that satisfy Property C6$_{25}$ but not Property C5$_{25}$: indeed, consider the (vacuous) set of desirable options D_{v} from Example 3, then by Proposition 56, $R_{D_{\mathrm{v}}}$ satisfies Property C6$_{25}$, but in Example 3 we have shown that it does not satisfy Property C5$_{25}$. This shows that the implication in Proposition 16$_{27}$ cannot be reversed.

Before we investigate the connection between choice models and desirability in more detail, we give the following example, which shows our earlier claim that it is indeed (and luckily!) possible for an option to be rejected, even though it is undominated in the option set:

Example 4. To support the claims after Proposition 35$_{44}$, consider the following choice function C_D, where D is a non-vacuous set of desirable options, implying that $u \in D$ for some $u \in \mathcal{V} \setminus \mathcal{V}_{>0}$. Then, by Proposition 55, $0 \in R_D(\{0, u\})$, or, in other words, $\{0\} \lhd_D \{u\}$, but $\{0\} \not\preceq \{u\}$. This example proves our earlier claim that it is indeed possible for an option to be rejected, even though it is undominated in the option set.

Furthermore, we now support the claim made after that same proposition, namely that not necessarily

(iii) if $A_1 \lhd A_2$ and $A_2 \preceq A_3$ then $A_1 \preceq A_3$;
(iv) if $A_1 \preceq A_2$ and $A_2 \lhd A_3$ then $A_1 \preceq A_3$,

for all A_1, A_2 and A_3 in $\mathcal{Q}$. For the first one, consider $A_1 \coloneqq \{0\}$, $A_2 \coloneqq \{u\}$ and $A_3 \coloneqq \{u, 2u\}$. Then $A_1 \lhd A_2$ and, since $A_2 \subseteq A_3$, by Proposition 33$_{43}$(i) therefore $A_2 \preccurlyeq A_3$, even though $A_1 \not\preccurlyeq A_3$ because 0 is not dominated by u nor by $2u$. For the second one, consider $A_1 \coloneqq A_2 \coloneqq \{0\}$ and $A_3 \coloneqq \{u\}$. Then $A_1 \preccurlyeq A_2$ and $A_2 \lhd A_3$, but $A_1 \not\preccurlyeq A_3$. ◊

Properties of the relation between choice functions and desirability

Since sets of desirable options represent only pairwise comparison, and are therefore generally less expressive than choice functions, we expect that going from a choice function to a compatible set of desirable options may lead to a loss of information, whereas going the opposite route does not. This is confirmed by Propositions 57 and 58, and in particular by their Corollary 59.

Proposition 57. *Consider any non-empty collection of coherent choice functions $\mathcal{C} \subseteq \overline{\mathbf{C}}$. Then $D_{\inf \mathcal{C}} = \inf\{D_C : C \in \mathcal{C}\}$ and $C_{\inf\{D_C : C \in \mathcal{C}\}} \sqsubseteq \inf \mathcal{C}$, and therefore also $C_{D_{\inf \mathcal{C}}} \sqsubseteq \inf \mathcal{C}$.*

Proof. Recall in advance that $\inf \mathcal{C}$ is a coherent choice function by Proposition 40$_{48}$, and that $\inf\{D_C : C \in \mathcal{C}\}$ is a coherent set of desirable options by Proposition 50$_{58}$.

For the first statement, consider any u in $\mathcal{V}$, and observe that

$$\begin{aligned} u \in D_{\inf \mathcal{C}} &\Leftrightarrow 0 \notin (\inf \mathcal{C})(\{0, u\}) \Leftrightarrow 0 \notin \bigcup_{C \in \mathcal{C}} C(\{0, u\}) \\ &\Leftrightarrow (\forall C \in \mathcal{C}) 0 \notin C(\{0, u\}) \\ &\Leftrightarrow (\forall C \in \mathcal{C}) u \in D_C \\ &\Leftrightarrow u \in \bigcap\{D_C : C \in \mathcal{C}\} = \inf\{D_C : C \in \mathcal{C}\}, \end{aligned}$$

where the first and fourth equivalences follow from Proposition 53$_{61}$.

For the second statement, consider any A in $\mathcal{Q}$ and u in $\mathcal{V}$ such that $u \in (\inf \mathcal{C})(A)$. Then $u \in C(A)$ for some C in $\mathcal{C}$, from which we infer that $(\exists C \in \mathcal{C})(\forall v \in A) u \in C(\{u, v\})$, by an immediate application of Axiom C3a$_{20}$ [use the contraposition with $\tilde{A}_1 \coloneqq \{u\}$, $\tilde{A}_2 \coloneqq \{u, v\}$ and $\tilde{A} \coloneqq A$]. By exchanging the quantifiers, we infer as an implication that $(\forall v \in A)(\exists C \in \mathcal{C}) u \in C(\{u, v\})$. Now recall that

$$\begin{aligned} u \in C(\{u, v\}) &\Leftrightarrow 0 \in C(\{0, v-u\}) && \text{by Axiom C4b}_{20} \\ &\Leftrightarrow v - u \notin D_C && \text{by Proposition 53}_{61}. \end{aligned}$$

This implies that $(\forall v \in A) v - u \notin \bigcap\{D_C : C \in \mathcal{C}\}$, which is equivalent to $u \in C_{\inf\{D_C : C \in \mathcal{C}\}}(A)$, by Proposition 54$_{62}$. The rest of the proof is now immediate. □

Proposition 58. *Consider any set of coherent sets of desirable options $\mathcal{D} \subseteq \overline{\mathbf{D}}$. Then $D_{C_\mathcal{D}} = \inf \mathcal{D}$. Moreover, $C_{\inf \mathcal{D}} \sqsubseteq C_\mathcal{D}$.*

Proof. Recall in advance that $C_\mathcal{D} = \inf\{C_D : D \in \mathcal{D}\}$ is a coherent choice function by Propositions 54$_{62}$ and 40$_{48}$, and that $\inf \mathcal{D}$ is a coherent set of desirable options by Proposition 50$_{58}$.

For the first statement, consider any u in $\mathcal{V}$ and observe that:

$$\begin{aligned}
u \in D_{\inf\{C_D : D \in \mathcal{D}\}} &\Leftrightarrow 0 \notin (\inf\{C_D : D \in \mathcal{D}\})(\{0,u\}) = \bigcup_{D \in \mathcal{D}} C_D(\{0,u\}) \\
&\Leftrightarrow (\forall D \in \mathcal{D}) 0 \notin C_D(\{0,u\}) \\
&\Leftrightarrow (\forall D \in \mathcal{D})(\exists v \in \{0,u\}) v \in D \\
&\Leftrightarrow (\forall D \in \mathcal{D}) u \in D \Leftrightarrow u \in \inf \mathcal{D}.
\end{aligned}$$

For the second statement, let $\mathcal{C} \coloneqq \{C_D : D \in \mathcal{D}\}$, then we infer from the first statement that $\{D_C : C \in \mathcal{C}\} = \mathcal{D}$. Now, use the second statement in Proposition 57. $\square$

From these two results we immediately infer the following:

Corollary 59. *Consider any coherent set of desirable options $D \in \overline{\mathbf{D}}$ and any coherent choice function $C \in \overline{\mathbf{C}}$. Then $D = D_{C_D}$ and $C_{D_C} \sqsubseteq C$.*

Proof. It is an immediate consequence of Propositions 58 and 57, by letting $\mathcal{D} \coloneqq \{D\}$ and $\mathcal{C} \coloneqq \{C\}$. $\square$

If we consider the maps

$$D_\bullet : \overline{\mathbf{D}} \to \overline{\mathbf{C}} : D \mapsto C_D \text{ as defined by Equation (2.18)}_{61},$$
$$C_\bullet : \overline{\mathbf{C}} \to \overline{\mathbf{D}} : C \mapsto D_C \text{ as defined in Equation (2.22)}_{62},$$

then Corollary 59 essentially states that $D_\bullet \circ C_\bullet = \mathrm{id}_{\overline{\mathbf{D}}}$, the identity map on $\overline{\mathbf{D}}$, while $(C_\bullet \circ D_\bullet)(C) \sqsubseteq C$ for all C in $\overline{\mathbf{C}}$. This observation helps us interpret Propositions 57 and 58 using the following commuting diagrams: Full lines

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\inf"] \arrow[d, "D_\bullet"'] & \inf\mathcal{C} \arrow[d, "D_\bullet"] & \\
\{D_C : C \in \mathcal{C}\} \arrow[r, "\inf"] & \inf\{D_C : C \in \mathcal{C}\} = D_{\inf\mathcal{C}} \arrow[r, "C_\bullet"] & C_{\inf\{D_C : C \in \mathcal{C}\}} = C_{D_{\inf\mathcal{C}}} \arrow[ul, dashed]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\inf"] \arrow[d, "C_\bullet"'] & \inf\mathcal{D} = D_{C_\mathcal{D}} \arrow[r, "C_\bullet"] & C_{\inf\mathcal{D}} = C_{D_{C_\mathcal{D}}} \arrow[dl, dashed] \\
\{C_D : D \in \mathcal{D}\} \arrow[r, "\inf"] & C_\mathcal{D} \arrow[u, "D_\bullet"'] &
\end{tikzcd}$$

Figure 2.3: Commuting diagrams for Propositions 57 and 58

indicate the maps ($D_\bullet$, $C_\bullet$ or inf), while dashed lines indicate an "is at most as informative as" ($\sqsubseteq$ for choice functions, $\subseteq$ for sets of desirable gambles) relation.

Example 10_{85} in Section 2.10_{81} further on will show that the inequalities in these results can be strict; this does not seem surprising for the inequality of Proposition 57_{66}, but is perhaps less intuitively obvious for the one in Proposition 58_{66}.

This also helps to see that coherent choice functions are indeed more informative than coherent sets of desirable options, in the sense that two different coherent choice functions ($C_{\inf\mathcal{D}}$ and $C_{\mathcal{D}}$) may determine the same coherent set of desirable options ($\inf\mathcal{D}$) when restricted to option sets of cardinality two. Thus, we need to move to this richer model $C_{\mathcal{D}}$ in order to fully represent the beliefs encompassed by $\{C_D : D \in \mathcal{D}\}$. Moreover, this happens even if each of the choice functions in the set $\mathcal{D}$ is fully determined by a coherent set of desirable options.

2.8.5 *Order-theoretic properties of purely binary choice functions*

As a consequence of the connections established in the previous section, infer the following basic result:

Proposition 60. *Consider any coherent choice functions C_1 and C_2. If $C_1 \sqsubseteq C_2$, then $D_{C_1} \subseteq D_{C_2}$. Conversely, consider any coherent sets of desirable options D_1 and D_2. Then $D_1 \subseteq D_2 \Leftrightarrow C_{D_1} \sqsubseteq C_{D_2}$.*

Proof. By Proposition 39_{47} it suffices therefore to show that $R_1 \sqsubseteq R_2 \Rightarrow D_{R_1} \subseteq D_{R_2}$ for all coherent rejection functions R_1 and R_2, and that $D_1 \subseteq D_2 \Leftrightarrow R_{D_1} \sqsubseteq R_{D_2}$ for all coherent sets of desirable options D_1 and D_2.

The first statement—that $R_1 \sqsubseteq R_2 \Rightarrow D_{R_1} \subseteq D_{R_2}$—is a direct consequence of the definition (Equation $(2.18)_{61}$) of D_R.

For the second statement—that $D_1 \subseteq D_2 \Leftrightarrow R_{D_1} \sqsubseteq R_{D_2}$—we start with necessity. Consider any coherent sets of desirable options D_1 and D_2 such that $D_1 \subseteq D_2$, and any A in $\mathcal{Q}$. We will show that then $R_{D_1}(A) \subseteq R_{D_2}(A)$. To establish this, consider any u in $R_{D_1}(A)$. Then $v - u \in D_1$ for some v in A, whence, since $D_1 \subseteq D_2$, therefore $v - u \in D_2$. This implies that $u \in R_{D_2}(A)$, whence indeed $R_{D_1} \sqsubseteq R_{D_2}$. For sufficiency, assume that $R_{D_1} \sqsubseteq R_{D_2}$. We have already shown in the first part of this proof that then $D_{R_{D_1}} \subseteq D_{R_{D_2}}$. By Corollary 59↖ we find that $D_{R_{D_1}} = D_1$ and $D_{R_{D_2}} = D_2$, which completes the proof. □

The converse of the first statement does not hold:

Example 5. This example serves as a counterexample to show that the converse statement of Proposition 60 does not hold—that $D_{C_1} \subseteq D_{C_2}$ does not imply $C_1 \sqsubseteq C_2$. Assume *ex absurdo* that this implication holds for all coherent choice functions C_1 and C_2. This would imply the following stronger condition:

$$D_{C_1} = D_{C_2} \Rightarrow C_1 = C_2 \text{ for all coherent choice functions } C_1 \text{ and } C_2.$$

To see this, if $D_{C_1} = D_{C_2}$, then $D_{C_1} \subseteq D_{C_2}$—and hence $C_1 \sqsubseteq C_2$—and $D_{C_2} \subseteq D_{C_1}$—and hence $C_2 \sqsubseteq C_1$. So we infer that both $C_1 \sqsubseteq C_2$ and $C_2 \sqsubseteq C_1$, and therefore indeed $C_1 = C_2$.

We will give an example of two different coherent choice functions C_1 and C_2 that have the same binary behaviour: $D_{C_1} = D_{C_2}$. We will work with the special vector space of gambles $\mathcal{V} = \mathcal{L}$ on a binary possibility space $\mathcal{X} = \{\mathrm{H},\mathrm{T}\}$, ordered by the standard point-wise ordering $\leq$. Let $C_1 := C_{D_1}$ where $D_1 := \mathcal{L}_{>0}$ is the vacuous set of desirable gambles, and let $C_2 := \inf\{C_{D_2}, C_{D'_2}\}$, where $D_2 := \mathcal{L}_{>0} \cup \{f \in \mathcal{L} : f(\mathrm{H}) > 0\}$ and $D'_2 := \mathcal{L}_{>0} \cup \{f \in \mathcal{L} : f(\mathrm{T}) > 0\}$ are two maximal sets of desirable gambles. Note that D_1, D_2 and D'_2 are coherent, and therefore so are C_1 and C_2. Furthermore, infer that $D_2 \cap D'_2 = D_1$. To establish this counterexample, it suffices to find that $D_{C_1} = D_{C_2}$ and $C_1 \neq C_2$.

For $D_{C_1} = D_{C_2}$, use Corollary 59$_{67}$ to infer that $D_{C_1} = D_1$ and use Proposition 58$_{66}$ to infer that $D_{C_2} = \inf\{D_2, D'_2\} = D_2 \cap D'_2 = D_1$, so indeed $D_{C_1} = D_{C_2}$.

To show that $C_1 \neq C_2$, it suffices to find one option set A in $\mathcal{Q}$ such that $0 \in C_1(A)$ and $0 \notin C_2(A)$. We state that $A := \{0, f, f'\}$ with $f = (f(\mathrm{H}), f(\mathrm{T})) = (1,-1)$ and $f' = (f'(\mathrm{H}), f'(\mathrm{T})) = (-1,1)$, is such an option set. Note that indeed $0 \in C_1(A)$ since $A \cap D_1 = \emptyset$. To see that $0 \notin C_2(A)$, note that f belongs to D_2 and f' to D'_2. Therefore $0 \notin C_{D_2}(A)$ and $0 \notin C_{D'_2}(A)$, whence indeed $0 \notin (\inf\{C_{D_2}, C_{D'_2}\})(A) = C_2(A)$. ◊

Let us investigate what the least informative coherent choice function C_{D_v} that is compatible with D_v, the vacuous set of desirable options, looks like.

Proposition 61. *The least informative coherent choice function C_{D_v} that is compatible with D_v is the vacuous choice function: $C_\mathrm{v} = C_{D_\mathrm{v}}$.*

Proof. By Proposition 54$_{62}$ and since $D_\mathrm{v} \in \overline{\mathbf{D}}$, we have that C_{D_v} is a coherent choice function. Since $D_\mathrm{v} = \mathcal{V}_{>0}$, it is given by $u \in C_{D_\mathrm{v}}(A) \Leftrightarrow (\forall v \in A) u \nprec v$ for all A in $\mathcal{Q}$ and u in A, and by Equation (2.14)$_{50}$ therefore indeed equal to C_v. □

Example 6. Consider, as a simple example, the case that the vector ordering is total, meaning that for any u and v in $\mathcal{V}$, either $u \prec v$, $v \prec u$ or $u = v$. It then follows from Proposition 61 that, for any coherent choice function C, $C(A) \subseteq C_\mathrm{v}(A) = \max A$ for all $A \in \mathcal{Q}$, where $\max A$ is the singleton containing the unique largest element of the finite option set A according to the strict total ordering $\prec$. But then Axiom C1$_{20}$ guarantees that $C(A) = C_\mathrm{v}(A) = \max A$ for all $A \in \mathcal{Q}$, so C_v is the *only* coherent choice function. ◊

So the vacuous set of desirable options D_v induces the vacuous choice function C_{D_v}. Is there a similar relationship for *maximal* sets of desirable gambles $\hat{D}$? Indeed, as it turns out, $C_{\hat{D}}$ is a maximal choice function:

Proposition 62. *Consider any maximal coherent set of desirable vectors $\hat{D}$ in $\hat{\mathbf{D}}$. Its least informative coherent compatible choice function $C_{\hat{D}}$ is a maximal coherent choice function. As a consequence, $\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\} \subseteq \hat{\mathbf{C}}$.*

Proof. Since $\hat{D}$ is a maximal set of desirable options, we find by Proposition 51_{59}, that

$$(\forall u \in \mathcal{V} \setminus \{0\})(u \in \hat{D} \text{ or } -u \in \hat{D}).$$

By Equation $(2.22)_{62}$ its least informative compatible choice function $C_{\hat{D}}$ is given by

$$\begin{aligned} C_{\hat{D}}(A) &= \{u \in A : (\forall v \in A) v - u \notin \hat{D}\} \\ &= \{u \in A : (\forall v \in A \setminus \{u\}) v - u \notin \hat{D}\} = \{u \in A : (\forall v \in A \setminus \{u\}) u - v \in \hat{D}\} \end{aligned}$$

for all A in $\mathcal{Q}$, where the second equality follows from the fact that $0 \notin \hat{D}$ by Axiom $\mathrm{D1}_{57}$ and the third one because $\hat{D}$ is maximal. As an intermediate result, we will show that $|C_{\hat{D}}(A)| = 1$ for any A in $\mathcal{Q}$. Consider any A in $\mathcal{Q}$ and assume *ex absurdo* that there are u_1 and u_2 in $\mathcal{V}$ such that u_1 and u_2 belong $C_{\hat{D}}(A)$ and $u_1 \neq u_2$. Then

$$(\forall v \in A \setminus \{u_1\}) u_1 - v \in \hat{D} \text{ and } (\forall v \in A \setminus \{u_2\}) u_2 - v \in \hat{D},$$

so in particular $u_1 - u_2 \in \hat{D}$ and $u_2 - u_1 \in \hat{D}$, whence $0 \in \hat{D}$ by Axiom $\mathrm{D4}_{57}$, contradicting Axiom $\mathrm{D1}_{57}$. This implies that $|C_{\hat{D}}(A)| = 1$ for all A in $\mathcal{Q}$, because $|C_{\hat{D}}(A)| = 0$ is impossible since $C_{\hat{D}}$ is coherent by Proposition 54_{62}.

We finish the proof by showing that then $C_{\hat{D}}$ is a maximal choice function. Assume *ex absurdo* that it is not, then $C_{\hat{D}} \sqsubset C$ for some C in $\overline{\mathbf{C}}$, so $C(A) \subseteq C_{\hat{D}}(A)$ for all A in $\mathcal{Q}$, and $C(A') \subset C_{\hat{D}}(A')$ for at least one A' in $\mathcal{Q}$. Since $|C_{\hat{D}}(A')| = 1$, therefore $|C(A')| = 0$, so $C(A') = \emptyset$, a contradiction with the coherence [Axiom $\mathrm{C1}_{20}$] of C. □

Recall that we have shown in Proposition 46_{52} that $\hat{\mathbf{C}} \neq \emptyset$, but we were unable to find explicit examples of maximal choice functions there. The maximal choice functions $\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\}$ are the first maximal choice functions we encounter, and are therefore interesting in their own right. An important question is the following: can *all* the maximal choice functions be written as $C_{\hat{D}}$ with $\hat{D}$ in $\hat{\mathbf{D}}$?

Interestingly, and related to this, Proposition 62 comes in handy in showing that the class of all choice functions is a belief structure.

Definition 25 (Belief structure [23, Definition 1])**.** *Consider any class of belief models $\mathbf{B}$, partially ordered by $\sqsubseteq_{\mathbf{B}}$, and the set of coherent ones $\overline{\mathbf{B}}$, inheriting the partial order $\sqsubseteq_{\mathbf{B}}$. The structure $(\mathbf{B};\overline{\mathbf{B}};\sqsubseteq_{\mathbf{B}})$ is called a* belief structure *if it satisfies the following three properties:*

B_1. $(\mathbf{B};\sqsubseteq_{\mathbf{B}})$ *is a complete lattice;*

B_2. $(\overline{\mathbf{B}};\sqsubseteq_{\mathbf{B}})$ *is an* intersection structure*, meaning that $\overline{\mathbf{B}}$ is closed under arbitrary infima: for any subset $\mathcal{B}$ of $\overline{\mathbf{B}}$, its infimum* $\inf \mathcal{B}$ *belongs to $\overline{\mathbf{B}}$;*

B_3. $(\overline{\mathbf{B}};\sqsubseteq_{\mathbf{B}})$ *has no top.*

Proposition 63. $(\mathbf{C};\overline{\mathbf{C}};\sqsubseteq)$ *is a belief structure.*

Proof. That $(\mathbf{C};\sqsubseteq)$ is a complete lattice follows from Proposition 38$_{47}$. That $(\overline{\mathbf{C}};\sqsubseteq)$ is an intersection structure follows from Proposition 40$_{48}$. Finally, that $(\overline{\mathbf{C}};\sqsubseteq)$ has no top follows from the fact that sets of desirable options are a belief structure[22] [31], so $(\overline{\mathbf{D}};\sqsubseteq)$ has no top. Since by Proposition 62 every maximal element of $\overline{\mathbf{D}}$ induces a different maximal element of $\overline{\mathbf{C}}$, therefore indeed $(\overline{\mathbf{C}};\sqsubseteq)$ has indeed no top either. □

2.8.6 *Connection with probability*

Desirability is more general than most of the imprecise-probabilistic models, and therefore also more general than (classical) probability. Below, we give a concise overview of the relationship between desirability and most of the more popular (imprecise-)probabilities models, which is summarised in Figure 2.4. We consider *lower previsions*, *sets of linear previsions*, and *probability mass functions*, and work with the option space $\mathcal{V} = \mathcal{L}(\mathcal{X})$ of gambles on the *finite* possibility space $\mathcal{X}$, ordered by the standard point-wise ordering $\leq$. There are other models, such as *full conditional probabilities*—that allow for conditioning on arbitrary events, just like sets of desirable gambles, as we will see in Chapter 6$_{205}$—which we do not consider here.

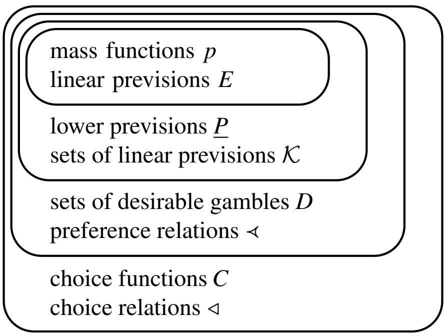


Figure 2.4: Schematic overview of the relationship between models we consider

Lower previsions

Sets of desirable gambles—and therefore also choice models—are connected with probability via lower previsions. A *lower prevision* $\underline{P}\colon\mathcal{L}(\mathcal{X}) \to \mathbb{R}$ and *upper prevision* $\overline{P}\colon\mathcal{L}(\mathcal{X}) \to \mathbb{R}$ are real-valued functionals whose domains are the set of all gambles. Consider any gamble f. Its lower prevision $\underline{P}(f)$ is

[22]They are even a *strong* belief structure, meaning that it is a belief structure that is moreover dually atomic.

the *subject's supremum buying price* for f, and its upper prevision $\overline{P}(f)$ is the *subject's infimum selling price* for f. Since selling f for a price $\mu \in \mathbb{R}$ is equivalent to buying $-f$ for a price $-\mu$, the lower prevision $\underline{P}$ and upper prevision $\overline{P}$ are related: $\overline{P}(f) = -\underline{P}(-f)$. We therefore can focus on either one of them; it is customary to focus on lower previsions.

Historically, lower previsions were first studied in 1975 by Willams [84]. Later on, they were addressed in much more detail in 1991 by Walley [82]. Since then, they are widely used as a very general imprecise-probabilistic model; see References [51, 52, 72] for an overview.

Given a set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X})$, we associate with it a lower prevision $\underline{P}_D$, defined as

$$\underline{P}_D(f) \coloneqq \sup\{\mu \in \mathbb{R} : f - \mu \in D\} \text{ for all } f \text{ in } \mathcal{L}(\mathcal{X}). \tag{2.24}$$

Since finding the gamble $f - \mu$ desirable is equivalent to preferring to buy f for the price μ, over the *status quo*—not buying nor selling anything—, this lower prevision indeed specifies the supremum buying price for any gamble. We call $\underline{P}$ coherent if there is some coherent set D of desirable gambles such that $\underline{P} = \underline{P}_D$.

Given a coherent choice function C on $\mathcal{L}(\mathcal{X})$, it induces through Equations (2.18)$_{61}$ and (2.24) a coherent lower prevision $\underline{P}_C \coloneqq \underline{P}_{D_C}$, which is given by

$$\underline{P}_C(f) = \sup\{\mu \in \mathbb{R} : 0 \notin C(\{0, f - \mu\})\} = \sup\{\mu \in \mathbb{R} : \{0\} \lhd_C \{f - \mu\}\} \tag{2.25}$$

for all f in $\mathcal{L}(\mathcal{X})$.

We are now looking for some kind of inverse operation: given a coherent lower prevision $\underline{P}$, we want to find a corresponding set of desirable gambles D that induces $\underline{P}$ (in other words, such that $\underline{P}_D = \underline{P}$), and such that D is as least informative as possible. Generally speaking, different sets of desirable gambles may induce the same lower prevision, as illustrated in Figure 2.5, where we have the binary possibility space $\{\mathrm{H}, \mathrm{T}\}$.
This is why in general coherent sets of desirable gambles are more informative than coherent lower previsions—and, as we will see below, therefore also more informative than (sets of) probability mass functions—as a belief model.

Given a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X})$, the least informative—smallest—coherent set of desirable gambles $D_{\underline{P}}$ that induces $\underline{P}$ is given by (see Reference [82, Section 3.8.1] who also introduces the term *set of strictly desirable gambles* for $D_{\underline{P}}$)

$$D_{\underline{P}} \coloneqq \mathcal{L}(\mathcal{X})_{>0} \cup \{f \in \mathcal{L}(\mathcal{X}) : \underline{P}(f) > 0\}, \tag{2.26}$$

and the least informative coherent choice function $C_{\underline{P}}$ that induces $\underline{P}$ through Equation (2.25) is by Equation (2.26) and Proposition 54$_{62}$ given by $C_{D_{\underline{P}}}$, or

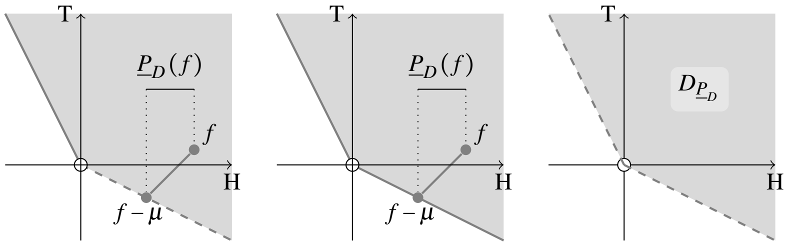


Figure 2.5: Left and middle: two different sets of desirable gambles inducing the same lower prevision. Closed border segments are indicated by full lines, and open ones by dashed lines. Right: set of strictly desirable gambles based on the lower prevision $\underline{P}_D$ from the left and middle figure.

explicitly, by

$$\begin{aligned} C_{\underline{P}}(A) &:= \{f \in A : (\forall g \in A) g - f \notin D_{\underline{P}}\} \\ &= \{f \in A : (\forall g \in A)(g \not> f \text{ and } \underline{P}(g-f) \le 0)\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X})). \end{aligned} \tag{2.27}$$

Probability mass functions and linear previsions

A *probability mass function* is an element of the *unit simplex* $\Sigma_{\mathcal{X}}$ in the linear space $\mathbb{R}^{\mathcal{X}}$ of all real-valued maps on $\mathcal{X}$:

$$\Sigma_{\mathcal{X}} := \{p \in \mathbb{R}^{\mathcal{X}} : (\forall x \in \mathcal{X}) p(x) \ge 0 \text{ and } \sum_{x \in \mathcal{X}} p(x) = 1\}.$$

A *linear prevision*—also called *expectation operator*—is an operator E on $\mathcal{L}(\mathcal{X})$ that satisfies:[23]

(i) $E(f) \ge \min f$;
(ii) $E(f+g) = E(f) + E(g)$;
(iii) $E(\lambda f) = \lambda E(f)$,

for all f and g in $\mathcal{L}(\mathcal{X})$ and λ in $\mathbb{R}$. We collect all linear previsions on $\mathcal{L}(\mathcal{X})$ in the set $\mathbb{P}_{\mathcal{X}}$. Every linear prevision is in particular a coherent lower prevision.

There is a connection between mass functions and linear previsions. With every mass function p, we let E_p be the corresponding linear prevision E_p be given by

$$E_p(f) := \sum_{x \in \mathcal{X}} p(x) f(x) \text{ for all } f \text{ in } \mathcal{L}(\mathcal{X}).$$

$E_p(f)$ is the weighted average of the values of f, according to the (probability) masses determined by p. The linear prevision E_p satisfies Conditions (i)–(iii).

[23] Actually, Property (iii) is a consequence of Properties (i) and (ii); see for instance Theorem 2.8.4 of Reference [82] and Theorem 4.16 of Reference [72].

Conversely, any linear prevision E determines a probability mass function p_E by means of indicators of elementary events:

$$p_E(x) \coloneqq E(\mathbb{I}_{\{x\}}) \text{ for all } x \text{ in } \mathcal{X}.$$

Using some basic algebra, we see that $E_{p_E} = E$ and $p_{E_p} = p$ for every linear prevision E and every probability mass function p. So we see that linear previsions and probability mass functions are in a one-to-one correspondence with each other.

Sets of linear previsions

Equivalent to lower previsions, we can consider closed and convex *sets of linear previsions* $\mathcal{K} \subseteq \mathbb{P}_{\mathcal{X}}$. Given a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X})$, we define the set of dominated linear previsions:

$$\mathcal{K}_{\underline{P}} \coloneqq \{P \in \mathbb{P}_{\mathcal{X}} : (\forall f \in \mathcal{L}(\mathcal{X}))P(f) \geq \underline{P}(f)\}.$$

This is a closed and convex non-empty subset of $\mathbb{P}_{\mathcal{X}}$. Conversely, given a closed and convex non-empty subset $\mathcal{K}$ of $\mathbb{P}_{\mathcal{X}}$, it can be shown (see Reference [82, Section 3.3.3]) that $\underline{P}_{\mathcal{K}}$, defined by

$$\underline{P}_{\mathcal{K}}(f) \coloneqq \min\{E(f) : E \in \mathcal{K}\} \text{ for all } f \text{ in } \mathcal{L}(\mathcal{X})$$

is a coherent lower prevision. Coherent lower previsions and closed and convex non-empty sets of linear previsions are in a one-to-one correspondence with each other. Moreover, due to the equivalence between linear previsions and probability mass functions, we can equivalently use a subset $\mathcal{M}$ of $\Sigma_{\mathcal{X}}$ to define a lower prevision $\underline{P}_{\mathcal{M}} \coloneqq \underline{P}_{\mathcal{K}_{\mathcal{M}}}$, where

$$\mathcal{K}_{\mathcal{M}} \coloneqq \{E_p : p \in \mathcal{M}\}.$$

Given a closed and convex set $\mathcal{K}$ of linear previsions, or a closed and convex set $\mathcal{M}$ of probability mass functions for that matter, the least informative coherent choice function compatible with it (induces $\underline{P}_{\mathcal{K}}$ or $\underline{P}_{\mathcal{M}}$ by Equation (2.25)$_{72}$) is the one given by Equation (2.27)$_{\curvearrowleft}$ that is based on the coherent lower prevision $\underline{P}_{\mathcal{K}}$, or $\underline{P}_{\mathcal{M}}$. In Section 2.10$_{81}$ we will see other coherent choice functions that are compatible with a given set of probability mass functions $\mathcal{M}$, and the assumption that $\mathcal{M}$ needs to be convex or closed can even be dropped. Interestingly, Seidenfeld et al.'s [67] coherent choice functions are in a one-to-one correspondence with (arbitrary) non-empty sets of probability mass functions and utility pairs.

2.8.7 *Archimedeanity revisited*

Seidenfeld et al.'s [67] extra rationality axiom besides Property C5$_{25}$, which allows them to prove their well-known representation result for coherent choice functions (that their coherent choice functions are dually atomic), is an Archimedean one, as already mentioned in Section 2.4$_{28}$. In Reference [86, Proposition 6], Zaffalon and Miranda show that the vacuous set of desirable options satisfies a pair-wise variant of Archimedeanity only in some pathological cases, and conclude from this that desirability is incompatible with Archimedeanity. We have adopted their arguments, ignoring the fact that the result in Reference [86, Proposition 6] holds for a pair-wise variant of Archimedeanity, and not necessarily for the version of Archimedeanity that Seidenfeld et al. [67] use.

Using the version of Archimedeanity that Seidenfeld et al. [67] consider, we set out to do something similar here in the context of choice functions, thereby strengthening our reasons for not adopting the Archimedean axiom. Translated from horse lotteries to option spaces, and from choice relations to rejection functions, Seidenfeld et al.'s [67] Archimedean axiom (see Section 2.4$_{28}$) can be written as:

For all A, A', A'', A'_i and A''_i (for i in $\mathbb{N}$) in $\mathcal{Q}$ such that the sequence A'_i converges point-wise to A' and the sequence A''_i converges point-wise to A'':

a. if $(\forall i \in \mathbb{N}) A''_i \subseteq R(A'_i \cup A''_i)$ and $A' \subseteq R(A \cup A')$ then $A'' \subseteq R(A \cup A'')$;
b. if $(\forall i \in \mathbb{N}) A''_i \subseteq R(A'_i \cup A''_i)$ and $A \subseteq R(A \cup A'')$ then $A \subseteq R(A \cup A')$.

Example 7. Consider the option space $\mathcal{V} = \mathcal{L}$ of gambles on a finite possibility space $\mathcal{X}$, and consider any maximal set of desirable gambles $\hat{D} \subseteq \mathcal{L}$. Let f be a gamble on the boundary of $\hat{D}$ that also belongs to $\hat{D}$, so that $f \in \hat{D}$ and $(\forall \varepsilon \in \mathbb{R}_{>0}) f - \varepsilon \notin \hat{D}$, and therefore, by Proposition 51$_{59}$, $(\forall \varepsilon \in \mathbb{R}_{>0}) -f + \varepsilon \in \hat{D}$. We let correspond with $\hat{D}$ its least informative coherent compatible rejection function $R_{\hat{D}}$, which, as we will see, is not Archimedean. Consider indeed the following option sets, for every i in $\mathbb{N}$:

$$A \coloneqq \{0\}, \quad A'_i \coloneqq \left\{-f + \frac{2}{i}\right\} \to A' \coloneqq \{-f\} \quad \text{and} \quad A''_i \coloneqq \left\{f + \frac{1}{i}\right\} \to A'' \coloneqq \{f\}.$$

Consider any i in $\mathbb{N}$. Then $A''_i \subseteq R_{\hat{D}}(A'_i \cup A''_i)$ since $-f + \frac{2}{i} - (f + \frac{1}{i}) = -2f + \frac{1}{i}$ belongs to $\hat{D}$. Furthermore, $A' \subseteq R_{\hat{D}}(A \cup A')$ since $0 - (-f) = f \in \hat{D}$. But $A'' \nsubseteq R_{\hat{D}}(A \cup A'')$, contradicting Archimedeanity. Indeed: $A'' \subseteq R_{\hat{D}}(A \cup A'')$ means $f \in R_{\hat{D}}(\{0, f\})$ or, in other words, $-f \in \hat{D}$, a contradiction. ◊

This shows that (at least some) maximal choice functions (based on binary choice) are not compatible with the Archimedean axioms considered in Seidenfeld et al. [67]. Since it uses the version of Archimedeanity that Seidenfeld et al. [67] use, this observation is, in our context, all the more a reason to let go off the Archimedeanity.

2.9 REJECTION SETS

As we have come to appreciate in the previous sections, one of the drawbacks of choice models is the technical difficulty in verifying the coherence axioms. In this section, we try to remedy this situation somewhat by providing an equivalent representation of choice functions in terms of those option sets that allow a subject to reject the zero option, which may be interpreted as those option sets that he should consider preferable to the status quo. We call them *rejection sets*. As we will see, this representation, in addition to capturing more intuitively the ideas underlying coherence, also helps simplify the verification of coherence in a number of particular cases.

2.9.1 *A representation in terms of rejection sets*

We give an equivalent representation of choice functions in terms of *rejection sets*.

Definition 26 (Rejection set)**.** *For any option u and any natural number i, let*

$$\mathbb{K}_{u,i} := \{A \in \mathcal{Q} : u \in R(A) \text{ and } |A| = i\} \text{ and } \mathbb{K}_u := \bigcup_{i \in \mathbb{N}} \mathbb{K}_{u,i}. \tag{2.28}$$

We call $\mathbb{K}_u$ *the* rejection set *of u.*

$\mathbb{K}_{u,i}$ is the collection of those option sets A *of cardinality i* such that u is rejected from A. Definition 26 provides an alternative representation for rejection functions R (or choice functions for that matter): Given any u in $\mathcal{V}$ and A in $\mathcal{Q}$ such that $u \in A$, the rejection set $\mathbb{K}_u$ determines whether or not u belongs to $R(A)$. Indeed, $u \in R(A) \Leftrightarrow A \in \mathbb{K}_{u,|A|} \Leftrightarrow A \in \mathbb{K}_u$, and checking this for every u in $\mathcal{V}$ and A in $\mathcal{Q}$ such that $u \in A$ fixes the entire rejection function.

We are going to characterise *coherent* choice functions in terms of these rejection sets. We can restrict our attention to the case $u = 0$:

Proposition 64. *Consider any choice function C and the family of rejection sets* $\{\mathbb{K}_u : u \in \mathcal{V}\}$ *it induces by means of Equation* (2.28). *Then*

$$C \text{ satisfies Axiom } \mathrm{C4b}_{20} \Leftrightarrow (\forall u \in \mathcal{V})\mathbb{K}_0 + \{u\} = \mathbb{K}_u.$$

Proof. For necessity, consider any option set A that includes 0. Then the option set $A + \{u\}$ includes u, and since by Axiom $\mathrm{R4b}_{20}$ it holds that $R(A + \{u\}) = R(A) + \{u\}$, we conclude that $A \in \mathbb{K}_0$ if and only if $A + \{u\} \in \mathbb{K}_f$.

Conversely, for sufficiency, consider any option set A and any option u in $\mathcal{V}$ and any $v \in R(A)$, then $A \in \mathbb{K}_v$, whence by assumption $A - \{v\} \in \mathbb{K}_0$ and as a consequence $A + \{-v + (u + v)\} = A + \{u\} \in \mathbb{K}_{u+v}$. Then indeed $u + v \in R(A + \{u\})$, so Axiom $\mathrm{C4b}_{20}$ holds. □

Taking this result into account, in what follows we will restrict our attention to rejection sets for which $\mathbb{K}_0 + \{u\} = \mathbb{K}_u$ for every u in $\mathcal{V}$. We can then simplify the notation above to

$$\mathbb{K}_i := \mathbb{K}_{0,i} = \{A \in \mathcal{Q} : 0 \in R(A) \text{ and } |A| = i\} \text{ and } \mathbb{K} := \mathbb{K}_0 = \{A \in \mathcal{Q} : 0 \in R(A)\}, \tag{2.29}$$

respectively. We can think of $\mathbb{K}$ as a straightforward generalisation of a set of desirable options D_C: just as every element u of D is strictly preferred to the zero option, similarly, for every element A of $\mathbb{K}$, the zero option is rejected from it. Note in particular the correspondence between $\mathbb{K}_2$ and sets of desirable options:

$$\mathbb{K}_2 = \{\{0,u\} : u \in \mathcal{V} \text{ and } 0 \in R(\{0,u\})\} = \{\{0,u\} : u \in D_C\}.$$

Our next result provides a characterisation of the different coherent axioms in terms of these sets:

Proposition 65. *Consider any choice function C that satisfies Axiom* C4b$_{20}$*, and consider the sets $\mathbb{K}_i$ and $\mathbb{K}$ defined in Equation* (2.29)*. Then C satisfies Axiom*

(i) C1$_{20}$ *if and only if* $(\forall A \in \mathcal{Q}_0)(\exists u \in A)A - \{u\} \notin \mathbb{K}$*;*
(ii) C2$_{20}$ *if and only if* $(\forall u \in \mathcal{V}_{>0})\{0,u\} \in \mathbb{K}_2$*;*
(iii) C3a$_{20}$ *if and only if* $(\forall A \in \mathbb{K}, A' \in \mathcal{Q}_0)(A \subseteq A' \Rightarrow A' \in \mathbb{K})$*;*
(iv) C3b$_{20}$ *if and only if* $(\forall A \in \mathbb{K}, u \in A \setminus \{0\})(A - \{u\} \in \mathbb{K} \Rightarrow A \setminus \{u\} \in \mathbb{K})$*;*
(v) C4a$_{20}$ *if and only if* $(\forall A \in \mathcal{Q}_0, \lambda \in \mathbb{R}_{>0})(A \in \mathbb{K} \Leftrightarrow \lambda A \in \mathbb{K})$.

Proof. (i) Taking Axiom C4b$_{20}$ into account, Axiom C1$_{20}$ holds if and only if $C(A) \neq \emptyset$ for every $A \in \mathcal{Q}_0$. This in turn is equivalent to $u \in C(A)$ for some u in A, which by Axiom C4b$_{20}$ is equivalent to $0 \in C(A - \{u\})$ or, in other words, to $A - \{u\} \notin \mathbb{K}$.
(ii) Under Axiom C4b$_{20}$, Axiom C2$_{20}$ is equivalent to $(\forall u \in \mathcal{V}_{>0})0 \notin C(\{0,u\})$, or, in other words, to $(\forall u \in \mathcal{V}_{>0})\{0,u\} \in \mathbb{K}_2$.
(iii) This is immediate, taking Proposition 24$_{38}$ into account.
(iv) This is immediate, taking Proposition 25$_{39}$ into account.
(v) This is immediate, taking Proposition 27$_{40}$ into account. □

An immediate consequence is:

Corollary 66. *A choice function C is coherent if and only if it satisfies Axiom* C4b$_{20}$ *and the rejection set $\mathbb{K}$ it induces by Equation* (2.29) *is increasing, scale invariant, contains $\{0,u\}$ for every $u \in \mathcal{V}_{>0}$ and satisfies the following two properties:*

(i) $(\forall A \in \mathcal{Q}_0)(\exists u \in A)A - \{u\} \notin \mathbb{K}$*;*
(ii) $(\forall A \in \mathbb{K}, \forall u \in A)(A - \{u\} \in \mathbb{K} \Rightarrow A \setminus \{u\} \in \mathbb{K})$.

The ‘at most as informative as’ relation $\sqsubseteq$ can be expressed very easily using rejection sets: let $\mathbb{K}$ be the rejection set of choice function C, and $\mathbb{K}'$ the rejection set of choice function C', then $C \sqsubseteq C' \Leftrightarrow \mathbb{K} \subseteq \mathbb{K}'$. Next we characterise the two additional Properties C5$_{25}$ and C6$_{25}$.

Proposition 67. *Consider any choice function C that satisfies Axiom* C4b$_{20}$. *Then C satisfies Property* C5$_{25}$ *if and only if*

$$(\forall A_1 \in \mathbb{K}, \forall A \in \mathcal{Q}_0)(A \subseteq A_1 \subseteq \operatorname{conv}(A) \Rightarrow A \in \mathbb{K}). \tag{2.30}$$

Moreover, C satisfies Property C6$_{25}$ *if and only if for all n in $\mathbb{N}$, all $u_1, \dots, u_n$ in $\mathcal{V}$ and all $\mu_1, \dots, \mu_n$ in $\mathbb{R}_{>0}$:*

$$\{0, u_1, u_2, \dots, u_n\} \in \mathbb{K} \Leftrightarrow \{0, \mu_1 u_1, \dots, \mu_n u_n\} \in \mathbb{K}.$$

Proof. The second statement is immediate, so it suffices to show the first statement. For necessity, application of Property C5$_{25}$ tells us that, whenever $A \subseteq A_1 \subseteq \operatorname{conv}(A)$, then $0 \in R(A_1)$ implies that $0 \in R(A)$, or, in other words, that $A_1 \in \mathbb{K}$ implies that $A \in \mathbb{K}$.

Conversely, for sufficiency, consider two option sets A and A_1 such that $A \subseteq A_1 \subseteq \operatorname{conv}(A)$. Consider any u in $C(A)$, then we want to show that $u \in C(A_1)$. From $u \in C(A)$, infer using Axiom C4b that $0 \in C(A - \{u\})$, or, in other words, that $A - \{u\} \notin \mathbb{K}$. On the other hand, $A \subseteq A_1 \subseteq \operatorname{conv}(A)$ implies $A - \{u\} \subseteq A_1 - \{u\} \subseteq \operatorname{conv}(A) - \{u\} = \operatorname{conv}(A - \{u\})$. So the assumption and $A - \{u\} \notin \mathbb{K}$ implies that $A_1 - \{u\} \notin \mathbb{K}$, or equivalently, that $u \in C(A_1)$. Hence indeed $C(A) \subseteq C(A_1)$. $\square$

2.9.2 *Particular cases*

In this section, we consider a number of particular cases of choice functions for which the representation in terms of rejection sets simplifies somewhat.

Purely binary choice functions

Let us characterise a purely binary choice function C_D, with D a set of desirable options. Quite interestingly, but not surprisingly, its representation in terms of rejection sets takes a simpler form—we only need the *binary* rejection set $\mathbb{K}_2$ of cardinality two:

Proposition 68. *Consider any coherent set of desirable options D and let $\mathbb{K}$ be the rejection set of C_D. Then $\mathbb{K}_2 = \{\{0,u\} : u \in D\}$ and*

$$\mathbb{K} = \{A \in \mathcal{Q}_0 : \{0,u\} \subseteq A \textit{ for some } u \in D\} = \{A \in \mathcal{Q}_0 : (\exists A_1 \in \mathbb{K}_2) A_1 \subseteq A\}.$$

Proof. Consider any option set A in $\mathbb{K}$. By Proposition 55$_{64}$, infer that $0 \in R_D(A)$ if and only if $A \cap D \neq \emptyset$ and $0 \in A$. If $|A| = 2$, then this implies that $A = \{0,u\}$ for some u in D, and as a consequence, infer already that $\mathbb{K}_2 \supseteq \{\{0,u\} : u \in D\}$. Conversely, consider any A' in $\mathbb{K}_2$. Then $A' = \{0,v\}$ for some v in $\mathcal{V}$. But since $0 \in R_D(A')$, we have $v \in D$, so $\mathbb{K}_2 \subseteq \{\{0,u\} : u \in D\}$, showing that indeed $\mathbb{K}_2 = \{\{0,u\} : u \in D\}$. If, on the other hand, $|A| \geq 3$, then $A \supseteq \{0,u\}$ for some u in D. But then $0 \in R_D(\{0,u\})$, so $A \supseteq A'$ for some A' in $\mathbb{K}_2$, and therefore indeed $\mathbb{K} = \{A \in \mathcal{Q}_0 : (\exists A_1 \in \mathbb{K}_2) A_1 \subseteq A\}$. $\square$

Coherent choice functions on binary spaces

Next, we consider coherent choice functions defined on *gambles on binary possibility spaces*. It turns out that, under Property $C6_{25}$, they are determined by rejection sets of cardinality two or three:

Proposition 69. *Consider any coherent choice function C on the set of gambles $\mathcal{L}(\mathcal{X})$ on a binary possibility space $\mathcal{X}$: $|\mathcal{X}| = 2$. If C satisfies Property* $C6_{25}$*, then*

$$\mathbb{K} = \big\{A \in \mathcal{Q}_0 : (\exists A_1 \in \mathbb{K}_2 \cup \mathbb{K}_3) A_1 \subseteq A\big\}.$$

Proof. Let us prove that for every A in $\mathbb{K}$ there is some A_1 in $\mathbb{K}_2 \cup \mathbb{K}_3$ for which $A_1 \subseteq A$.

Consider any A in $\mathbb{K}$. By applying Proposition 31_{42} to the option set $A \cap \mathcal{L}_{\leq 0}$, we find, since $\max(A \cap \mathcal{L}_{\leq 0}) = \{0\}$ and $(A \cap \mathcal{L}_{\leq 0}) \setminus \max(A) = A \cap \mathcal{L}_{<0}$, that $A \cap \mathcal{L}_{<0} \subseteq R(A \cap \mathcal{L}_{\leq 0})$, so Axiom $R3a_{20}$ implies that then $A \cap \mathcal{L}_{<0} \subseteq R(A)$. Since $A \in \mathbb{K}$ and therefore also $0 \in R(A)$, by Axiom $R3b_{20}$ [with $\tilde{A} := A \cap \mathcal{L}_{<0}$, $\tilde{A}_1 := \{0\} \cup (A \cap \mathcal{L}_{<0})$ and $\tilde{A}_2 := A$, so $\tilde{A}_1 \setminus \tilde{A} = \{0\}$ and $\tilde{A}_2 \setminus \tilde{A} = A \cap \mathcal{L}_{<0}^c$] we find that then $0 \in R(A \cap \mathcal{L}_{<0}^c)$ and hence also $A \cap \mathcal{L}_{<0}^c \in \mathbb{K}$, so we can assume without loss of generality that $A \cap \mathcal{L}_{<0} = \emptyset$. There are two possibilities.

If $A \cap \mathcal{L}_{>0} \neq \emptyset$, then for any f in $A \cap \mathcal{L}_{>0}$, Axiom $C2_{20}$ implies that $0 \in R(\{0, f\})$, whence the set $\{0, f\} \subseteq A$ belongs to $\mathbb{K}_2$. So we find indeed that $A_1 := \{0, f\}$ in $\mathbb{K}_2$ and $A_1 \subseteq A$.

If $A \cap \mathcal{L}_{>0} = \emptyset$, then we have $A = \{f_1, \dots, f_n, g_1, \dots, g_m\}$ for some $n \geq 0$ and $m \geq 0$ but $\max\{m, n\} \geq 1$, where f_i belongs to the second quadrant (i.e., $f_i(a) < 0 < f_i(b)$) for every i in $\{1, \dots, n\}$ and g_j belongs to the fourth quadrant (i.e., $g_j(a) > 0 > g_j(b)$) for every j in $\{1, \dots, m\}$. Let $\lambda_i := \frac{-1}{f_i(a)} > 0$ and $\mu_j := \frac{1}{g_j(a)} > 0$ for every i in $\{1, \dots, n\}$ and j in $\{1, \dots, m\}$. Then, applying Property $C6_{25}$, we find that

$$0 \in R(B) \text{ with } B := \{0, \lambda_1 f_1, \dots, \lambda_n f_n, \mu_1 g_1, \dots, \mu_m g_m\}.$$

Observe that $\lambda_i f_i(a) = -1$ for every i in $\{1, \dots, n\}$. Letting $i^* := \arg\max\{\lambda_i f_i(b) : i \in \{1, \dots, n\}\}$, we infer that

$$\lambda_k f_k(b) < \lambda_{i^*} f_{i^*}(b) \Rightarrow \lambda_k f_k \in R(\{\lambda_k f_k, \lambda_{i^*} f_{i^*}\}) \Rightarrow \lambda_k f_k \in R(B) \text{ for all } k \text{ in } \{1, \dots, n\},$$

where the first implication follows from Axiom $R2_{20}$ and the last implication follows from Axiom $R3a_{20}$. Similarly, $\mu_j g_j(a) = 1$ for every j in $1, \dots, m$, and letting $j^* := \arg\max\{\mu_j g_j(b) : j \in \{1, \dots, m\}\}$, we infer that

$$\begin{aligned} \mu_\ell g_\ell(b) < \mu_{j^*} g_{j^*}(b) &\Rightarrow \mu_\ell g_\ell \in R(\{\mu_\ell g_\ell, \mu_{j^*} g_{j^*}\}) \\ &\Rightarrow \mu_\ell g_\ell \in R(B) \qquad \text{for all } \ell \text{ in } \{1, \dots, m\}, \end{aligned}$$

where again the first implication follows from Axiom $R2_{20}$ and the last implication follows from Axiom $R3a_{20}$. If we now apply $R3b_{20}$, we deduce that $0 \in R(\{0, \lambda_{i^*} f_{i^*}, \mu_{j^*} g_{j^*}\})$, whence $0 \in R(\{0, f_{i^*}, g_{j^*}\})$, applying Property $R6_{25}$. Thus, we have found a subset $\{0, f_{i^*}, g_{j^*}\}$ of A with cardinality three that also belongs to $\mathbb{K}$. $\square$

Proposition 69 depends crucially on the assumption that $|\mathcal{X}| = 2$, as our next example shows. This example is heavily based on Reference [53]

Example 8. Consider a ternary possibility space $\mathcal{X}$, some n in $\mathbb{N}$, and let f_k be the gamble given by $f_k \coloneqq \left(-1, \frac{k}{n}, -\frac{k^2}{n^2}\right)$, for all k in $\{1,\dots,n\}$. Let us first show that for each k in $\{1,\dots,n\}$ we can find an expectation operator E_k that satisfies $E_k(f_k) > 0 > E_k(f_j)$ for every j in $\{1,\dots,n\} \setminus \{k\}$.

To find such expectation operators, let E be the expectation operator associated with the probability mass function $\left(0, \frac{2k}{n+2k}, \frac{n}{n+2k}\right)$. Then $E(f_k - f_j) = \frac{k-j}{n(n+2k)}(2k-(k+j)) = \frac{(k-j)^2}{n(n+2k)} \geq 0$, whence $E(f_k - f_j) > 0$ if $k \neq j$. Moreover, $E(f_k) = \frac{k^2}{n(n+2k)} > 0$.

If we now consider any λ in $(0,1)$ and define E_k as the expectation operator associated with the probability mass function $\left(\lambda, (1-\lambda)\frac{2k}{n+2k}, (1-\lambda)\frac{n}{n+2k}\right)$, we obtain $E_k(f_k - f_j) = (1-\lambda)E(f_k - f_j) > 0$ whenever $k \neq j$. Moreover, $E_k(f_k) = -\lambda + (1-\lambda)E(f_k)$ and $E_k(f_j) = -\lambda + (1-\lambda)E(f_j)$, so[24]

$$E_k(f_k) > 0 \Leftrightarrow \lambda < \frac{E(f_k)}{1+E(f_k)} \quad \text{and} \quad E_k(f_j) < 0 \Leftrightarrow \lambda > \frac{E(f_j)}{1+E(f_j)}.$$

Since, for every j in $\{1,\dots,n\} \setminus \{k\}$, $\frac{E(f_j)}{1+E(f_j)} < \frac{E(f_k)}{1+E(f_k)}$ because $E(f_j) < E(f_k)$, we let λ be an arbitrary element of

$$\left(\max_{j \in \{1,\dots,n\} \setminus \{k\}} \frac{E(f_j)}{1+E(f_j)}, \frac{E(f_k)}{1+E(f_k)}\right),$$

and for this λ we find that $E_k(f_k) > 0 > E_k(f_j)$ for every $j \neq k$.

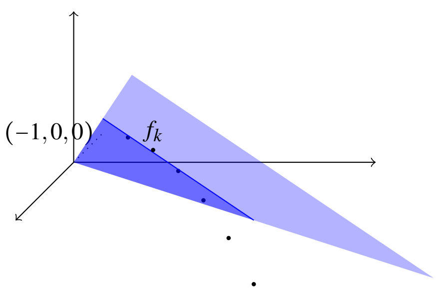


Figure 2.6: Illustration of the gambles involved in Example 8

[24] $E(f_k) > 0$ implies that $1 + E(f_k) \neq 0$; to prove that $1 + E(f_j) \neq 0$, note that $E(f_j) = \frac{j(2k-j)}{n(n+2k)}$, so $1 + E(f_j) = \frac{j(2k-j)+n(n+2k)}{n(n+2k)}$ is equal to 0 if and only if $j(2k-j) + n(n+2k) = 0$, because the denominator of $1+E(f_j)$ is strictly positive. Observe that $j(2k-j)+n(n+2k) = j^2 - 2kj - 2kn + n^2$ is equal to 0 if and only if $j = \frac{2k \pm 2(k+n)}{2} = k \pm (k+n)$, which is impossible since j belongs to $\{1,\dots,n\}$.

Now, let D_k be the coherent set of gambles given by $D_k \coloneqq \{f \in \mathcal{L} : E_k(f) > 0\}$, and let C_{D_k} be its least informative compatible coherent choice function. Then, by Proposition 40_{48}, the choice function $C \coloneqq \inf\{C_{D_k} : k \in \{1,\dots,n\}\}$ is also coherent, and by Proposition 56_{65} it satisfies Property $\mathrm{C6}_{25}$. If we now consider the option set $A = \{0, f_1,\dots,f_n\}$, we find that $C_{D_k}(A) = \{f_k\}$ for every k in $\{1,\dots,n\}$, since $E_k(f_k) > 0 > E_k(f_j)$ implies that both f_k and $f_k - f_j$ belong to D_k, for every j in $\{1,\dots,n\} \setminus \{k\}$. As a consequence, we find that $C(A) = \{f_1,\dots,f_n\}$, whence $A \in \mathbb{K}$. However, for every k in $\{1,\dots,n\}$ it holds that $C_{D_k}(A \setminus \{f_k\}) = \{0\}$, using again that $E_k(f_j) < 0$ for every $j \neq k$, therefore $C(A \setminus \{f_k\}) = A \setminus \{f_k\}$. Therefore A has no proper subset that also belongs to the rejection set $\mathbb{K}$. ◊

It is a consequence of coherence that a choice function is uniquely determined by those option sets that allow us to reject the zero option, i.e., those that are considered preferable to the status quo. In this section, we have investigated the structure of these sets and shown that the coherence axioms can be expressed rather elegantly and intuitively in terms of these sets. In addition, we have shown that all the necessary information is given by option sets of cardinality two for purely binary choice functions, and with cardinality two or three when the possibility space is binary and the choice function satisfies Property $\mathrm{C6}_{25}$. Moreover, we have shown that this last result does not extend to larger possibility spaces: determining an analogous representation for arbitrary spaces is an important open problem.

2.10 EXAMPLES OF COHERENT CHOICE FUNCTIONS

Given a coherent set of desirable options, we can define a coherent choice function by selecting undominated elements as in Equation $(2.22)_{62}$. However, these are not the only possible coherent choice functions: for instance, any infimum of such coherent choice functions (see Definition 24_{65}) is still coherent. This observation gives a procedure to define coherent choice functions complying with a sensitivity analysis interpretation, where the underlying uncertainty model is that of coherent sets of desirable options: we may consider a set $\mathcal{D}$ of possible models and then the set of coherent choice functions they determine; by taking the infimum of this set we end up choosing those options that are considered acceptable by at least one of the possible models.

To end this chapter about coherent choice functions, let us give some interesting examples of coherent choice functions that are sometimes considered in the literature.

Example 9. Consider again the two-dimensional option space $\mathcal{V} = \mathcal{L}$ of gambles on the binary possibility space $\mathcal{X} = \{\mathrm{H},\mathrm{T}\}$, and let D_1 an D_2 be the coherent sets of desirable options given by

$$D_1 = \{f \in \mathcal{L}(\mathcal{X}) : f(\mathrm{H}) + f(\mathrm{T}) > 0\} \text{ and } D_2 = \{f \in \mathcal{L}(\mathcal{X}) : 2f(\mathrm{H}) + f(\mathrm{T}) > 0\}.$$

Then the choice function $C = \inf\{C_{D_1}, C_{D_2}\}$ is coherent. If we denote arbitrary gambles f in $\mathcal{L}(\mathcal{X})$ as $f = (f(\mathrm{H}), f(\mathrm{T}))$, and if we consider for instance the option set $A = \{(4,-4),(2,-1),(0,0),(-1,2)\}$, we obtain

$$C(A) = \inf\{C_{D_1}, C_{D_2}\}(A) = C_{D_1}(A) \cup C_{D_2}(A) = \{(2,-1),(-1,2),(4,-4)\}.$$

The first two options are elements of $C_{D_1}(A)$ and the last one is an element of $C_{D_2}(A)$. ◊

As special cases, we may consider 'infimum of purely binary' choice functions where some additional condition is imposed on the coherent sets of desirable options. We will investigate two such situations below, although others are possible. In Definition 27, we focus only on sets of *maximal* coherent sets of desirable options, which we introduced in Section 2.6$_{46}$, and found first instances of in Section 2.8$_{55}$. Their interest lies in the fact that they are related to probability mass functions, as discussed quite thoroughly in References [13, 31, 58] in the context of gambles.

Definition 27 (M-admissible choice function)**.** *If $\mathcal{D} \subseteq \hat{\mathbf{D}}$ is a set of* maximal *coherent set of desirable options, the coherent choice function $C_{\mathcal{D}}$ is called* M-admissible. *We will also denote it by $C^{M}_{\mathcal{D}}$ as a reminder that the infimum is taken over maximal sets.*

In particular, we can consider the M-admissible choice functions for the set $\mathcal{D} = \hat{\mathbf{D}}_D$ of all maximal coherent set of desirable options that include a coherent set of desirable options D. In order not to overburden the notation, we let

$$C^{\mathrm{M}}_{D} \coloneqq C^{\mathrm{M}}_{\hat{\mathbf{D}}_D} = \inf\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}} \text{ and } D \subseteq \hat{D}\}, \tag{2.31}$$

and similarly to what we did before, we introduce the map

$$C^{\mathrm{M}}_{\bullet} : \mathcal{D} \to \mathcal{C} : D \mapsto C^{\mathrm{M}}_{D} \text{ as defined in Equation } (2.31)_{82}.$$

The following result can be regarded as a particular case of Proposition 58$_{66}$, where all the coherent sets of desirable options are maximal ones. As we have seen there too, the diagram below commutes if we focus on sets of desirable options, but this is no longer the case if we consider the more informative model of coherent choice functions.

Proposition 70. *Consider any coherent set of desirable options $D' \in \overline{\mathbf{D}}$. Then $D' = D_{C^{M}_{D'}}$ and $C_{D'} \sqsubseteq C^{M}_{D'}$.*

Proof. Consider any u in $\mathcal{V}$, then

$$\begin{aligned}
u \in D_{C^{\mathrm{M}}_{D'}} &\Leftrightarrow 0 \notin C^{\mathrm{M}}_{D'}(\{0,u\}) && \text{by Proposition } 53_{61}\\
&\Leftrightarrow (\forall \hat{D} \in \hat{\mathbf{D}}_{D'})\, 0 \notin C_{\hat{D}}(\{0,u\}) && \text{by Definition } 27_{82}\\
&\Leftrightarrow (\forall \hat{D} \in \hat{\mathbf{D}}_{D'})\, u \in \hat{D} && \text{by Proposition } 53_{61}\\
&\Leftrightarrow u \in \bigcap \hat{\mathbf{D}}_{D'} \Leftrightarrow u \in D' && \text{by Proposition } 52_{59},
\end{aligned}$$

$$\begin{tikzcd}
D' \arrow[d, shift right, "C^{\mathrm{M}}_{\bullet}"'] \arrow[dr, "\hat{C}"] & \\
C^{\mathrm{M}}_{D'} \arrow[u, shift right, "\hat{D}"'] & C_{D'} \arrow[l, dashed]
\end{tikzcd}$$

Figure 2.7: Commuting diagram for M-admissible choice functions

proving the first statement. It then follows from Proposition 53_{61} that D' is compatible with $C^{\mathrm{M}}_{D'}$, and therefore from Proposition 54_{62} that $C_{D'} = \inf \overline{\mathbf{C}}_{D'} \sqsubseteq C^{\mathrm{M}}_{D'}$. □

The inequality in Proposition 70 can be strict—meaning that $C_{D'} \sqsubset C^{\mathrm{M}}_{D'}$ for some coherent set of desirable options D'—as we will show in Example 11_{86} below.

As another special case, we consider choice functions associated with Levi's notion of E-admissibility [48, Chapter 5]. They are based on a non-empty *set of probability mass functions*. Consider a *finite* possibility space $\mathcal{X}$.

With any probability mass function p, we associate (as a special case of Equation $(2.26)_{72}$) a set of desirable gambles

$$D_p := \mathcal{L}_{>0} \cup \{f \in \mathcal{L} : E_p(f) > 0\} \tag{2.32}$$

and a choice function C_p defined for all A in $\mathcal{Q}$ by

$$C_p(A) := \{f \in A : (\forall g \in A)(E_p(f) \geq E_p(g) \text{ and } f \nless g)\} \text{ for all } A \text{ in } \mathcal{Q}. \tag{2.33}$$

Proposition 71. *The set of desirable gambles D_p and the choice function C_p are coherent and compatible, and moreover $C_p = C_{D_p}$.*

Proof. By Proposition 54_{62}, it suffices to prove (a) that D_p is coherent; and (b) that $C_p = C_{D_p}$.

(a) is proven in Reference [82, Appendix F], for instance. In order to make this dissertation more self-contained, we provide an explicit proof here. That Axiom $\mathrm{D1}_{57}$ holds, follows from $0 \notin \mathcal{L}_{>0}$ and $E_p(0) = 0$. Axiom $\mathrm{D2}_{57}$ holds by definition. For Axiom $\mathrm{D3}_{57}$, consider any f in D_p and real λ in $\mathbb{R}_{>0}$, then $0 < f$ and therefore $0 < \lambda f$, or $E_p(f) > 0$ and therefore $E_p(\lambda f) = \lambda E_p(f) > 0$, whence indeed $\lambda f \in D_p$. For Axiom $\mathrm{D4}_{57}$, consider any f and g in D_p, then there are three possibilities. The first is that both f and g belong to $\mathcal{L}_{>0}$, and therefore also $f + g \in \mathcal{L}_{>0}$. The second is that both $E_p(f) > 0$ and $E_p(g) > 0$, and therefore also $E_p(f+g) = E_p(f) + E_p(g) > 0$. And the third is that, without loss of generality, $E_p(f) > 0$ and $g \in \mathcal{L}_{>0}$, whence $E_p(g) \geq 0$ and therefore $E_p(f+g) = E_p(f) + E_p(g) > 0$. In all cases therefore indeed $f + g \in D_p$.

For (b), consider any A in $\mathcal{Q}$, use Equation $(2.22)_{62}$ to find that $C_{D_p}(A) = \{f \in A : (\forall g \in A) g - f \notin D_p\}$ and Equation (2.32) to find that $g - f \notin D_p \Leftrightarrow (g - f \notin \mathcal{L}_{>0} \text{ and } E_p(g-f) \leq 0)$. It then follows from Equation (2.33) that indeed $C_p(A) = C_{D_p}(A)$. □

This result allows us to introduce the following, second special case of 'infimum of purely binary' choice functions.

Definition 28 (E-admissible choice function)**.** *With any non-empty set of probability mass functions $\mathcal{M}$,*[25] *we associate the corresponding* E-admissible *choice function* $C^{\mathrm{E}}_{\mathcal{M}} := \inf\{C_p : p \in \mathcal{M}\} = C_{\{D_p : p \in \mathcal{M}\}}$.

Proposition 72. *Given any non-empty set of probability mass functions $\mathcal{M}$, we have for all A in $\mathcal{Q}$ that*

$$C^{\mathrm{E}}_{\mathcal{M}}(A) = \Big\{ f \in A : (\exists p \in \mathcal{M}) E_p(f) \in \operatorname*{arg\,max}_{g \in A} E_p(g) \Big\} \cap C_{\mathrm{v}}(A).$$

Proof. We infer from Definition 28 and Proposition 71↶ that

$$C^{\mathrm{E}}_{\mathcal{M}}(A) = \bigcup_{p \in \mathcal{M}} C_{D_p}(A) = \bigcup_{p \in \mathcal{M}} \{ f \in A : (\forall g \in A) g - f \notin D_p \},$$

where the last equality follows from Proposition 54_{62}. Now

$$\begin{aligned}(\forall g \in A) g - f \notin D_p &\Leftrightarrow (\forall g \in A)(f \nless g \text{ and } E_p(g-f) \leq 0) \\ &\Leftrightarrow f \in C_{\mathrm{v}}(A) \text{ and } (\forall g \in A) E_p(g-f) \leq 0),\end{aligned}$$

where the first equivalence follows from Equation (2.32)↶, and the second from Proposition 61_{69}. Hence indeed

$$\begin{aligned}C^{\mathrm{E}}_{\mathcal{M}}(A) &= \bigcup_{p \in \mathcal{M}} \{ f \in A : (\forall g \in A) E_p(g-f) \leq 0 \} \cap C_{\mathrm{v}}(A) \\ &= \{ f \in A : (\exists p \in \mathcal{M})(\forall g \in A) E_p(g) \leq E_p(f) \} \cap C_{\mathrm{v}}(A).\end{aligned}$$ □

The following proposition establishes a connection between M-admissible and E-admissible choice functions.

Proposition 73. *For any non-empty set of probability mass functions $\mathcal{M}$, $C^{\mathrm{E}}_{\mathcal{M}} \sqsubseteq C^{M}_{\hat{\mathbf{D}}_{\mathcal{M}}}$, where $\hat{\mathbf{D}}_{\mathcal{M}} := \bigcup_{p \in \mathcal{M}} \hat{\mathbf{D}}_{D_p} \subseteq \hat{\mathbf{D}}$.*

Proof. We consider any p in $\mathcal{M}$ and prove that $C_{D_p} \sqsubseteq C^{\mathrm{M}}_{\hat{\mathbf{D}}_{\{p\}}}$. The proof follows by taking the infimum $\inf_{p \in \mathcal{M}}$ over $p \in \mathcal{M}$ on both sides of this inequality. Consider any A in $\mathcal{Q}$ and assume that $f \in C^{\mathrm{M}}_{\hat{\mathbf{D}}_{\{p\}}}(A)$, so there is some $\hat{D}$ in $\hat{\mathbf{D}}_{\{p\}}$ such that $f \in C_{\hat{D}}(A)$, or equivalently, $(\forall g \in A) g - f \notin \hat{D}$. Hence $(\forall g \in A) g - f \notin D_p$, because $D_p \subseteq \hat{D}$, and therefore indeed $f \in C_{D_p}(A)$. □

[25] Although Levi's notion of E-admissibility was originally concerned with *convex closed* sets of probability mass functions [48, Chapter 5], we impose no such requirement here on the set $\mathcal{M}$.

The key for this result is that, for any probability mass function p, there is in general more than one coherent set of desirable options D that is associated with it by means of the formula

$$E_p(f) = \sup\{\mu \in \mathbb{R} : f - \mu \in D\}. \tag{2.34}$$

Among all the coherent sets of desirable options satisfying Equation (2.34) with respect to a fixed p, the least informative one is the one given by Equation $(2.32)_{83}$, which is usually referred to as the set of *strictly desirable* gambles associated with p within the imprecise probabilities literature. This in turn gives rise to a coherent choice function that will be less informative than one determined by a maximal set of options that is compatible with p by means of Equation (2.34).

Thus, the choice between E-admissible and M-admissible coherent choice functions can be made by considering our attitude towards imprecision, that determines the use of strictly desirable or maximal sets of options: the former are as conservative as possible, and make a choice only when it is implied by the probability mass function p; while the latter are as informative as it can be considering the axioms of coherence and the probability mass function p.

The following examples show why choice functions are more powerful than sets of desirable options as uncertainty representations, and elucidate the difference between E-admissible and M-admissible choice functions.

Example 10. Consider the situation where you have a coin with two *identical* sides of unknown type:[26] either both sides are heads (H), or both sides are tails (T). The uncertain variable that represents the outcome of a coin flip assumes a value in the finite possibility space $\mathcal{X} := \{\mathrm{H}, \mathrm{T}\}$. The options we consider are gambles: real-valued functions on $\mathcal{X}$, which constitute the two-dimensional vector space $\mathbb{R}^{\mathcal{X}}$, ordered by the pointwise order. We model this situation using (a) coherent sets of desirable gambles, (b) M-admissible choice functions, and (c) E-admissible choice functions. In all three cases we start from two simple models: one that describes practical certainty of H and another that describes practical certainty of T, and we take their infimum—the most informative model that is still at most as informative as either—as a candidate model for the coin problem.

For (a), we use two coherent sets of desirable gambles D_{H} and D_{T}, expressing practical certainty of H and T, respectively, given by the *maximal* sets of desirable gambles $D_{\mathrm{H}} := \mathcal{L}_{>0} \cup \{f \in \mathcal{L} : f(\mathrm{H}) > 0\}$ and $D_{\mathrm{T}} := \mathcal{L}_{>0} \cup \{f \in \mathcal{L} : f(\mathrm{T}) > 0\}$. The model for the coin with two identical sides is then $D_{\mathrm{H}} \cap D_{\mathrm{T}} = \mathcal{L}_{>0}$. This vacuous model D_{v} is incapable of distinguishing between this situation and the one where we are completely ignorant about the coin.

[26] The example can be trivially reformulated to consider an uncertain variable taking values in a binary possibility space, with only one of those elements occurring; however we think the use of a coin adds some intuition.

For an approach (b) that distinguishes between these two situations, we draw inspiration from Proposition 57_{66}: instead of working with the sets of desirable gambles themselves, we move to the corresponding choice functions $C_{\mathrm{H}} \coloneqq C_{D_{\mathrm{H}}}$ and $C_{\mathrm{T}} \coloneqq C_{D_{\mathrm{T}}}$, where

$$\begin{aligned}
C_{\mathrm{H}}(A) &= \{f \in A : (\forall g \in A) g - f \notin D_{\mathrm{H}}\} = \arg\max\{f(\mathrm{H}) : f \in A\} \cap C_{\mathrm{v}}(A) \\
&= \arg\max\{g(\mathrm{T}) : g \in \arg\max\{f(\mathrm{H}) : f \in A\}\} \\
C_{\mathrm{T}}(A) &= \arg\max\{f(\mathrm{T}) : f \in A\} \cap C_{\mathrm{v}}(A) \\
&= \arg\max\{g(\mathrm{H}) : g \in \arg\max\{f(\mathrm{T}) : f \in A\}\}
\end{aligned}$$

for all A in $\mathcal{Q}$. We infer that $|C_{\mathrm{H}}(A)| = |C_{\mathrm{T}}(A)| = 1$ for every A in $\mathcal{Q}$: for instance in the case of C_{H}, note that amongst all the options attaining the maximum value on heads, exactly one of them is undominated. The M-admissible choice function we are looking for is $C^{\mathrm{M}}_{\{D_{\mathrm{H}},D_{\mathrm{T}}\}} = \inf\{C_{\mathrm{H}}, C_{\mathrm{T}}\}$, which selects at most two options from each option set. It is given by

$$C^{\mathrm{M}}_{\{D_{\mathrm{H}},D_{\mathrm{T}}\}}(A) = (\arg\max\{f(\mathrm{H}) : f \in A\} \cup \arg\max\{f(\mathrm{T}) : f \in A\}) \cap C_{\mathrm{v}}(A)$$

for all A in $\mathcal{Q}$, and differs from the vacuous choice function C_{v}. Indeed, consider the particular option set $A = \{f, g, h\}$, where $f = (1, 0)$, $g = (0, 1)$ and $h = (1/2, 1/2)$. Then $C^{\mathrm{M}}_{\{D_{\mathrm{H}},D_{\mathrm{T}}\}}(A) = \{f, g\} \neq A = C_{\mathrm{v}}(A)$.

For (c), the set of probability mass functions $\mathcal{M}$ consists of the two degenerate probability mass functions: $\mathcal{M} = \{p_{\mathrm{H}}, p_{\mathrm{T}}\}$, where $p_{\mathrm{H}} = (1, 0)$ and $p_{\mathrm{T}} = (0, 1)$. The corresponding expectations $E_{\mathrm{H}} \coloneqq E_{p_{\mathrm{H}}}$ and $E_{\mathrm{T}} \coloneqq E_{p_{\mathrm{T}}}$ satisfy $E_{\mathrm{H}}(f) = f(\mathrm{H})$ and $E_{\mathrm{T}}(f) = f(\mathrm{T})$ for all f in $\mathcal{L}$. So we see that $C_{p_{\mathrm{H}}} = C_{\mathrm{H}}$ and $C_{p_{\mathrm{T}}} = C_{\mathrm{T}}$, and therefore this approach leads to the same choice function as the previous one: $C^{\mathrm{E}}_{\{p_{\mathrm{H}},p_{\mathrm{T}}\}} = C^{\mathrm{M}}_{\{D_{\mathrm{H}},D_{\mathrm{T}}\}} = \inf\{C_{\mathrm{H}}, C_{\mathrm{T}}\}$. ◊

The example above shows that the correspondence between desirability and choice functions is not a complete inf-homomorphism.

Example 11. In this example, we illustrate the difference between E-admissible and M-admissible choice functions. We consider the same finite possibility space $\mathcal{X} \coloneqq \{\mathrm{H}, \mathrm{T}\}$ as in Example $10_{\curvearrowleft}$, with the same option space and vector ordering. For both E-admissibility and M-admissibility, we each time consider the least informative choice functions: the E-admissible choice function $C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}}$ associated with set of all probability mass functions $\mathcal{M} = \Sigma_{\mathcal{X}}$, and the M-admissible choice function $C^{\mathrm{M}}_{D_{\mathrm{v}}}$ associated with the set of all maximal sets of desirable gambles $\hat{\mathbf{D}}_{D_{\mathrm{v}}} = \hat{\mathbf{D}}$. Since $C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}}$ and $C^{\mathrm{M}}_{D_{\mathrm{v}}}$ are the most conservative E-admissible, respectively M-admissible choice functions, we wonder about the relationship between them, as well as their relationship with the vacuous choice function C_{v}. We find that $C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}} \sqsubseteq C^{\mathrm{M}}_{D_{\mathrm{v}}}$. Indeed, consider any A in $\mathcal{Q}$ and any f in $C^{\mathrm{M}}_{D_{\mathrm{v}}}(A)$, being equivalent to $0 \in C^{\mathrm{M}}_{D_{\mathrm{v}}}(A - \{f\})$ by Proposition 27_{40}

whence

$$f \in C^{\mathrm{M}}_{D_{\mathrm{v}}}(A) \Leftrightarrow (\exists \hat{D} \in \hat{\mathbf{D}})(\forall g \in A - \{f\}) g \notin \hat{D} \Leftrightarrow (\exists \hat{D} \in \hat{\mathbf{D}}) A - \{f\} \cap \hat{D} = \varnothing.$$

Since for every $\hat{D}$ in $\hat{\mathbf{D}}$, there is some probability mass function p in $\Sigma_{\mathcal{X}}$ such that $D_p \subseteq \hat{D}$ [it suffices to consider the probability mass function p corresponding with $\hat{D}$], we find that

$$f \in C^{\mathrm{M}}_{D_{\mathrm{v}}}(A) \Rightarrow (\exists p \in \Sigma_{\mathcal{X}}) A - \{f\} \cap D_p = \varnothing \Leftrightarrow (\exists p \in \Sigma_{\mathcal{X}}) f \in C_{D_p}(A).$$

or in other words, $f \in C^{\mathrm{M}}_{D_{\mathrm{v}}}(A)$ implies that $f \in C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}}(A)$, whence $C^{\mathrm{M}}_{D_{\mathrm{v}}}(A) \subseteq C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}}(A)$. By the definition of the vacuous choice function, we have as an intermediate result that $C_{\mathrm{v}} \sqsubseteq C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}} \sqsubseteq C^{\mathrm{M}}_{D_{\mathrm{v}}}$.

Both inequalities are strict; to show that $C_{\mathrm{v}} \neq C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}}$, consider the option set $A \coloneqq \{0, f, g\}$, where $f = (1, -1/4)$ and $g = (-1/4, 1)$. Because all options in A are point-wise undominated in A, we find that $C_{\mathrm{v}}(A) = A$, and in particular, that $0 \in C_{\mathrm{v}}(A)$. On the other hand, it follows from Proposition 7284 that

$$0 \in C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}}(A) \Leftrightarrow (\exists p \in \Sigma_{\mathcal{X}})(0 \geq p(\mathrm{H}) - \frac{1}{4} p(\mathrm{T}) \text{ and } 0 \geq -\frac{1}{4} p(\mathrm{H}) + p(\mathrm{T})),$$

which would imply that $(\exists p \in \Sigma_{\mathcal{X}})(0 \geq \frac{3}{4} p(\mathrm{H}) + \frac{3}{4} p(\mathrm{T}))$, which is impossible.

More importantly, we also have that $C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}} \neq C^{\mathrm{M}}_{D_{\mathrm{v}}}$. Consider the option set $A \coloneqq \{0, f, -f\}$, where $f = (1, -1)$. Then for the specific probability mass function $p \coloneqq (1/2, 1/2) \in \Sigma_{\mathcal{X}}$, we find that $0 \in C_p(A)$, whence $0 \in C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}}(A)$. To show that $0 \in C^{\mathrm{E}}_p(A)$, infer that $0 = E_p(0) = E_p(f) = E_p(-f)$, and use Proposition 7284 as a characterisation for the E-admissible choice functions. On the other hand, $0 \in C^{\mathrm{M}}_{D_{\mathrm{v}}}(A)$ is equivalent to $f \in \hat{D}$ and $-f \notin \hat{D}$ for some $\hat{D}$ in $\hat{\mathbf{D}}$. But $f \notin \hat{D}$ and $-f \notin \hat{D}$ implies that $-f \in \hat{D}$ and $f \in \hat{D}$ by Proposition 5159, a contradiction. So $0 \notin C^{\mathrm{M}}_{D_{\mathrm{v}}}(A)$, whence $C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}} \neq C^{\mathrm{M}}_{D_{\mathrm{v}}}$.

This example shows that $C_{\mathrm{v}} = C_{D_{\mathrm{v}}} = C_{\overline{\mathbf{D}}} \sqsubset C^{\mathrm{E}}_{\Sigma_{\mathcal{X}}} \sqsubset C_{\hat{\mathbf{D}}} = C^{\mathrm{M}}_{D_{\mathrm{v}}}$. ◊

We can interpret the example above in terms of choice relations, in the following manner: in the case of a complete preference relation we always have that $\{0\} \lhd_C \{u, -u\}$ for every option u. This is not the case for those induced by sets of strictly desirable gambles, such as the coherent choice function C_p in Example 11, which therefore cannot be obtained as infima of a family of complete choice relations (as are those given by M-admissibility).

There are other coherent sets of desirable gambles that can be associated with a probability mass function and that are intermediate between the strictly desirable and the maximal ones. One example are the so-called *lexicographic* sets of desirable gambles, which we will investigate in detail in Chapter 4125.

Taking into account Proposition 40_{48}, we can also define coherent choice functions by taking the infimum of a family of coherent choice functions determined by such lexicographic sets. As we will see, this provides another example of coherent choice function that admits an axiomatic characterisation in some cases.

To conclude this section, we want to mention that there are other popular choice rules besides maximality and E-admissibility, such as, amongst others, Γ-*maximin*, Γ-*maximax* and *interval dominance* [71]. However, they are not coherent: none of them is guaranteed to satisfy, amongst others, Axiom $C4b_{20}$.

3

Natural extension

In the previous chapter, we have seen what a coherent choice function is, and how it relates to other theories of uncertainty, through the connection with desirability. We took for granted that the choice function is given, and coherent.

However, it is somewhat unrealistic to assume that the subject always specifies an entire choice function C: this means that he would have to specify for every A in $\mathcal{Q}$ and every u in A whether or not he rejects u in A—whether or not u belongs to $R(A)$, or equivalently, whether or not u belongs to $C(A)$—and this in a coherent fashion such that his assessment satisfies Axioms $\mathrm{C1}_{20}$–$\mathrm{C4}_{20}$. Rather, a subject will typically specify a choice function only partially, by specifying the rejection of *some* u from *some* A. We call this partial specification of a choice function his *assessment*. Such an assessment can consist of an *arbitrary* number of rejection statements; we do not want to rule out here the possibility that the subject's assessment consists of an uncountable collection of rejection statements. What does such an assessment imply about the choice for other options and other option sets? In other words, given such an assessment, what is the implied choice between other option sets, using *only* the consequences of coherence?

We will define the natural extension, when it exists, as the least committal coherent choice function that 'extends' a given assessment. In the following section, we describe in more detail what assessments can look like, and what we mean by 'extending' an assessment.

3.1 ASSESSMENTS

The subject is typically allowed to make statements of the form "I reject u from the option set B",[1] meaning that his rejection function R should satisfy $u \in R(B)$—and equivalently, that his choice function C should satisfy $u \notin C(B)$. He can express such a statement for several B in $\mathcal{Q}$, and within one given B, for several u in B. We do not require that, for a given B in $\mathcal{Q}$, the subject specifies *the complete* set $R(B)$, nor that he considers *all* the option sets in $\mathcal{Q}$ to reject options from. Instead, we allow that, for *some* option sets B, he provides a *subset* of $R(B)$—or equivalently that he precludes *some* options in *some* B from being an element from $C(B)$. From this discussion, it is already apparent that it is more meaningful or natural to give assessments in terms of rejection functions than in terms of choice functions.

Indeed, we cannot reverse the role of C and R here: it makes no sense that the subject provides us with a subset of $C(B)$, since assessing *less*—specifying a smaller subset of $C(B)$—would correspond to a *more informative* and less conservative choice. Therefore, he should either provide us with a superset A of $C(B)$ such that $A \subseteq B$, or with a subset of $R(B)$. Both approaches are equivalent, but the latter is more directly interpretable, which is why we will assume here that the subject specifies his beliefs using rejection statements.

Formally, then, an assessment is a subset of

$$\{(A,v) : A \in \mathcal{Q}, v \in A\},$$

whose elements are couples (B,u) whose interpretation is "the subject rejects u from B". At this point, we will require that the assessment should satisfy Axiom $\mathrm{R4b}_{20}$: u is rejected from B if and only if 0 is rejected from $B - \{u\}$. It therefore can be assumed that the subject gives his assessment directly in the form of "I reject 0 from B". Any assessment can therefore be reduced to a subset of

$$\{(A,0) : A \in \mathcal{Q}_0\},$$

or, equivalently and more directly, *any assessment is simply a subset of* $\mathcal{Q}_0$. The subject collects option sets from which he rejects 0 in his assessment $\mathcal{B} \subseteq \mathcal{Q}_0$, which we regard as an incomplete specification of the subject's choice function. The interpretation is, again, that he rejects 0 from every B in $\mathcal{B}$.

Ideally, we want to 'extend' the assessment $\mathcal{B}$ to a coherent rejection function R that reflects the information in $\mathcal{B}$, in the sense that $0 \in R(B)$ for every B in $\mathcal{B}$:

Definition 29 (Extending an assessment)**.** *Given any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$ and any rejection function R on $\mathcal{Q}$, we say that R* extends the assessment *$\mathcal{B}$ if $0 \in R(B)$ for every B in $\mathcal{B}$.*

[1] Throughout, when such a statement is given, we silently assume that u belongs to B.

Under certain conditions, it is possible to find an extension that is coherent, as we will see in the following sections. At this point, we can already mention that the property of extending an assessment is closed under (arbitrary) infima:

Proposition 74. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$ and any non-empty collection $\mathcal{R}$ of rejection functions that extend $\mathcal{B}$. Then* $\inf \mathcal{R}$ *extends $\mathcal{B}$.*

Proof. Since $\mathcal{R}$ extends $\mathcal{B}$, we have that $(\forall B \in \mathcal{B})(\forall R \in \mathcal{R}) 0 \in R(B)$, and therefore also that $(\forall B \in \mathcal{B}) 0 \in \bigcap_{R \in \mathcal{R}} R(B) = (\inf \mathcal{R})(B)$, so $\inf \mathcal{R}$ indeed extends $\mathcal{B}$. □

Some assessments are stronger than others—or, in other words, imply others. For instance, a subset of some assessment is again an assessment, which is implied by—and therefore weaker than—the former one. Let us formalise this.

Definition 30. *Given any two assessments $\mathcal{B}_1 \subseteq \mathcal{Q}_0$ and $\mathcal{B}_2 \subseteq \mathcal{Q}_0$, we say that $\mathcal{B}_1$ is* at least as strong as *$\mathcal{B}_2$ if $(\forall B_2 \in \mathcal{B}_2)(\exists B_1 \in \mathcal{B}_1) B_1 \lessdot B_2$.*

Let us verify whether a subset of an assessment is indeed at least as weak as its original assessment. If $\mathcal{B}_2 \subseteq \mathcal{B}_1$, then $(\forall B_2 \in \mathcal{B}_2)(\exists B_1 \in \mathcal{B}_1) B_1 = B_2$, so by Proposition 33₄₃(i) and Definition 30, $\mathcal{B}_1$ is indeed at least as strong as $\mathcal{B}_2$. The importance of this *at least as strong as* relation for assessments lies in the following proposition, which will be important for the proof of Corollary 88₁₀₆.

Proposition 75. *Consider any two assessments $\mathcal{B}_1 \subseteq \mathcal{Q}_0$ and $\mathcal{B}_2 \subseteq \mathcal{Q}_0$ such that $\mathcal{B}_1$ is at least as strong as $\mathcal{B}_2$, and any coherent rejection function R on $\mathcal{Q}$. If R extends $\mathcal{B}_1$, then R extends $\mathcal{B}_2$.*

Proof. Consider any B_2 in $\mathcal{B}_2$. Then $B_1 \lessdot B_2$ for some B_1 in $\mathcal{B}_1$, and therefore, because R extends $\mathcal{B}_1$, we have that $0 \in R(B_1)$. By Proposition 34₄₄ then also $0 \in R(B_2)$. Since the choice of B_2 was arbitrary in $\mathcal{B}_2$, we infer that $0 \in R(B_2)$ for all B_2 in $\mathcal{B}_2$, so R indeed extends $\mathcal{B}_2$. □

3.2 DEFINING THE NATURAL EXTENSION

Now that we have discussed how a subject may specify an incomplete assessment $\mathcal{B}$ of his rejection function, we are ready to define the natural extension of this assessment.

Definition 31 (Natural extension)**.** *Given any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$, the* natural extension *of $\mathcal{B}$ is the rejection function*

$$\mathcal{E}(\mathcal{B}) := \inf\{R \in \overline{\mathbf{R}} : (\forall B \in \mathcal{B}) 0 \in R(B)\} = \inf\{R \in \overline{\mathbf{R}} : R \textit{ extends } \mathcal{B}\},$$

where we let $\inf \varnothing$ *be equal to* $\mathrm{id}_{\mathcal{Q}}$*, the identity rejection function that maps every option set to itself.*

We can equivalently define the natural extension as a *choice function*—or a *choice relation* for that matter—instead of a rejection function, but that turns out to be notationally more involved, which is why we have decided to use *rejection functions*. The translation to the other types of choice models is straightforward, also for the remaining results in this section.

Corollary 76. *Consider any two assessments $\mathcal{B}_1 \subseteq \mathcal{Q}_0$ and $\mathcal{B}_2 \subseteq \mathcal{Q}_0$ such that $\mathcal{B}_1$ is at least as strong as $\mathcal{B}_2$. Then $\mathcal{E}(\mathcal{B}_2) \sqsubseteq \mathcal{E}(\mathcal{B}_1)$.*

Proof. By Proposition $75_{\curvearrowleft}$, $\{R \in \overline{\mathbf{R}} : R \text{ extends } \mathcal{B}_1\} \subseteq \{R \in \overline{\mathbf{R}} : R \text{ extends } \mathcal{B}_2\}$, and therefore indeed $\mathcal{E}(\mathcal{B}_2) = \inf\{R \in \overline{\mathbf{R}} : R \text{ extends } \mathcal{B}_2\} \sqsubseteq \inf\{R \in \overline{\mathbf{R}} : R \text{ extends } \mathcal{B}_1\} = \mathcal{E}(\mathcal{B}_1)$. □

Definition $31_{\curvearrowleft}$ is not very useful for practical inference purposes: it does not provide an explicit constructive expression for $\mathcal{E}(\mathcal{B})$. To try and remedy this, consider the special rejection function $R_{\mathcal{B}}$ based on the assessment $\mathcal{B}$, defined as:

$$R_{\mathcal{B}}(A) \coloneqq \Big\{u \in A : (\exists A' \in \mathcal{Q})\Big(A' \supseteq A \text{ and } (\forall v \in \{u\} \cup (A' \setminus A))$$
$$\big((A' - \{v\}) \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \preccurlyeq A'\big)\Big)\Big\} \tag{3.1}$$

for all A in $\mathcal{Q}$.

Although the expression for $R_{\mathcal{B}}$ seems involved and, admittedly, inelegant, its interpretation should be clear. An option u can be rejected from A—is an element of $R_{\mathcal{B}}(A)$—in two ways: by considering the option set A itself (corresponding to choosing $A' = A$ in Equation (3.1)), or some strictly larger option set A' (corresponding to choosing $A' \supset A$ in Equation (3.1)).

Let us focus first on the case that $A' = A$. Then there are again two ways to reject an option u from A: trivially, meaning that $u \prec v$ for some v in A (and equivalently $(A - \{u\}) \cap \mathcal{V}_{>0} \neq \emptyset$, or equivalently $u \notin \max A$), or using the assessment $\mathcal{B}$, meaning that $\{u\} + \mu B \preccurlyeq A$ for some B in $\mathcal{B}$ and μ in $\mathbb{R}_{>0}$. Taking into account coherence, and more particularly Axiom R4_{20}, $0 \in R_{\mathcal{B}}(B)$ (and therefore also $B \in \mathcal{B}$) implies $u \in R_{\mathcal{B}}(\{u\} + \mu B)$ for all μ in $\mathbb{R}_{>0}$, whence, by Proposition 34_{44}, $\{u\} + \mu B \preccurlyeq A$ implies that $u \in R_{\mathcal{B}}(A)$.

If on the other hand $A' \supset A$, then an option u is rejected from A if u is rejected from A' and, additionally, all the supplementary options in $A' \setminus A$ are rejected from A', using at least one of the two ways as before. The idea is that then Axiom R3b_{20} guarantees that u is rejected from A.

The rejection function $R_{\mathcal{B}}$ satisfies some interesting properties. For instance, it always satisfies the rationality Axioms R2_{20}–R4_{20}.

Lemma 77. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$. Then $R_{\mathcal{B}}$ satisfies Axioms* R2_{20}–R4_{20}.

Proof. We check Axioms $\mathrm{R2}_{20}$–$\mathrm{R4}_{20}$ in the following order: Axiom $\mathrm{R2}_{20}$, $\mathrm{R4a}_{20}$ and $\mathrm{R4b}_{20}$, and only then $\mathrm{R3a}_{20}$, and $\mathrm{R3b}_{20}$, because Propositions 24_{38} and 25_{39} then allow us to prove easier equivalent variants of Axiom $\mathrm{R3}_{20}$.

For Axiom $\mathrm{R2}_{20}$, consider any u and v in $\mathcal{V}$ such that $u \prec v$. We need to show that $u \in R_{\mathcal{B}}(\{u,v\})$, so consider the option set $A \coloneqq \{u,v\}$, and let A' be equal to A in Equation (3.1). Since $u \prec v$, also $0 \prec v-u$, so $(A' - \{u\}) \cap \mathcal{V}_{\succ 0} = \{0, v-u\} \cap \mathcal{V}_{\succ 0} \neq \emptyset$. Therefore indeed $0 \in R_{\mathcal{B}}(\{u,v\})$.

For Axiom $\mathrm{R4a}_{20}$, we show that $R_{\mathcal{B}}$ satisfies the equivalent version $(\mathrm{R4a.1})_{21}$. So consider any A in $\mathcal{Q}$, λ in $\mathbb{R}_{>0}$ and u in A, and assume that $u \in R_{\mathcal{B}}(A)$. Then there is some $A' \supseteq A$ in $\mathcal{Q}$ such that

$$(\forall v \in \{u\} \cup (A' \setminus A))\big((A' - \{v\}) \cap \mathcal{V}_{\succ 0} \neq \emptyset \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \preceq A'\big). \tag{3.2}$$

We need to prove that then $\lambda u \in R_{\mathcal{B}}(\lambda A)$, so consider the option set λA. Note that $\lambda A' \supseteq \lambda A$, and consider any v' in $\{\lambda u\} \cup (\lambda A' \setminus \lambda A)$—so $v' = \lambda v$ for some v in $\{u\} \cup (A' \setminus A)$. Infer that $(A' - \{v\}) \cap \mathcal{V}_{\succ 0} \neq \emptyset$ implies that $(\lambda A' - \{v'\}) \cap \mathcal{V}_{\succ 0} \neq \emptyset$. Also, by Proposition 33_{43}(vi), $(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \preceq A'$ implies that $(\exists B' \in \mathcal{B}, \exists \mu' \in \mathbb{R}_{>0})\{v'\} + \mu' B' \preceq \lambda A'$ [consider $B' = B$ and $\mu' = \lambda\mu > 0$]. Therefore indeed $\lambda u \in R_{\mathcal{B}}(\lambda A)$.

For Axiom $\mathrm{R4b}_{20}$, we show that $R_{\mathcal{B}}$ satisfies the equivalent version $(\mathrm{R4b.1})_{21}$. So consider any A in $\mathcal{Q}$, and u and w in A, and assume that $u \in R_{\mathcal{B}}(A)$. Then there is some $A' \supseteq A$ such that Equation (3.2) holds. We need to prove that then $u + w \in R_{\mathcal{B}}(A + \{w\})$, so consider the option set $A + \{w\}$. Note that $A' + \{w\} \supseteq A + \{w\}$, and consider any v' in $\{u+w\} \cup ((A' + \{w\}) \setminus (A + \{w\}))$—so $v' = v + w$ for some v in $\{u\} \cup (A' \setminus A)$. Infer that $A' + \{w\} - \{v+w\} = A' - \{v\}$—so $(A' - \{v\}) \cap \mathcal{V}_{\succ 0} \neq \emptyset \Leftrightarrow (A' + \{w\} - \{v+w\}) \cap \mathcal{V}_{\succ 0} \neq \emptyset$. Also, by Proposition 33_{43} (v), $\{v\} + \mu B \preceq A'$ implies that $\{v'\} + \mu B \preceq A' + \{w\}$—so $(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \preceq A'$ implies that $(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v'\} + \mu B \preceq A' + \{w\}$. Therefore indeed $u + w \in R_{\mathcal{B}}(A + \{w\})$.

For Axiom $\mathrm{R3a}_{20}$, due to Proposition 24_{38}, and because we just have shown that $R_{\mathcal{B}}$ satisfies Axiom $\mathrm{R4b}_{20}$, it suffices to show that $0 \in R_{\mathcal{B}}(A) \Rightarrow 0 \in R_{\mathcal{B}}(A \cup \{u\})$ for all A in $\mathcal{Q}$ and u in $\mathcal{V}$. So consider any A in $\mathcal{Q}$ and u in $\mathcal{V}$, and assume that $0 \in R_{\mathcal{B}}(A)$, meaning that there is some $A' \supseteq A$ in $\mathcal{Q}$ such that

$$(\forall v \in \{0\} \cup (A' \setminus A))\big((A' - \{v\}) \cap \mathcal{V}_{\succ 0} \neq \emptyset \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \preceq A'\big). \tag{3.3}$$

We need to prove that then $0 \in R_{\mathcal{B}}(A \cup \{u\})$: we need to find some $A'' \supseteq A \cup \{u\}$ in $\mathcal{Q}$ such that

$$\begin{aligned}&(\forall v \in \{0\} \cup (A'' \setminus (A \cup \{u\})))\\ &\qquad\qquad\big((A'' - \{v\}) \cap \mathcal{V}_{\succ 0} \neq \emptyset \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \preceq A''\big).\end{aligned}$$

If $u \in A$, then the proof is done: it suffices to consider $A'' = A'$, so assume that $u \notin A$. We state that the particular choice $A'' \coloneqq A' \cup \{u\} \supseteq A \cup \{u\}$ satisfies the equation above. To prove this, infer first that $\{0\} \cup (A'' \setminus (A \cup \{u\})) \subseteq \{0\} \cup (A' \setminus A)$ [indeed, if $u \notin A'$, then $\{0\} \cup (A'' \setminus (A \cup \{u\})) = \{0\} \cup (A' \setminus A)$; if on the other hand $u \in A'$, then $\{0\} \cup (A'' \setminus (A \cup \{u\})) = \{0\} \cup (A' \setminus (A \cup \{u\})) \subseteq \{0\} \cup (A' \setminus A)$]. Consider any v in $\{0\} \cup (A'' \setminus (A \cup \{u\}))$. Then $v \in \{0\} \cup (A' \setminus A)$, whence, by Equation (3.3) $(A' - \{v\}) \cap \mathcal{V}_{\succ 0} \neq \emptyset$—implying that $(A'' - \{v\}) \cap \mathcal{V}_{\succ 0} \neq \emptyset$ since $A' \subseteq A''$—or $(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \preceq$

A'—by Proposition 33$_{43}$(i) implying that $(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A''$. Therefore indeed $0 \in R_{\mathcal{B}}(A \cup \{u\})$.

For Axiom R3b$_{20}$, due to Proposition 25$_{39}$, and because we have already shown that $R_{\mathcal{B}}$ satisfies Axiom R4b$_{20}$, it suffices to show that $0 \in R_{\mathcal{B}}(A) \Rightarrow 0 \in R_{\mathcal{B}}(A \setminus \{u\})$ for all A in $\mathcal{Q}$ and u in $R_{\mathcal{B}}(A) \setminus \{0\}$. So consider any A in $\mathcal{Q}$ and u in $R_{\mathcal{B}}(A) \setminus \{0\}$, and assume that $0 \in R_{\mathcal{B}}(A)$, meaning that there is some $A' \supseteq A$ in $\mathcal{Q}$ such that Equation (3.3)$_{\curvearrowleft}$ holds. Because also $u \in R_{\mathcal{B}}(A)$, there is some $A'_u \supseteq A$ in $\mathcal{Q}$ such that

$$(\forall v \in \{u\} \cup (A'_u \setminus A))\big((A'_u - \{v\}) \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A'_u\big). \tag{3.4}$$

We need to prove that then $0 \in R_{\mathcal{B}}(A \setminus \{u\})$: we need to find some $A'' \supseteq A \setminus \{u\}$ in $\mathcal{Q}$ such that

$$(\forall v \in \{0\} \cup (A'' \setminus (A \setminus \{u\})))$$
$$\big((A'' - \{v\}) \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A''\big).$$

We state that the particular choice $A'' := A' \cup A'_u \supseteq A \setminus \{u\}$ satisfies the equation above. To prove this, infer first that $\{0\} \cup (A'' \setminus (A \setminus \{u\})) = \big(\{0\} \cup (A' \setminus (A \setminus \{u\}))\big) \cup \big(\{0\} \cup (A'_u \setminus (A \setminus \{u\}))\big) = \big(\{0\} \cup (A' \setminus A)\big) \cup \big(A'_u \setminus (A \setminus \{u\})\big) = \big(\{0\} \cup (A' \setminus A)\big) \cup \big(\{u\} \cup (A'_u \setminus A)\big)$. Consider any v in $\{0\} \cup (A'' \setminus (A \setminus \{u\}))$. Then $v \in \{0\} \cup (A' \setminus A)$ or $v \in \{u\} \cup (A'_u \setminus A)$. If $v \in \{0\} \cup (A' \setminus A)$, then by Equation (3.3)$_{\curvearrowleft}$ $(A' - \{v\}) \cap \mathcal{V}_{>0} \neq \emptyset$—implying that $(A'' - \{v\}) \cap \mathcal{V}_{>0} \neq \emptyset$ since $A' \subseteq A''$—or $(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A'$—by Proposition 33$_{43}$(i) implying that $(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A''$. If $v \in \{u\} \cup (A'_u \setminus A)$, then by Equation (3.4) $(A'_u - \{v\}) \cap \mathcal{V}_{>0} \neq \emptyset$—implying that $(A'' - \{v\}) \cap \mathcal{V}_{>0} \neq \emptyset$ since $A'_u \subseteq A''$—or $(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A'_u$—by Proposition 33$_{43}$(i) implying that $(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A''$. Therefore indeed $0 \in R_{\mathcal{B}}(A \setminus \{u\})$. □

A second interesting property of $R_{\mathcal{B}}$ is that it extends $\mathcal{B}$. This is necessary if we want to use $R_{\mathcal{B}}$ as a more constructive expression for the natural extension.

Lemma 78. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$. Then $R_{\mathcal{B}}$ extends $\mathcal{B}$.*

Proof. By Definition 29$_{90}$, we need to prove that $0 \in R_{\mathcal{B}}(B)$ for all B in $\mathcal{B}$. So consider any B in $\mathcal{B}$. By Equation (3.1)$_{92}$, it suffices to prove that there is some $A \supseteq B$ in $\mathcal{Q}$ such that

$$(\forall v \in \{0\} \cup (A \setminus B))\big((A - \{v\}) \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists B' \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B' \lessdot B\big).$$

We will show that the particular choice $A := B$ satisfies this condition. Indeed, $\{0\} \cup (A \setminus B) = \{0\}$, so we need only consider $v = 0$. Let $B' := B$ and $\mu := 1$, then $\{v\} + \mu B' = \{0\} + 1B = B \lessdot B$, using Proposition 33$_{43}$(i) in the last step. So we have found an option set $A \supseteq B$ such that, for every v in $\{0\} \cup (A \setminus B)$, we have $\{v\} + \mu B' \lessdot B$ for some B' in $\mathcal{B}$ and μ in $\mathbb{R}_{>0}$, whence indeed $0 \in R_{\mathcal{B}}(B)$. □

So now we know already that $R_{\mathcal{B}}$ satisfies the rationality Axioms R2$_{20}$–R4$_{20}$ and extends $\mathcal{B}$. In general, there are other rejection functions with these properties, but we are interested in the *least informative* one. The following proposition guarantees that $R_{\mathcal{B}}$ is the least informative rejection function with these properties.

Proposition 79. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$. Then $R_{\mathcal{B}}$ is the least informative rejection function that satisfies Axioms* R2$_{20}$–R4$_{20}$ *and extends $\mathcal{B}$.*

Proof. We already know from Lemma 77$_{92}$ and Lemma 78 that $R_{\mathcal{B}}$ satisfies Axioms R2$_{20}$–R4$_{20}$ and extends $\mathcal{B}$, so it suffices to show that $R_{\mathcal{B}}$ is the least informative such rejection function. Consider any rejection function R' that satisfies Axioms R2$_{20}$–R4$_{20}$ and extends $\mathcal{B}$. We will show that $R_{\mathcal{B}} \sqsubseteq R'$, or, in other words, that $R_{\mathcal{B}}(A) \subseteq R'(A)$ for all A in $\mathcal{Q}$. Since both $R_{\mathcal{B}}$ and R' satisfy Axiom R4b$_{20}$, it suffices to show that $0 \in R_{\mathcal{B}}(A) \Rightarrow 0 \in R'(A)$ for all A in $\mathcal{Q}$. So consider any A in $\mathcal{Q}$ and assume that $0 \in R_{\mathcal{B}}(A)$. Infer already that then $0 \in A$. By Equation (3.1)$_{92}$, there is some $A' \supseteq A$ in $\mathcal{Q}$ such that

$$(\forall v \in \{0\} \cup (A' \setminus A))\big((A' - \{v\}) \cap \mathcal{V}_{>0} \neq \varnothing \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A'\big).$$

Consider any v in $\{0\} \cup (A' \setminus A)$. Then, by the equation above, $(A' - \{v\}) \cap \mathcal{V}_{>0} \neq \varnothing$—and therefore $v \prec u$ for some u in A', whence by Axiom R2$_{20}$, $v \in R'(\{u, v\})$, so by Axiom R3a$_{20}$, $v \in R'(A')$—or $\{v\} + \mu B \lessdot A'$ for some B in $\mathcal{B}$ and μ in $\mathbb{R}_{>0}$—and therefore, since R' extends $\mathcal{B}$, $0 \in R'(B)$, so by Axiom R4a$_{20}$, we have that $0 \in R'(\mu B)$, and using Axiom R4b$_{20}$, that $v \in R'(\{v\} + \mu B)$, and therefore finally, using Proposition 34$_{44}$, we infer that $v \in R'(A')$. So we have shown that $v \in R'(A')$ for every v in $\{0\} \cup (A' \setminus A)$, and therefore $\{0\} \cup (A' \setminus A) \subseteq R'(A')$. Use Axiom R3b$_{20}$ [with $\tilde{A} := A' \setminus A$, $\tilde{A}_1 := \{0\} \cup (A' \setminus A)$ and $\tilde{A}_2 := A'$; then $\tilde{A}_1 \setminus \tilde{A} = \{0\}$ since $0 \in A \subseteq A'$ and $\tilde{A}_2 \setminus \tilde{A} = A$ since $A \subseteq A'$] to infer that then indeed $0 \in R'(A)$. □

3.3 ASSESSMENTS AVOIDING COMPLETE REJECTION

To investigate under which conditions the natural extension of an assessment is coherent, we need to know whether the assessment can be extended to a coherent rejection function. The question we need to answer, is: "When is an assessment $\mathcal{B}$ extendible to a coherent rejection rejection function?"

After inspection of the rationality axioms R1$_{20}$–R4$_{20}$, we see that all axioms but the first are *productive*, in the sense that application of these axioms allows us to identify new rejected options within, possibly, new option sets. Axiom R1$_{20}$ however is a *destructive* one: it indicates how far our rejections can go, and where the inferences should stop. Indeed, it requires that, within a given option set A, not every element of A may be rejected. In other words, it requires that, for any given option set, we should choose at least one of its elements. Therefore we need to be careful and avoid assessments that lead to a violation of Axiom R1$_{20}$, or to a complete rejection of some option set.

Definition 32 (Avoiding complete rejection)**.** *Given any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$, we say that $\mathcal{B}$* avoids complete rejection *when $R_{\mathcal{B}}$ satisfies Axiom* R1$_{20}$.

The reason why we decided to call this property *avoiding complete rejection* is clear: given any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$ that avoids complete rejection, the special rejection function $R_{\mathcal{B}}$ satisfies Axiom R1$_{20}$—in other words, $R_{\mathcal{B}}$ never returns the complete option set.

Example 12. To give an example of an assessment $\mathcal{B}$ that *does not avoid complete rejection*, consider first $\mathcal{B} \coloneqq \{\{0\}\} \subseteq \mathcal{Q}_0$. By Proposition $79_{\curvearrowleft}$ further on, $R_{\mathcal{B}}$ extends $\mathcal{B}$, and therefore $0 \in R_{\mathcal{B}}(\{0\})$: $R_{\mathcal{B}}$ does not satisfy Axiom $\mathrm{R1}_{20}$, so $\mathcal{B}$ does not avoid complete rejection.

Actually, this assessment leads to the trivial rejection function $R_{\mathcal{B}} = \mathrm{id}_{\mathcal{Q}}$ that, evaluated in any option set A, returns the complete A. Indeed, consider any A in $\mathcal{Q}$; we will show that $u \in R_{\mathcal{B}}(A)$ for every u in A. By Proposition $79_{\curvearrowleft}$ further on, $R_{\mathcal{B}}$ satisfies Axioms $\mathrm{R2}_{20}$–$\mathrm{R4}_{20}$, so in particular, by Axiom $\mathrm{R4b}_{20}$, $u \in R_{\mathcal{B}}(\{u\})$ for every u in A. Therefore, by Axiom $\mathrm{R3a}_{20}$ [with $\tilde{A}_1 \coloneqq \{u\}$, $\tilde{A}_2 \coloneqq \{u\}$ and $\tilde{A} \coloneqq A$; then $\tilde{A}_2 \subseteq \tilde{A}$ since $u \in A$], we find that indeed $u \in R_{\mathcal{B}}(A)$ for every u in A.

As a second example of an assessment that does not avoid complete rejection, consider $\mathcal{B} \coloneqq \{\{0,u\},\{0,-u\}\} \subseteq \mathcal{Q}_0$ for an arbitrary u in $\mathcal{V}$. Again by Proposition $79_{\curvearrowleft}$ further on, $R_{\mathcal{B}}$ extends $\mathcal{B}$—so $0 \in R_{\mathcal{B}}(\{0,u\})$ and $0 \in R_{\mathcal{B}}(\{0,-u\})$—and satisfies Axioms $\mathrm{R2}_{20}$–$\mathrm{R4}_{20}$. By Axiom $\mathrm{R4b}_{20}$, from $0 \in R_{\mathcal{B}}(\{0,-u\})$ we infer that $u \in R_{\mathcal{B}}(\{0,u\})$. Using that $0 \in R_{\mathcal{B}}(\{0,u\})$, we infer that $\{0,u\} = R_{\mathcal{B}}(\{0,u\})$, contradicting Axiom $\mathrm{R1}_{20}$. Therefore $\mathcal{B}$ does not avoid complete rejection. ◊

There is an interesting characterisation for 'avoiding complete rejection'. This characterisation is important: it will for instance help us in Chapter 7_{221} to study the natural extension of other types of assessments, namely *structural* assessments. It also significantly simplifies the task of checking whether an assessment avoids complete rejection.

Lemma 80. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$. Then $\mathcal{B}$ does not avoid complete rejection if and only if there is some A' in $\mathcal{Q}$ such that*

$$\max A' = A' \text{ and } 0 \in A' \text{ and } (\forall v \in A')(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A'. \tag{3.5}$$

Proof. Use Lemma 77_{92} to infer that $R_{\mathcal{B}}$ satisfies Axioms $\mathrm{R2}_{20}$–$\mathrm{R4}_{20}$. Therefore, by Corollary 26_{39}, $\mathcal{B}$ does not avoid complete rejection if and only if $0 \in R_{\mathcal{B}}(\{0\})$. Use Equation $(3.1)_{92}$ to infer that $0 \in R_{\mathcal{B}}(\{0\})$ if and only if there is some A' in $\mathcal{Q}$ such that $0 \in A'$ and

$$(\forall v \in A')\big((A' - \{v\}) \cap \mathcal{V}_{>0} \neq \varnothing \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A'\big). \tag{3.6}$$

We show that this is equivalent to Equation (3.5). Sufficiency is fairly immediate. If Equation (3.5) holds, then by definition there is some A' in $\mathcal{Q}$ such that in particular $0 \in A'$ and $(\forall v \in A')(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A'$, and therefore Equation (3.6) will definitively hold.

Therefore, it suffices to show necessity. So assume that there is some A' in $\mathcal{Q}$ such that $0 \in A'$ and that satisfies Equation (3.6). Let $A'' \coloneqq \max A'$, which is non-empty because of Proposition 31_{42}. Then $(\forall v \in A'')(A'' - \{v\}) \cap \mathcal{V}_{>0} = \varnothing$, so since $A'' \subseteq A'$, Equation (3.6) implies that then $(\forall v \in A'')(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A'$, and using Proposition 33_{43}(ii)&(iv) therefore $(\forall v \in A'')(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A''$. If $0 \in A''$, then the proof is finished. So assume that $0 \notin A''$. Consider any u in A'' and let

$\tilde{A} \coloneqq A'' - \{u\}$. Then $\max \tilde{A} = \tilde{A}$, $0 \in \tilde{A}$ and $(\forall v \in \tilde{A})(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v+u\} + \mu B \lessdot A''$. Using Proposition 33_{43}(v) therefore indeed $(\forall v \in \tilde{A})(\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \lessdot A'' - \{u\} = \tilde{A}$. □

3.4 NATURAL EXTENSION OF ASSESSMENTS THAT AVOID COMPLETE REJECTION

The discussion after Equation $(3.1)_{92}$ tells us that the rejection function $R_{\mathcal{B}}$ seems to serve, loosely speaking, as the least committal coherent extension of the assessment $\mathcal{B}$. Let us formalise this idea, and show the connection between Equation $(3.1)_{92}$ and the natural extension $\mathcal{E}(\mathcal{B})$.

Theorem 81 (Natural extension). *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$. Then the following statements are equivalent:*

(i) *$\mathcal{B}$ avoids complete rejection;*

(ii) *There is a coherent extension of $\mathcal{B}$: $(\exists R \in \overline{\mathbf{R}})(\forall B \in \mathcal{B}) 0 \in R(B)$;*

(iii) *$\mathcal{E}(\mathcal{B}) \neq \mathrm{id}_{\mathcal{Q}}$;*

(iv) *$\mathcal{E}(\mathcal{B}) \in \overline{\mathbf{R}}$;*

(v) *$\mathcal{E}(\mathcal{B})$ is the least informative rejection function that is coherent and extends $\mathcal{B}$.*

When any (and hence all) of these equivalent statements hold, then $\mathcal{E}(\mathcal{B}) = R_{\mathcal{B}}$.

Proof. This proof is structured as follows: we first show that (ii)⇔(iii)⇔(iv)⇔(v); subsequently that whenever any (and hence all) of these four equivalent conditions hold, then $\mathcal{E}(\mathcal{B}) = R_{\mathcal{B}}$, and finally, that (i)⇔(ii).

For the first part, that (ii)⇔(iii)⇔(iv)⇔(v), we will show the circular chain of implications (ii)⇒(iv)⇒(iii)⇒(ii), after which we will prove that (iv) and (v) are equivalent.

To show that (ii) implies (iv), since there is a coherent extension of $\mathcal{B}$, the set $\{R \in \overline{\mathbf{R}} : (\forall B \in \mathcal{B}) 0 \in R(B)\}$ of coherent rejection functions that extend $\mathcal{B}$ is a non-empty. Use Proposition 40_{48} to infer that its infimum is indeed a coherent rejection function.

That (iv) implies (iii) is an immediate consequence of the fact that the rejection function $\mathrm{id}_{\mathcal{Q}}$ fails to satisfy Axiom $\mathrm{R1}_{20}$ [indeed, $\mathrm{id}_{\mathcal{Q}}(A) = A$ for every A in $\mathcal{Q}$], and is therefore not coherent.

To show that (iii) implies (ii), assume that $\mathcal{E}(\mathcal{B}) \neq \mathrm{id}_{\mathcal{Q}}$ and *ex absurdo* that there is no coherent extension of $\mathcal{B}$. Then $\{R \in \overline{\mathbf{R}} : (\forall B \in \mathcal{B}) 0 \in R(B)\} = \emptyset$, so by Definition 31_{91} $\mathcal{E}(\mathcal{B}) = \mathrm{id}_{\mathcal{Q}}$, a contradiction.

We finish the first part by showing that (iv)⇔(v). We clearly only have to show that (iv) implies (v), since (v) states, amongst other things, that $\mathcal{E}(\mathcal{B})$ is a coherent rejection function, and it therefore implies (iv). So assume that (iv) holds—that $\mathcal{E}(\mathcal{B})$ is a coherent rejection function. We first show that $\mathcal{E}(\mathcal{B})$ extends $\mathcal{B}$. Since $\mathcal{E}(\mathcal{B}) \in \overline{\mathbf{R}}$, we have that $\{R \in \overline{\mathbf{R}} : R \text{ extends } \mathcal{B}\}$ is non-empty, and therefore, by Proposition 74_{91}, its infimum $\mathcal{E}(\mathcal{B})$ extends $\mathcal{B}$. Assume *ex absurdo* that (v) does not hold—so $\mathcal{E}(\mathcal{B})$ is not the least informative rejection function that is coherent and extends $\mathcal{B}$. We know already that $\mathcal{E}(\mathcal{B})$ is coherent [by (iv)] and that it extends $\mathcal{B}$. So the only possibility left, is that there is a less informative coherent rejection function $R' \sqsubset \mathcal{E}(\mathcal{B})$ that extends $\mathcal{B}$. Therefore $0 \notin R'(A)$ and $0 \in \mathcal{E}(\mathcal{B})(A)$ for some A in $\mathcal{Q}$. Infer that $0 \in \mathcal{E}(\mathcal{B})(A) \Leftrightarrow 0 \in \bigcap_{R \in \overline{\mathbf{R}}, (\forall B \in \mathcal{B}) 0 \in R(B)} R(A) \Leftrightarrow (\forall R \in \overline{\mathbf{R}})\big(((\forall B \in \mathcal{B}) 0 \in R(B)) \Rightarrow 0 \in R(A)\big)$, so in particular for the coherent rejection function R', we have that $((\forall B \in \mathcal{B}) 0 \in R'(B)) \Rightarrow 0 \in R'(A)$. Since R' extends $\mathcal{B}$, the implicant is true, so $0 \in R'(A)$, a contradiction. Therefore indeed (v)$_{\curvearrowleft}$ holds.

Subsequently, we prove that $\mathcal{E}(\mathcal{B}) = R_{\mathcal{B}}$ whenever any (and hence all) of the equivalent statements (ii)$_{\curvearrowleft}$–(v)$_{\curvearrowleft}$ hold. By Proposition 79_{95}, $R_{\mathcal{B}}$ is the least informative rejection function that satisfies Axioms R2_{20}–R4_{20} and extends $\mathcal{B}$. Therefore $R_{\mathcal{B}} \sqsubseteq \mathcal{E}(\mathcal{B})$, so $R_{\mathcal{B}}(A) \subseteq \mathcal{E}(\mathcal{B})(A) \subset A$ for all A in $\mathcal{Q}$, where the second set inclusion follows by the coherence of $\mathcal{E}(\mathcal{B})$. This implies that $R_{\mathcal{B}}(A) \subset A$ for all A in $\mathcal{Q}$, so $R_{\mathcal{B}}$ satisfies Axiom R1_{20}. We conclude that $R_{\mathcal{B}}$ is the least informative rejection function that is coherent and extends $\mathcal{B}$, so by (v)$_{\curvearrowleft}$ indeed $R_{\mathcal{B}} = \mathcal{E}(\mathcal{B})$.

We finish the proof by showing that (i)$_{\curvearrowleft}$$\Leftrightarrow$(ii)$_{\curvearrowleft}$. For necessity, by Proposition 79_{95} we know that $R_{\mathcal{B}}$ satisfies Axioms R2_{20}–R4_{20} and extends $\mathcal{B}$. Since $\mathcal{B}$ avoids complete rejection, by Definition 32_{95} we furthermore know that $R_{\mathcal{B}}$ satisfies Axiom R1_{20}, and hence it is a coherent rejection function that extends $\mathcal{B}$. So we showed that (i)$_{\curvearrowleft}$$\Rightarrow$(ii)$_{\curvearrowleft}$.

For sufficiency, we already know that (ii)$_{\curvearrowleft}$ implies that $R_{\mathcal{B}} = \mathcal{E}(\mathcal{B})$, and that $\mathcal{E}(\mathcal{B})$ is a coherent rejection function (see (iv)$_{\curvearrowleft}$), so $R_{\mathcal{B}}$ is a coherent rejection function, and hence in particular it satisfies Axiom R1_{20}. Therefore $\mathcal{B}$ indeed avoids complete rejection. □

If an assessment avoids complete rejection, then so does any weaker assessment:

Corollary 82. *Consider any two assessments $\mathcal{B}_1 \subseteq \mathcal{Q}_0$ and $\mathcal{B}_2 \subseteq \mathcal{Q}_0$ such that $\mathcal{B}_1$ is at least as strong as $\mathcal{B}_2$. If $\mathcal{B}_1$ avoids complete rejection, then so does $\mathcal{B}_2$.*

Proof. Since $\mathcal{B}_1$ avoids complete rejection, by Theorem 81$_{\curvearrowleft}$ there is a coherent extension R of $\mathcal{B}_1$. Because $\mathcal{B}_1$ is at least as strong as $\mathcal{B}_2$, by Proposition 75_{91} R extends also $\mathcal{B}_2$, whence, again by Theorem 81$_{\curvearrowleft}$ $\mathcal{B}_2$ avoids complete rejection. □

Similarly as for other imprecise probabilities models such as desirability, we will impose no requirements on the completeness of the model; see References [18, 82] for reasons why: even after extending an assessment using the natural extension, the resulting rejection function is not guaranteed to be *exhaustive*. With a rejection function being not necessarily 'exhaustive' we mean that it is a non-exhaustive description of the subject's beliefs. Further elicitation may very well result in additional rejections of the zero option, but the subject may be unwilling or incapable to specify them. Hence, we will not require a rejection function to be exhaustive, nor will we interpret it in this way. See Example 14_{106} for an illustration of a similar idea.

Using Theorem 81$_{\curvearrowleft}$, we can prove a counterpart of Reference [31, Theorem 3] explaining the relationship between avoiding complete rejection and maximal choice models:

Proposition 83. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$. Then $\mathcal{B}$ avoids complete rejection if and only if there is some maximal rejection function in $\hat{\mathbf{R}}$ that extends $\mathcal{B}$.*

Proof. Sufficiency follows readily from Theorem 81_{97}, using that $\hat{\mathbf{R}} \subseteq \overline{\mathbf{R}}$.

For necessity, assume that $\mathcal{B}$ avoids complete rejection. Infer from Theorem 81_{97} that $\mathcal{E}(\mathcal{B})$ is coherent and extends $\mathcal{B}$. Use Proposition 46_{52} to infer that $\mathcal{E}(\mathcal{B})$ is dominated by a maximal rejection function $\hat{R}$ that dominates $\mathcal{E}(\mathcal{B})$, and therefore indeed extends $\mathcal{B}$. □

3.5 NATURAL EXTENSION AND DESIRABILITY

Let us compare our discussion of natural extension with the case of binary preferences and desirability, which we introduced in Section 2.8_{55}. A *desirability assessment* $B \subseteq \mathcal{V}$ is usually (see for instance Section 1.2 of Reference [57], and also Reference [31]) a set of options that the agent finds desirable—strictly prefers to the zero option. Of course, any desirability assessment $B \subseteq \mathcal{V}$ can be transformed into an assessment for rejection functions: we simply assess that 0 is rejected in the binary choice between 0 and u, for every option u in B. The assessment based on B is therefore given by $\mathcal{B}_B \coloneqq \{\{0,u\} : u \in B\}$; clearly B and $\mathcal{B}_B$ are in a one-to-one correspondence: given an assessment $\mathcal{B}_B$ that consist of an arbitrary family of binary option sets, we retrieve B as $B = \bigcup(\mathcal{B}_B \setminus \{0\}) = (\bigcup \mathcal{B}_B) \setminus \{0\}$.

Given any desirability assessment $B \subseteq \mathcal{V}$ and any set of desirable options $D \subseteq \mathcal{V}$, we say that *D extends B* if $B \subseteq D$.

Proposition 84. *Consider any desirability assessment $B \subseteq \mathcal{V}$ and any set of desirable options $D \subseteq \mathcal{V}$. Then D extends B if and only if R_D extends $\mathcal{B}_B$.*

Proof. Consider the following equivalences:

$$\begin{aligned} D \text{ extends } B \Leftrightarrow B \subseteq D &\Leftrightarrow (\forall u \in B) u \in D \\ &\Leftrightarrow (\forall u \in B) 0 \in R_D(\{0,u\}) \\ &\Leftrightarrow (\forall B' \in \mathcal{B}_B) 0 \in R_D(B') \Leftrightarrow R_D \text{ extends } \mathcal{B}_B. \end{aligned}$$ □

For desirability, the Axioms D2_{57}–D4_{57} are the productive ones, while the only destructive axiom is Axiom D1_{57}. The property for desirability that corresponds to avoiding complete rejection for choice models, is *avoiding non-positivity*, commonly formulated as (see for instance Reference [31, Definition 1])[2]

$$\operatorname{posi}(B) \cap \mathcal{V}_{\leq 0} = \emptyset \tag{3.7}$$

[2] If the vector space $\mathcal{V}$ is the set of all gambles, the condition is called *avoiding partial loss*; see Reference [57]. A very similar but slightly weaker condition is called *avoiding sure loss*; see also for instance Reference [82, Section 3.7].

for the desirability assessment $B \subseteq \mathcal{V}$. The interpretation is clear: an assessment must never imply, using scaling and combination—Axioms D3$_{57}$ and D4$_{57}$,[3] and hence by applying posi to the assessment B—the desirability of an option in $\mathcal{V}_{\leq 0}$.

Theorem 81$_{97}$ is the equivalent for choice models of the natural extension theorem for desirability,[4] whose proof inspired our proof of Theorem 81$_{97}$. To make this thesis more self-contained, let us state this natural extension theorem for desirability.

Theorem 85 (Natural extension for desirability [31, Theorem 1])**.** *Consider any desirability assessment $B \subseteq \mathcal{V}$, and define its natural extension as*[5]

$$\mathcal{E}^{\mathbf{D}}(B) \coloneqq \inf\{D \in \overline{\mathbf{D}} : B \subseteq D\}.$$

Then the following statements are equivalent:
- (i) *B avoids non-positivity;*
- (ii) *B is included in some coherent set of desirable options;*
- (iii) $\mathcal{E}^{\mathbf{D}}(B) \neq \mathcal{V}$*;*
- (iv) $\mathcal{E}^{\mathbf{D}}(B) \in \overline{\mathbf{D}}$*;*
- (v) *$\mathcal{E}^{\mathbf{D}}(B)$ is the least informative set of desirable options that is coherent and includes B.*

When any (and hence all) of these equivalent statements hold, then $\mathcal{E}^{\mathbf{D}}(B) = \operatorname{posi}(\mathcal{V}_{>0} \cup B)$.

Let us go back to our natural extension Theorem 81$_{97}$ (for choice models) and consider a desirability assessment $B \subseteq \mathcal{V}$, and its completely binary (choice models) assessment $\mathcal{B}_B$. If B avoids non-positivity, then we wonder whether we can retrieve, using Theorem 81$_{97}$, the formula $\mathcal{E}^{\mathbf{D}}(B) = \operatorname{posi}(\mathcal{V}_{>0} \cup B)$, as Theorem 85 indicates.

Theorem 86. *Consider any desirability assessment $B \subseteq \mathcal{V}$. Then B avoids non-positivity if and only if $\mathcal{B}_B$ avoids complete rejection, and if this is the case, then $\mathcal{E}(\mathcal{B}_B) = R_{\mathcal{E}^{\mathbf{D}}(B)}$.*

Proof. We start with the first part, that B avoids non-positivity if and only if $\mathcal{B}_B$ avoids complete rejection. For necessity, since B avoids non-positivity, by Theorem 85, we have that $B \subseteq D$ for some coherent set of desirable options D. Consider the coherent rejection function R_D. Since $(\forall u \in B) u \in D$, we find that $(\forall u \in B) 0 \in R_D(\{0, u\})$, and therefore $(\forall B \in \mathcal{B}_B) 0 \in R_D(B)$. So we have found a coherent rejection function R_D that extends $\mathcal{B}_B$, and therefore, by Theorem 81$_{97}$, $\mathcal{B}_B$ indeed avoids complete rejection.

[3] We leave Axiom D2$_{57}$ out of this discussion since applying it can never result in the additional desirability of options in $\mathcal{V}_{\leq 0}$.

[4] For a very general form, see Reference [31, Theorem 1].

[5] We let $\inf \varnothing = \mathcal{V}$.

For sufficiency, since $\mathcal{B}_B$ avoids complete rejection, by Theorem 81_{97} $(\forall u \in B) 0 \in R(\{0,u\})$ for some coherent rejection function R. Consider the coherent set of desirable options $D_R = \{u \in \mathcal{V} : 0 \in R(\{0,u\})\}$. Since $(\forall u \in B) 0 \in R(\{0,u\})$, we find that $B \subseteq D_R$. So we have found a coherent set of desirable options D_R for which $B \subseteq D_R$, and therefore, by Theorem 85, B indeed avoids non-positivity.

We now prove the second part, that $\mathcal{E}(\mathcal{B}_B) = R_{\mathcal{E}^{\mathbf{D}}(B)}$. Since B avoids non-positivity, by Theorem 85, its natural extension is given by the coherent set of desirable options $\mathcal{E}^{\mathbf{D}}(B) = \operatorname{posi}(\mathcal{V}_{>0} \cup B)$. Similarly, since $\mathcal{B}_B$ avoids complete rejection, by Theorem 81_{97}, its natural extension is given by the coherent rejection function $\mathcal{E}(\mathcal{B}_B) = R_{\mathcal{B}_B}$. So it suffices to show that $R_{\mathcal{B}_B} = R_{\operatorname{posi}(\mathcal{V}_{>0} \cup B)}$. Since both rejection functions are coherent, what we should prove can be even further reduced: by Axiom R4b_{20} it suffices to prove that $0 \in R_{\mathcal{B}_B}(A) \Leftrightarrow 0 \in R_{\operatorname{posi}(\mathcal{V}_{>0} \cup B)}(A)$ for all A in $\mathcal{Q}$.

We first prove that $0 \in R_{\mathcal{B}_B}(A) \Rightarrow 0 \in R_{\operatorname{posi}(\mathcal{V}_{>0} \cup B)}(A)$ for all A in $\mathcal{Q}$. So consider any A in $\mathcal{Q}$ such that $0 \in R_{\mathcal{B}_B}(A)$. Then, by Equation $(3.1)_{92}$, there is some $A' \supseteq A$ in $\mathcal{Q}$ such that

$$(\forall v \in \{0\} \cup (A' \setminus A))\big((A' - \{v\}) \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists w \in B, \exists \mu \in \mathbb{R}_{>0})\{v\} + \{0, \mu w\} \preceq A'\big). \tag{3.8}$$

Without loss of generality, let $A \coloneqq \{0, u_1, \dots, u_k\}$ for some k in $\mathbb{N}$, and $A_1 \coloneqq A \cup \{v_1, \dots, v_\ell\}$ for some ℓ in $\mathbb{Z}_{\geq 0}$. Then $\{0\} \cup (A' \setminus A) = \{0, v_1, \dots, v_\ell\}$, and therefore, by Equation (3.8),

$$\underbrace{\big(A' \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists w \in B, \exists \mu \in \mathbb{R}_{>0})\{0, \mu w\} \preceq A'\big)}_{\text{(i)}} \text{ and}$$

$$(\forall i \in \{1, \dots, \ell\}) \underbrace{\big((A' - \{v_i\}) \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists w \in B, \exists \mu \in \mathbb{R}_{>0})\{v_i, v_i + \mu w\} \preceq A'\big)}_{\text{(ii)}}$$

For clarity, as indicated we abbreviate the two expressions involved as (i) and (ii), and find implications for each of them. Observe that Expression (i) is equivalent to

$$(\exists w \in B, \exists \mu \in \mathbb{R}_{>0}, \exists w' \in A')(0 \prec w' \text{ or } \mu w \leq 0 \text{ or } \mu w \leq w'),$$

taking into account that $0 \in A'$. Since B avoids non-positivity, and therefore $w \not\leq 0$ for every w in B, this is equivalent to

$$(\exists w \in B, \exists \mu \in \mathbb{R}_{>0}, \exists w' \in A')(0 \prec w' \text{ or } \mu w \leq w'),$$

which by Lemma 4_{12} is in turn equivalent to $(\exists w \in B, \exists w' \in A') w' \in \operatorname{posi}(\mathcal{V}_{>0} \cup \{w\})$. Therefore, since $\operatorname{posi}(\mathcal{V}_{>0} \cup \{w\}) \subseteq \operatorname{posi}(\mathcal{V}_{>0} \cup B)$ for every w in B, this implies that

$$A' \cap \operatorname{posi}(\mathcal{V}_{>0} \cup B) \neq \emptyset. \tag{3.9}$$

Observe using Proposition 33_{43}(v) that Expression (ii) is equivalent to

$$(A' - \{v_i\}) \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists w \in B, \exists \mu \in \mathbb{R}_{>0})\{0, \mu w\} \preceq A' - \{v_i\},$$

and therefore, since $0 \in A' - \{v_i\}$, this is also equivalent to

$$(A' - \{v_i\}) \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists w \in B, \exists \mu \in \mathbb{R}_{>0})\{\mu w\} \preceq A' - \{v_i\}.$$

Taking into account that $A' = \{0, u_1, \dots, u_k, v_1, \dots, v_\ell\}$, and therefore also that 0 belongs to $A' - \{v_i\}$, we find another equivalent expression:

$$(\exists w \in B, \exists \mu \in \mathbb{R}_{>0}, \exists w' \in A')(0 < w' - v_i \text{ or } \mu w \leq w' - v_i),$$

which by Lemma 4_{12} is in turn equivalent to $(\exists w \in B, \exists \mu \in \mathbb{R}_{>0}, \exists w' \in A')w' - v_i \in \operatorname{posi}(\mathcal{V}_{>0} \cup \{w\})$. Therefore, since $\operatorname{posi}(\mathcal{V}_{>0} \cup \{w\}) \subseteq \operatorname{posi}(\mathcal{V}_{>0} \cup B)$ for every w in B, this implies that

$$A' \cap (\operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_i\}) \neq \emptyset. \tag{3.10}$$

Combining the two Implications $(3.9)_{\curvearrowleft}$ and (3.10) of Expressions (i) and (ii) respectively, we find that Equation $(3.8)_{\curvearrowleft}$ implies that

$$A' \cap \operatorname{posi}(\mathcal{V}_{>0} \cup B) \neq \emptyset \text{ and } (\forall i \in \{1, \dots, \ell\}) A' \cap (\operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_i\}) \neq \emptyset. \tag{3.11}$$

We now prove that this implies that $A \cap \operatorname{posi}(\mathcal{V}_{>0} \cup B) \neq \emptyset$. To this end, infer that, by Equation (3.11), in particular $A' \cap \operatorname{posi}(\mathcal{V}_{>0} \cup B) \neq \emptyset$, so $w_1 \in \operatorname{posi}(\mathcal{V}_{>0} \cup B)$ for some w_1 in A'. If $w_1 \in A$, the proof is done, so assume that $w_1 \in A' \setminus A$ and therefore $w_1 = v_{j_1}$ for some j_1 in $\{1, \dots, \ell\}$.

By Equation (3.11), $A' \cap (\operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_1}\}) \neq \emptyset$, so $w_2 \in \operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_1}\}$ for some w_2 in A'. Remark that $\operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_1}\} \subseteq \operatorname{posi}(\mathcal{V}_{>0} \cup B)$, because v_{j_1} belongs to $\operatorname{posi}(\mathcal{V}_{>0} \cup B)$, and therefore the proof is done if $w_2 \in A$, so assume that $w_2 \in A' \setminus A$ and therefore $w_2 = v_{j_2}$ for some j_2 in $\{1, \dots, \ell\}$. If $j_2 = j_1$, then $v_{j_1} = v_{j_2} \in \operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_1}\}$ and therefore $0 \in \operatorname{posi}(\mathcal{V}_{>0} \cup B)$, contradicting the avoiding non-positivity of B, so $j_2 \in \{1, \dots, \ell\} \setminus \{j_1\}$.

By another application of Equation (3.11), we find that $A' \cap (\operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_2}\}) \neq \emptyset$, so $w_3 \in \operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_2}\} \subseteq \operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_1}\} \subseteq \operatorname{posi}(\mathcal{V}_{>0} \cup B)$ for some w_3 in A'. If $w_3 \in A$ the proof is done, so assume that $w_3 = v_{j_3}$ for some j_3 in $\{1, \dots, \ell\}$. If $j_3 \in \{j_1, j_2\}$, then $0 \in \operatorname{posi}(\mathcal{V}_{>0} \cup B)$, contradicting the avoiding non-positivity of B, so $j_3 \in \{1, \dots, \ell\} \setminus \{j_1, j_2\}$.

We can go on in the same vein until after ℓ steps we have shown that $A \cap \operatorname{posi}(\mathcal{V}_{>0} \cup B) \neq \emptyset$—and then the proof is done—or have found some j_ℓ in $\{1, \dots, \ell\} \setminus \{j_1, \dots, j_{\ell-1}\}$ such that $v_{j_\ell} \in \operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_{\ell-1}}\} \subseteq \operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_{\ell-2}}\} \subseteq \dots \subseteq \operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_1}\}$. Equation (3.11) then tells us that $A' \cap \operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_\ell}\} \neq \emptyset$, so $w_\ell \in \operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_\ell}\} \subseteq \operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_{\ell-1}}\} \subseteq \dots \subseteq \operatorname{posi}(\mathcal{V}_{>0} \cup B) + \{v_{j_1}\}$ for some w_ℓ in A'. Then $w_\ell \neq v_j$ for all j in $\{1, \dots, \ell\}$, because, otherwise, $0 \in \operatorname{posi}(\mathcal{V}_{>0} \cup B)$, contradicting the avoiding non-positivity of B. Hence $w_\ell \in A$, so indeed $A \cap \operatorname{posi}(\mathcal{V}_{>0} \cup B) \neq \emptyset$.

Since $0 \in A$, by Proposition 55_{64} therefore indeed $0 \in R_{\operatorname{posi}(\mathcal{V}_{>0} \cup B)}(A)$.

We now prove that $0 \in R_{\operatorname{posi}(\mathcal{V}_{>0} \cup B)}(A) \Rightarrow 0 \in R_{\mathcal{B}_B}(A)$ for all A in $\mathcal{Q}$. So consider any A in $\mathcal{Q}$ such that $0 \in R_{\operatorname{posi}(\mathcal{V}_{>0} \cup B)}(A)$, whence, by Proposition 55_{64}, $0 \in A$ and $A \cap \operatorname{posi}(\mathcal{V}_{>0} \cup B) \neq \emptyset$, and therefore $u \in \operatorname{posi}(\mathcal{V}_{>0} \cup B)$ for some u in A. Then $u > 0$ or $u \geq \sum_{i=1}^k \lambda_k v_k$ for some k in $\mathbb{N}$, $\lambda_1, \dots, \lambda_k$ in $\mathbb{R}_{>0}$, and $v_1, \dots, v_k$ in B. Because we already know from Theorem 81_{97} that $R_{\mathcal{B}_B}$ is a coherent rejection function, if $u > 0$ then by Axiom R1_{20}, $0 \in R_{\mathcal{B}_B}(\{0, u\})$. On the other hand, if $u \geq \sum_{i=1}^k \lambda_k v_k$, by the assessment $\mathcal{B}_B$, we have that $0 \in R_{\mathcal{B}_B}(\{0, v_i\})$ for all i in $\{1, \dots, k\}$, and therefore by Axiom R4b_{20} also that $0 \in R_{\mathcal{B}_B}(\{0, \lambda_i v_i\})$ for all i in $\{1, \dots, k\}$. Use Proposition 32_{43} to infer that then also $0 \in R_{\mathcal{B}_B}(\{0, \sum_{i=1}^k \lambda_i v_i\})$, and therefore, by Proposition 30_{41}(ii) also $0 \in R_{\mathcal{B}_B}(\{0, u\})$. So in any of the two cases, we find that $0 \in R_{\mathcal{B}_B}(\{0, u\})$, whence by Axiom R3a_{20} [with $\tilde{A} \coloneqq \{0\}, \tilde{A}_1 \coloneqq \{0, u\}$ and $\tilde{A}_2 \coloneqq A$] indeed $0 \in R_{\mathcal{B}_B}(A)$. $\square$

Actually, Theorem 86_{100} consists of three remarkable statements: The first statement is that the natural extension of a purely binary assessment $\mathcal{B}_B$, for some $B \subseteq \mathcal{V}$, is a rejection function that is purely binary itself; the second one is that its (binary) behaviour is exactly described by the set of desirable options $\operatorname{posi}(\mathcal{V}_{>0} \cup B)$; both statements are conditional on $\mathcal{B}_B$ avoiding complete rejection—which is furthermore, as a third statement, equivalent to B avoiding non-positivity.

Focussing on this second statement, for any desirability assessment $B \subseteq \mathcal{V}$ that avoids non-positivity, the natural extension $\mathcal{E}(\mathcal{B}_B)$ (for choice models) induces the binary choice $D_{\mathcal{E}(\mathcal{B}_B)}$ reflected by $\operatorname{posi}(\mathcal{V}_{>0} \cup B)$. To see this, Theorem 86_{100} guarantees that $\mathcal{E}(\mathcal{B}_B) = R_{\operatorname{posi}(\mathcal{V}_{>0} \cup B)}$, where by Theorem 85_{100}, $\operatorname{posi}(\mathcal{V}_{>0} \cup B)$ is a coherent set of desirable options and by Proposition 58_{66}, therefore indeed $D_{\mathcal{E}(\mathcal{B}_B)} = \operatorname{posi}(\mathcal{V}_{>0} \cup B)$.

To summarise these statements, consider the commuting diagram in Figure $3.1_{\curvearrowright}$, where we have used the maps

$$
\begin{aligned}
&\mathcal{E}^{\mathbf{D}} : \mathcal{P}(\mathcal{V}) \to \mathbf{D} : B \mapsto \mathcal{E}^{\mathbf{D}}(B),\\
&\mathcal{B}_{\bullet} : \mathcal{P}(\mathcal{V}) \to \mathcal{Q}_0 : B \mapsto \mathcal{B}_B \coloneqq \{\{0,u\} : u \in B\},\\
&\mathcal{E} : \mathcal{P}(\mathcal{Q}_0) \to \mathbf{R} : \mathcal{B} \mapsto \mathcal{E}(\mathcal{B}),\\
&D_{\bullet} : \mathbf{R} \to \mathbf{D} : R \mapsto D_R \coloneqq \{u \in \mathcal{V} : 0 \in R(\{0,u\})\},\\
&R_{\bullet} : \mathbf{D} \to \mathbf{R} : D \mapsto R_D,
\end{aligned}
$$

with $\mathcal{E}^{\mathbf{D}}(B)$ defined in Theorem 85_{100}, $\mathcal{E}(\mathcal{B})$ in Definition 31_{91} and, as usual, R_D given by $R_D(A) = \{u \in A : (\forall v \in A) v - u \notin D\}$ for all A in $\mathcal{Q}$. Start with a desirability assessment $B \subseteq \mathcal{V}$ that avoids non-positivity. Taking the natural extension for desirability commutes with taking the corresponding assessment (for choice models), then the natural extension, and eventually going back to the set of desirable options corresponding to this natural extension. Furthermore, taking the natural extension of the corresponding assessment (for choice models) commutes with taking the natural extension for desirability, and then going to the corresponding rejection function.

In my opinion, one of the most important consequences of Theorem 86_{100} is that the natural extension (for choice models) of a purely binary assessment that avoids complete rejection, is a rejection function that is purely binary itself, in the sense that it is derived from a set of desirable options. If the assessment $\mathcal{B}$ is not binary, then nothing guarantees that $R_{\mathcal{B}}$ is derived from a set of desirable options. Moreover, it might be that $R_{\mathcal{B}}$ is not even an infimum of such rejection functions: in Example 16_{109}, using a non-binary assessment, we construct such a coherent rejection function. This shows that our notion of coherent choice allows for a richer theory than a theory that would model choice using *sets of sets of desirable options*.

But what can we say about the binary part $D_{\mathcal{E}(\mathcal{B})}$ of the implications of a non-binary assessment $\mathcal{B} \subseteq \mathcal{Q}_0$? It turns out that the following collection of

$$\begin{tikzcd}
B \arrow[rr, "\mathcal{E}^{\mathbf{D}}"] \arrow[d, "\mathcal{B}_{\bullet}"'] & & \mathcal{E}^{\mathbf{D}}(B) = D_{\mathcal{E}(\mathcal{B}_B)} \arrow[d, shift right, "R_{\bullet}"'] \\
\mathcal{B}_B \arrow[rr, "\mathcal{E}"] & & \mathcal{E}(\mathcal{B}_B) = R_{\mathcal{E}^{\mathbf{D}}(B)} \arrow[u, shift right, "D_{\bullet}"']
\end{tikzcd}$$

Figure 3.1: Commuting diagram for the natural extension for binary assessments

desirability assessments is important:[6]

$$\mathcal{A}_{\mathcal{B}} \coloneqq \{\{u_B : B \in \mathcal{B}\} : (\forall B \in \mathcal{B}) u_B \in B \setminus \{0\}\}.$$

Note that each element of $\mathcal{A}_{\mathcal{B}}$ has cardinality $|\mathcal{B}|$.

Example 13. To gain a feel for what $\mathcal{A}_{\mathcal{B}}$ is, consider the assessment $\mathcal{B} \coloneqq \{\{0, u, -u\}\} \subseteq \mathcal{Q}_0$. Then $\mathcal{A}_{\mathcal{B}} = \{\{u\}, \{-u\}\}$, and we interpret each element of $\mathcal{A}_{\mathcal{B}}$ as a desirability assessment. Similarly, consider the assessment $\mathcal{B}' \coloneqq \{\{0, u_1, u_2\}, \{0, v\}\} \subseteq \mathcal{Q}_0$. Then $\mathcal{A}_{\mathcal{B}'} = \{\{u_1, v\}, \{u_2, v\}\}$. $\Diamond$

Proposition 87. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$. Then*[7]

$$\begin{aligned} R_{\mathcal{B}} &\sqsubseteq \inf\{R_{\mathcal{E}^{\mathbf{D}}(A)} : A \in \mathcal{A}_{\mathcal{B}} \text{ and } A \text{ avoids non-positivity}\} \\ &= \inf\{R_{\operatorname{posi}(\mathcal{V}_{>0} \cup A)} : A \in \mathcal{A}_{\mathcal{B}} \text{ and } A \text{ avoids non-positivity}\}. \end{aligned}$$

As a consequence, $D_{R_{\mathcal{B}}} \subseteq D_{\mathcal{B}}$, where we let[8]

$$\begin{aligned} D_{\mathcal{B}} &\coloneqq \inf\{\mathcal{E}^{\mathbf{D}}(A) : A \in \mathcal{A}_{\mathcal{B}} \text{ and } A \text{ avoids non-positivity}\} \\ &= \bigcap_{\substack{A \in \mathcal{A}_{\mathcal{B}} \\ \operatorname{posi}(A) \cap \mathcal{V}_{\leq 0} = \emptyset}} \operatorname{posi}(\mathcal{V}_{>0} \cup A). \end{aligned}$$

Proof. Consider any A in $\mathcal{A}_{\mathcal{B}}$. We will first show that the assessment $\mathcal{B}_A = \{\{0, u\} : u \in A\}$ is at least as strong as $\mathcal{B}$. By Definition 30$_{91}$ and Proposition 33$_{43}$(i), it suffices to show that $(\forall B \in \mathcal{B})(\exists B' \in \mathcal{B}_A) B' \preceq B$, so consider any B in $\mathcal{B}$. Then by definition, there is an element u_B of B that belongs to A, and therefore, $B' \coloneqq \{0, u_B\} \in \mathcal{B}_A$. Since

---

[6]We assume that $|B| \geq 2$—so that B contains at least one option different from 0—, for every B in $\mathcal{B}$. If this is not the case, then $\mathcal{B}$ does not avoid complete rejection, since it assesses that 0 should be rejected from $\{0\}$. This assumption is weaker than avoiding complete rejection, and it guarantees that $\mathcal{A}_{\mathcal{B}}$ is well defined.

[7]Here we let $\inf \emptyset = \mathrm{id}_{\mathcal{Q}}$.

[8]Here we let $\inf \emptyset = \bigcap \emptyset = \mathcal{V}$.

both 0 and $u_{\mathcal{B}}$ belong to B, therefore $B' \lessdot B$, and we have shown that $\mathcal{B}_A$ is indeed at least as strong as $\mathcal{B}$. Infer already that, by Corollary 76_{92}, therefore $\mathcal{E}(\mathcal{B}) \sqsubseteq \mathcal{E}(\mathcal{B}_A)$.

There are two possibilities: either (i) $\{A \in \mathcal{A}_{\mathcal{B}} : A \text{ avoids non-positivity}\} \neq \emptyset$, or (ii) $\{A \in \mathcal{A}_{\mathcal{B}} : A \text{ avoids non-positivity}\} = \emptyset$.

If (i) $\{A \in \mathcal{A}_{\mathcal{B}} : A \text{ avoids non-positivity}\} \neq \emptyset$, we will show that then $\mathcal{B}$ avoids complete rejection. Consider any A in $\mathcal{A}_{\mathcal{B}}$ that avoids non-positivity. By Theorem 86_{100}, then $\mathcal{B}_A$ avoids complete rejection. Since we have already shown that the assessment $\mathcal{B}_A$ is at least as strong as the assessment $\mathcal{B}$, use Proposition 75_{91} to infer that $\mathcal{B}$ indeed avoids complete rejection.

Since A avoids non-positivity, by Theorem 86_{100}, therefore $\mathcal{E}(\mathcal{B}_A) = R_{\mathcal{E}^{\mathbf{D}}(A)}$. Because $\mathcal{B}$ avoids complete rejection, by Theorem 81_{97} $\mathcal{E}(\mathcal{B}) = R_{\mathcal{B}}$, so $R_{\mathcal{B}} \sqsubseteq R_{\mathcal{E}^{\mathbf{D}}(A)}$. Since the choice of A in $\mathcal{A}_{\mathcal{B}}$ was arbitrary—as long as A avoids non-positivity—, therefore indeed $R_{\mathcal{B}} \sqsubseteq \inf\{R_{\mathcal{E}^{\mathbf{D}}(A)} : A \in \mathcal{A}_{\mathcal{B}} \text{ and } A \text{ avoids non-positivity}\}$. By Theorem 85_{100} $\mathcal{E}^{\mathbf{D}}(A) = \operatorname{posi}(\mathcal{V}_{>0} \cup A)$, so indeed also $R_{\mathcal{B}} \sqsubseteq \inf\{R_{\operatorname{posi}(\mathcal{V}_{>0} \cup A)} : A \in \mathcal{A}_{\mathcal{B}} \text{ and } A \text{ avoids non-positivity}\}$, proving the first part of the proposition. This also shows that

$$D_{\mathcal{B}} = \bigcap_{\substack{A \in \mathcal{A}_{\mathcal{B}} \\ \operatorname{posi}(A) \cap \mathcal{V}_{\leq 0} = \emptyset}} \operatorname{posi}(\mathcal{V}_{>0} \cup A).$$

Using Propositions 60_{68} and 58_{66}, we infer from $R_{\mathcal{B}} \sqsubseteq R_{\mathcal{E}^{\mathbf{D}}(A)}$ that $D_{R_{\mathcal{B}}} \subseteq \mathcal{E}^{\mathbf{D}}(A)$. Again, since the choice of A in $\mathcal{A}_{\mathcal{B}}$ was arbitrary—as long as A avoids non-positivity—, therefore indeed $D_{R_{\mathcal{B}}} \subseteq D_{\mathcal{B}}$, proving the second part of the proposition.

If (ii) $\{A \in \mathcal{A}_{\mathcal{B}} : A \text{ avoids non-positivity}\} = \emptyset$, then the statements become vacuous: $D_{R_{\mathcal{B}}} \subseteq \mathcal{V}$ and $R_{\mathcal{B}} \sqsubseteq \mathrm{id}_{\mathcal{Q}}$, and therefore indeed true. □

Proposition 87 provides an outer—more informative—approximation for the natural extension, that is especially useful for its pairwise behaviour, captured by its corresponding set of desirable options. The inequalities can be strict—meaning that $D_{R_{\mathcal{B}}} \subset D_{\mathcal{B}}$ and similarly for the expression that involves $R_{\mathcal{B}}$—as we will show in Example 18_{114}.

Note also that the (non-strict) inequalities are tight, in the sense that they can become equalities. This is for instance the case when the assessment $\mathcal{B}$ is a purely binary assessment $\mathcal{B}_B$, derived from a desirability assessment $B \subseteq \mathcal{V} \setminus \{0\}$.[9] To see this, Theorem 86_{100} implies that $\mathcal{B}_B$ avoids complete rejection if and only if B avoids non-positivity, and, if this is the case, then $R_{\mathcal{B}_B} = R_{\operatorname{posi}(\mathcal{V}_{>0} \cup B)}$, what, since $\mathcal{A}_{\mathcal{B}_B} = \{B\}$, is indeed equal to $\inf\{R_{\operatorname{posi}(\mathcal{V}_{>0} \cup A)} : A \in \mathcal{A}_{\mathcal{B}_B} \text{ and } A \text{ avoids non-positivity}\}$. The equality $D_{R_{\mathcal{B}_B}} = D_{\mathcal{B}_B}$ follows then at once.

A consequence of Proposition 87—and therefore of Theorem 86_{100}—is the following sufficient condition for an assessment to avoid complete rejection,

[9]The restriction that 0 cannot belong to B is because then $\mathcal{B}_B$ consists of binary sets, and it therefore guarantees that $\mathcal{A}_{\mathcal{B}_B}$ is well defined; see Footnote 6. This is a weaker requirement that avoiding non-positivity: if 0 would belong to B, then B would not avoid non-positivity.

that is easier to check:[10]

Corollary 88. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$. If $(\exists D \in \overline{\mathbf{D}})(\forall B \in \mathcal{B}) B \cap D \neq \emptyset$, then $\mathcal{B}$ avoids complete rejection.*

Proof. Since $(\forall B \in \mathcal{B}) B \cap D \neq \emptyset$ for some D in $\overline{\mathbf{D}}$, there is an element A' of $\mathcal{A}_{\mathcal{B}}$ such that $A' \subseteq D$. Therefore, by Theorem 85_{100}, A avoids non-positivity, so $\{A \in \mathcal{A}_{\mathcal{B}} : A \text{ avoids non-positivity}\} \neq \emptyset$ and hence $\inf\{R_{\mathcal{E}^{\mathrm{D}}(A)} : A \in \mathcal{A}_{\mathcal{B}} \text{ and } A \text{ avoids non-positivity}\}$ is a coherent rejection function. Using Proposition 87_{104}, $R_{\mathcal{B}} \subseteq \inf\{R_{\mathcal{E}^{\mathrm{D}}(A)} : A \in \mathcal{A}_{\mathcal{B}} \text{ and } A \text{ avoids non-positivity}\}$, so it satisfies Axiom $\mathrm{R1}_{20}$. Since by Proposition 79_{95} it also satisfies Axioms $\mathrm{R2}_{20}$–$\mathrm{R4}_{20}$ and extends $\mathcal{B}$, $R_{\mathcal{B}}$ is a coherent rejection function. Hence there is a coherent extension of $\mathcal{B}$, so by Theorem 81_{97}, therefore indeed $\mathcal{B}$ avoids complete rejection. □

3.6 EXAMPLES

Let us gain more insight in some aspects of the natural extension of a given assessment, by means of some examples.

The first example is related to the idea of the non-exhaustive interpretation[11] that we generally adopt. When the subject gives an assessment $\mathcal{B} \subseteq \mathcal{Q}_0$, almost always[12] there will be option sets B for which $0 \in R_{\mathcal{B}}(B)$ that do not belong to $\mathcal{B}$. This is as expected, as $R_{\mathcal{B}}$ is a coherent extension of $\mathcal{B}$, and due to some combination of different sets in $\mathcal{B}$, we can infer the rejection of 0 from other option sets.

In the example we will consider an assessment $\mathcal{B}$ that avoids complete rejection and that consists of only one option set $B \in \mathcal{Q}_0$, and examine whether this is also possible within *this single assessment*: are there non-zero options u in B such that $u \in R_{\mathcal{B}}(B)$? Clearly, if $\max(B \setminus \{0\}) \neq B \setminus \{0\}$, then, since $R_{\mathcal{B}}$ is coherent, by Proposition 31_{42}, this is the case, so we will additionally assume that $\max(B \setminus \{0\}) = B \setminus \{0\}$. At this point, bear in mind that such additional rejections are impossible for *single* purely binary—or desirability—assessments.

Example 14 (Exhaustive interpretation)**.** We will work with the special vector space of gambles $\mathcal{V} = \mathcal{L}$ on a binary possibility space $\mathcal{X} = \{\mathrm{H}, \mathrm{T}\}$, ordered by the standard point-wise ordering $\leq$.

The assessment we consider is $\mathcal{B} := \{B\}$, where $B := \{0, f, g\}$ with $f = (f(\mathrm{H}), f(\mathrm{T})) := (-1, \frac{1}{5})$ and $g := (-\frac{3}{2}, 2)$. Note that, by Corollary 88, this assessment avoids complete rejection: for instance, the coherent set of desirable

[10] It is an open question whether this condition is also necessary.

[11] See Section 3.4_{97}.

[12] That is, unless the rejection function R defined as $(\forall B \in \mathcal{Q})(\forall u \in \mathcal{V}) u \in R(B) \Leftrightarrow B - \{u\} \in \mathcal{B}$ is coherent.

options $D \coloneqq \operatorname{posi}(\mathcal{V}_{>0} \cup \{f\})$ satisfies $D \cap B = \{f, g\} \neq \emptyset$. We will show that—in addition to $0 \in R_{\mathcal{B}}(B)$ by the assessment—also $f \in R_{\mathcal{B}}(B)$, or equivalently, by Axiom R4b$_{20}$, $0 \in R_{\mathcal{B}}(B - \{f\})$, being an implication of the assessment, but not directly assessed as such. To this end, by Equation (3.1)$_{92}$, we need to show that there is some $A \supseteq B - \{f\}$ in $\mathcal{Q}$ such that

$$(\forall h \in \{0\} \cup (A \setminus (B - \{f\})))((A - \{h\}) \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\exists \mu \in \mathbb{R}_{>0})\{h\} + \mu B \lessdot A).$$

We will show that the particular choice $A = B - \{f\}$ satisfies the equation above. Indeed, then $\{0\} \cup (A \setminus (B - \{f\})) = \{0\}$, so we need only consider $h = 0$. It suffices to show that $\{0, \mu f, \mu g\} = \mu B \lessdot B - \{f\} = \{-f, 0, g - f\}$ for some μ in $\mathbb{R}_{>0}$. Clearly, 0 in μB is dominated by 0 in $B - \{f\}$. Also, since $f(\mathrm{H}) < 0$ while $\mu f(\mathrm{H}) > 0$ and $\mu g(\mathrm{H}) > 0$, the only possibility for μf and μg to be dominated, is $\mu f \leq g - f$ and $\mu g \leq g - f$. So if we can find some μ in $\mathbb{R}_{>0}$ such that $(-\mu, \frac{\mu}{5}) = \mu f \leq g - f = (-\frac{1}{2}, \frac{9}{5})$ and $(-\frac{3}{2}\mu, 2\mu) = \mu g \leq g - f = (-\frac{1}{2}, \frac{9}{5})$, then we have shown that $f \in R_{\mathcal{B}}(B)$. For instance the choice $\mu = \frac{1}{2}$ satisfies the two inequalities: indeed, $(-\frac{1}{2}, \frac{1}{5}) \leq (-\frac{1}{2}, \frac{9}{5})$ and $(-\frac{3}{4}, 1) \leq (-\frac{1}{2}, \frac{9}{5})$.

In order to visualise what happens in this example, consider Figure 3.2, where the gambles involved—0, f and g—are indicated. The assessment $\mathcal{B}$ is $\{\{0, f, g\}\}$; f is indicated in dark because it is the gamble of importance, for which we find that $f \in R_{\mathcal{B}}(\{0, f, g\})$. ◊

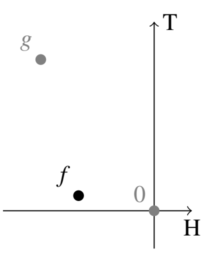


Figure 3.2: Illustration of the gambles involved in Example 14

As a more technical example, we investigate whether in Equation (3.1)$_{92}$ the superset $A' \supseteq A$ can be replaced by A itself: does it suffice to always consider the particular choice $A' = A$? We are motivated by the fact that this would simplify the expression for $R_{\mathcal{B}}$ significantly, and we feel strengthened by the observation that in many cases this clearly suffices—see, for instance, the proofs of Lemmas 77$_{92}$ and 78$_{94}$, and Example 14.

Example 15. Again, we will work with the special vector space of gambles $\mathcal{V} = \mathcal{L}$ on a binary possibility space $\mathcal{X} = \{\mathrm{H}, \mathrm{T}\}$, ordered by the standard pointwise ordering $\leq$.

The assessment we consider is $\mathcal{B} \coloneqq \{B\}$, where $B \coloneqq \{f, 0, g\}$ with $f = (f(\mathrm{H}), f(\mathrm{T})) \coloneqq (-1,1)$ and $g \coloneqq (2,-2)$. Note that, by Corollary 88_{106}, this assessment avoids complete rejection: for instance, the coherent set of desirable options $D \coloneqq \mathrm{posi}(\mathcal{V}_{>0} \cup \{f\})$ satisfies $D \cap B = \{f\} \neq \emptyset$. We are interested in the particular option set $A \coloneqq \{(-\frac{3}{2}, \frac{3}{2}), 0, g\}$, and want to know whether we can derive that $0 \in R_{\mathcal{B}}(A)$, or in other words, whether there are $A' \supseteq A$ in $\mathcal{Q}$ such that

$$(\forall h \in \{0\} \cup (A' \setminus A))\big((A' - \{h\}) \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists \mu \in \mathbb{R}_{>0})\{h\} + \mu B \preccurlyeq A'\big).$$

We will first show that this is indeed the case. Let $A' \coloneqq B \cup \{(-\frac{3}{2}, \frac{3}{2})\} = \{(-\frac{3}{2}, \frac{3}{2}), (-1,1), 0, (2,-2)\} \supseteq A$. Then $\{0\} \cup (A' \setminus A) = \{(-1,1), 0\} = \{f, 0\}$, so it suffices to check that

$$\big(A' \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists \mu \in \mathbb{R}_{>0})\mu B \preccurlyeq A'\big) \text{ and }$$
$$\big((A' - \{f\}) \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists \mu \in \mathbb{R}_{>0})\{f\} + \mu B \preccurlyeq A'\big).$$

To see that this holds, since $B \subseteq A'$ and by Proposition 33_{43}(i) we have that $B \preccurlyeq A'$. Furthermore, infer that $\{f\} + \frac{1}{2}B = \{(-\frac{3}{2}, \frac{3}{2}), (-1,1), 0\} \subseteq A'$, and therefore, again by Proposition 33_{43}(i), indeed $\{f\} + \frac{1}{2}B \preccurlyeq A'$. So we conclude that $0 \in R_{\mathcal{B}}(A)$.

If we only consider $A' = A$, then $\{0\} \cup (A' \setminus A) = \{0\}$, so we only need to check whether $\mu B \preccurlyeq A$ for some μ in $\mathbb{R}_{>0}$, or in other words, whether $\{(-\mu, \mu), 0, (2\mu, -2\mu)\} = \{\mu f, 0, \mu g\} \preccurlyeq \{(-\frac{3}{2}, \frac{3}{2}), 0, g\} = \{(-\frac{3}{2}, \frac{3}{2}), 0, (2,-2)\}$ for some μ in $\mathbb{R}_{>0}$. For every non-zero element of μB there correspond exactly one element in A in the same quadrant, so the condition becomes $\mu \geq \frac{3}{2}$, $\mu \leq \frac{3}{2}$, $\mu \leq 1$ and $\mu \geq 1$ for some μ in $\mathbb{R}_{>0}$, which is impossible. So by only considering the particular choice $A' = A$, we would incorrectly conclude that $0 \notin R_{\mathcal{B}}(A)$, and therefore, it does not suffice to only consider the particular choice $A' = A$ in Equation $(3.1)_{92}$.

In order to visualise what happens in this example, consider Figure 3.3, where the gambles involved—0, f, g and $(-\frac{3}{2}, \frac{3}{2})$—are indicated. The assessment is indicated by circles (○), the option set A by grey disks (•), and the additional gamble that defines A' by a dark disk (•). ◊

Many important choice functions—or rejection functions or choice relations for that matter—are infima of purely binary choice models: consider, for instance, the E-admissible or M-admissible choice functions. It is an important question whether *all* the coherent choice functions are infima of purely binary choice functions, since, if this question answered positively, this would immediately imply a representation theorem of coherent choice functions in terms of purely binary ones. If this question is answered in the negative, choice functions would constitute a theory that is more general than sets of desirable gambles in two ways: not only because it allows for more than binary choice,

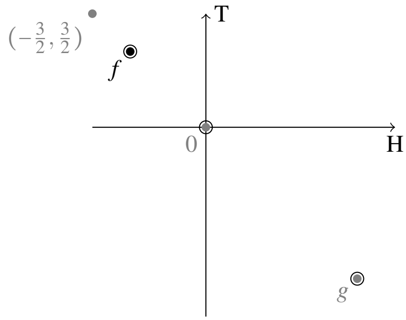


Figure 3.3: Illustration of the gambles involved in Example 15107

also because it is capable of expressing preferences that can never be retrieved as an infimum of purely binary preferences. From this discussion, it should be clear that this question is a very important one.[13]

Below we will answer this question in the negative: we will define a special rejection function $R_{\mathcal{B}}$, based on some particular assessment $\mathcal{B} \subseteq \mathcal{Q}_0$, and prove that it is no infimum of purely binary rejection functions.[14]

Example 16 (Is every coherent choice function an infimum of purely binary choice functions?)**.** As in the previous examples, we will work with the special vector space of gambles $\mathcal{V} = \mathcal{L}$ on a binary possibility space $\mathcal{X} = \{\mathrm{H}, \mathrm{T}\}$, ordered by the standard point-wise ordering $\leq$.

We consider a *single* assessment $\mathcal{B} \coloneqq \{B\}$, where B consists of a gamble and one scaled variant of it, together with 0: The assessment we consider is $B \coloneqq \{0, f, \lambda f\}$ with f a gamble and λ an element of $\mathbb{R}_{>0}$ and different from 1. If $f \in \mathcal{L}_{\leq 0}$ then $\mathcal{B}$ does not avoid complete rejection, and if f belongs to $\mathcal{L}_{>0}$, then the assessment is trivial, and hence $R_{\mathcal{B}} = R_{\mathrm{v}}$. So assume that f belongs to $(\mathcal{L}_{\leq 0} \cup \mathcal{L}_{>0})^c$; for convenience assume that $f(\mathrm{H}) < 0 < f(\mathrm{T})$.[15] Without loss of generality, we may assume that $\lambda > 1$.[16] The idea is that B consists of 0 and

[13] In fact, during my research for this dissertation, the question whether there are coherent choice functions that are no infima of purely binary choice functions, arose naturally on different occasions, and was crucial for several properties. It was only when we found the expression of the natural extension that it became possible for us to answer it.

[14] This example can be related to Example 3 in Reference [61], where Schervish et al. show that in the context of E-admissibility, a rejected option may be undominated by any chosen one in a pairwise comparison (and even by any option in the convex hull of the chosen options), provided that the set of probabilities is not closed.

[15] The other possibility is $f(\mathrm{T}) < 0 < f(\mathrm{H})$, but the conclusions are analogous.

[16] If $\lambda < 1$, then we can relabel f and λf: we let $\tilde{f} \coloneqq \lambda f$ and $\tilde{\lambda} \coloneqq \frac{1}{\lambda} > 1$, so $\mathcal{B} = \{0, f, \lambda f\} = \{0, \tilde{\lambda}\tilde{f}, \tilde{f}\}$.

two gambles that lie on the same line through 0, and on the same side of that line; see Figure 3.4 for an illustration of the assessment.

Note that, by Corollary 88_{106}, this assessment indeed avoids complete rejection: for instance, the coherent set of desirable options $D \coloneqq \operatorname{posi}(\mathcal{V}_{>0} \cup \{f\})$ satisfies $D \cap B = \{f, \lambda f\} \neq \emptyset$. Therefore, $R_{\mathcal{B}}$ is a coherent rejection function. To prove that $R_{\mathcal{B}}$ is no infimum of purely binary rejection functions, we first show the intermediate result that $0 \notin R_{\mathcal{B}}(A)$, where $A \coloneqq \{0, f\}$. To prove this, assume *ex absurdo* that $0 \in R_{\mathcal{B}}(A)$, and infer using Equation $(3.1)_{92}$ that then there would be some $A' \supseteq A$ in $\mathcal{Q}$ such that

$$(\forall h \in \{0\} \cup (A' \setminus A))\big((A' - \{h\}) \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\exists \mu \in \mathbb{R}_{>0})\{h\} + \mu B \lessdot A'\big). \tag{3.12}$$

At this point, remark already that $A' \neq A$: indeed, if *ex absurdo* $A' = A$, then $\{0\} \cup (A' \setminus A) = \{0\}$, so we need only consider $h = 0$. Infer that $A' \cap \mathcal{L}_{>0} = \emptyset$ and $(\forall \mu \in \mathbb{R}_{>0})\{0, \mu f, \mu\lambda f\} \not\lessdot \{0, f\}$, a contradiction, and therefore $A' \supset A$.

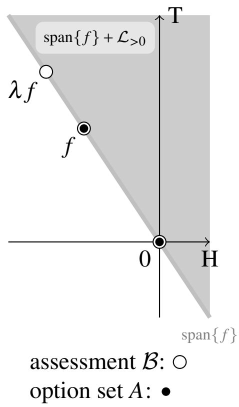


Figure 3.4: Illustration of the assessment and some relevant sets of gambles for Example 16

Without loss of generality, we let $A' \coloneqq \{0, f, h_1, \dots, h_n\} \supset A$ where n belongs to $\mathbb{N}$ and $h_1, \dots, h_n$ to $\mathcal{L}$, so $\{0\} \cup (A' \setminus A) = \{0, h_1, \dots, h_n\}$. Then, by Lemma 89_{112}, we find that $(\max A') \cap \{0, h_1, \dots, h_n\} \neq \emptyset$. As an intermediate result, we show that $(\max A') \cap (\operatorname{span}\{f\} + \mathcal{L}_{>0}) = \emptyset$. To see this, since $\{0, f\} \cap (\operatorname{span}\{f\} + \mathcal{L}_{>0}) = \emptyset$, infer that $(\max A') \cap (\operatorname{span}\{f\} + \mathcal{L}_{>0}) \subseteq \{h_1, \dots, h_n\}$, and assume *ex absurdo* that $(\max A') \cap (\operatorname{span}\{f\} + \mathcal{L}_{>0}) \neq \emptyset$. Let h be an element of $\arg\max\{g(\mathrm{T}) : g \in (\max A') \cap (\operatorname{span}\{f\} + \mathcal{L}_{>0})\}$, then $h(\mathrm{T}) + \mu\lambda f(\mathrm{T}) > h(\mathrm{T})$, so $h + \mu\lambda f \in \{h\} + \mu B$ is undominated in $(\max A') \cap (\operatorname{span}\{f\} + \mathcal{L}_{>0})$ whence $\{h\} + \mu B \not\lessdot (\max A') \cap (\operatorname{span}\{f\} + \mathcal{L}_{>0})$ for all μ in $\mathbb{R}_{>0}$. Note that, since h belongs to $\operatorname{span}\{f\} + \mathcal{L}_{>0}$, also $h + \mu\lambda f$ belongs to $\operatorname{span}\{f\} + \mathcal{L}_{>0}$ for every μ in

$\mathbb{R}_{>0}$. Therefore, since an element of $\operatorname{span}\{f\}+\mathcal{L}_{>0}$ can never be dominated by an element of $(\operatorname{span}\{f\}+\mathcal{L}_{>0})^c = \operatorname{span}\{f\}+\mathcal{L}_{\leq 0}$,[17] also $\{h\}+\mu B \not\lessdot \max A'$ for all μ in $\mathbb{R}_{>0}$. By Proposition 33_{43}(ii)&(iv), therefore also $\{h\}+\mu B \not\lessdot A'$ for all μ in $\mathbb{R}_{>0}$. Since h belongs to $\max A'$, also $A'-\{h\}\cap\mathcal{L}_{>0}=\emptyset$, a contradiction with Equation (3.12). So we have that $(\max A')\cap(\operatorname{span}\{f\}+\mathcal{L}_{>0})=\emptyset$, and therefore, again because an element of $\operatorname{span}\{f\}+\mathcal{L}_{>0}$ can never be dominated by an element of $\operatorname{span}\{f\}+\mathcal{L}_{\leq 0}$, also $A'\cap(\operatorname{span}\{f\}+\mathcal{L}_{>0})=\emptyset$.

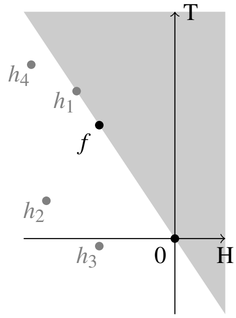


Figure 3.5: Illustration of a specific option set $A' \coloneqq \{0, f, h_1, \dots, h_4\}$, for which we know that $A'\cap(\operatorname{span}\{f\}+\mathcal{L}_{>0})=\emptyset$

Now that we know something about the shape of A'—namely, that $A'\cap(\operatorname{span}\{f\}+\mathcal{L}_{>0})=\emptyset$—, we go back to Equation (3.12), and consider first $h=0$. Then $A'\cap\mathcal{L}_{>0}\neq\emptyset$ or $(\exists\mu\in\mathbb{R}_{>0})\mu B\lessdot A'$. Since $A'\cap(\operatorname{span}\{f\}+\mathcal{L}_{>0})=\emptyset$, in particular $A'\cap\mathcal{L}_{>0}=\emptyset$, so the only possibility left is $(\exists\mu\in\mathbb{R}_{>0})\mu B\lessdot A'$, or, in other words, $\{0,\mu f,\mu\lambda f\}\lessdot\{0,f,h_1,\dots,h_n\}$ for some μ in $\mathbb{R}_{>0}$. There are three possibilities: if (i) $\mu=1$, then $h_i\geq\lambda f$—and therefore, since $A'\cap(\operatorname{span}\{f\}+\mathcal{L}_{>0})=\emptyset$, necessarily $h_i=\lambda f$—for some i in $\{1,\dots,n\}$; if (ii) $\mu=\frac{1}{\lambda}$ then $h_j\geq\frac{1}{\lambda}f$—and therefore, since $A'\cap(\operatorname{span}\{f\}+\mathcal{L}_{>0})=\emptyset$, necessarily $h_j=\frac{1}{\lambda}f$—for some j in $\{1,\dots,n\}$; and finally, if (iii) $\mu\notin\{\frac{1}{\lambda},1\}$, then $h_k\geq\mu f$ and $h_\ell\geq\mu\lambda f$—and therefore, since $A'\cap(\operatorname{span}\{f\}+\mathcal{L}_{>0})=\emptyset$, necessarily $h_k=\mu f$ and $h_\ell=\mu\lambda f$—for some k and ℓ in $\{1,\dots,n\}$. In any case, we find that $\{h_1,\dots,h_n\}\cap\operatorname{posi}\{f\}\neq\emptyset$. Without loss of generality, let h_1 be the unique gamble in $\{h_1,\dots,h_n\}\cap\operatorname{posi}\{f\}$ with highest value in T: $\{h_1\}=\arg\max\{g(\mathrm{T}):g\in\{h_1,\dots,h_n\}\cap\operatorname{posi}\{f\}\}$. Then, since $h_1\in\{0\}\cup(A'\setminus A)$, by Equation (3.12), we have that $(A'-\{h_1\})\cap\mathcal{L}_{>0}\neq\emptyset$ or $(\exists\mu\in\mathbb{R}_{>0})\{h_1\}+\mu B\lessdot A'$. Since $A'\cap(\operatorname{span}\{f\}+\mathcal{L}_{>0})=\emptyset$ and $h_1\in\operatorname{posi}\{f\}$, we find in particular $A'\cap(\{h_1\}+\mathcal{L}_{>0})=\emptyset$, whence, using Lemma 3_{12}, $(A'-\{h_1\})\cap\mathcal{L}_{>0}=\emptyset$. There-

[17]To see this, note that $(\operatorname{span}\{f\}+\mathcal{L}_{>0})$ is a coherent set of desirable gambles, whence, by Axioms $\mathrm{D2}_{57}$ and $\mathrm{D4}_{57}$, any gamble dominating some desirable gamble is desirable itself.

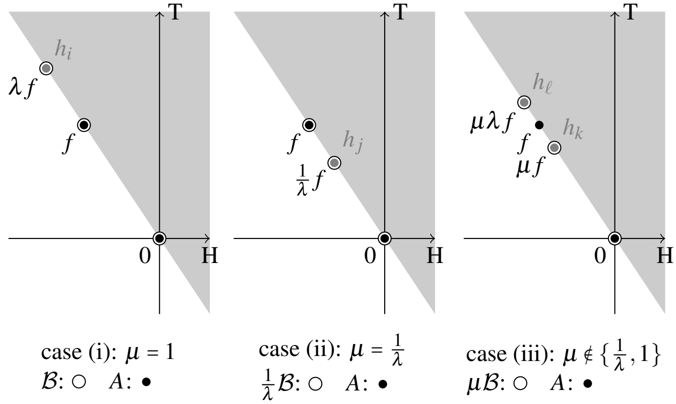


Figure 3.6: Illustration of the three different cases mentioned

fore necessarily $\{h_1, h_1 + \mu f, h_1 + \mu\lambda f\} = \{h_1\} + \mu B \lessdot A'$ for some μ in $\mathbb{R}_{>0}$. Note that both $h_1 + \mu f$ and $h_1 + \mu\lambda f$ belong to $\operatorname{posi}\{f\}$, and have a value in T that is strictly higher than $h_1(\mathrm{T})$. But at least one of $h_1 + \mu f$ or $h_1 + \mu\lambda f$ is not equal to f, and therefore an element of $\{h_1, \ldots, h_n\} \cap \operatorname{posi}\{f\}$, a contradiction with the fact that $h_1 \in \arg\max\{g(\mathrm{T}) : g \in \{h_1, \ldots, h_n\} \cap \operatorname{posi}\{f\}\}$. Therefore indeed $0 \notin R_{\mathcal{B}}(A)$. For an illustration of the argument, see Figure 3.7.

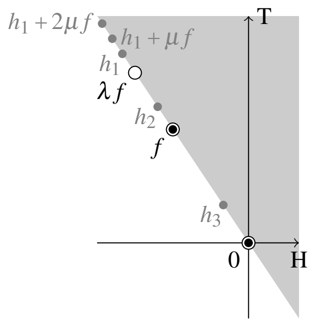


Figure 3.7: Illustration of the contradiction, using h_1, the gamble in $\{h_1, \ldots, h_n\} \cap \operatorname{span}\{f\}$ with the highest value in T: both $h_1 + \mu f$ and $h_1 + \mu\lambda f$ have a higher value in T, and at least one of them is not equal to f

So we have found a rejection function $R_{\mathcal{B}}$ that does not satisfy Axiom R6$_{25}$ [$0 \in R_{\mathcal{B}}(\{0, f, \lambda f\})$ but $0 \notin R_{\mathcal{B}}(\{0, f\})$], and therefore, by Proposition 56$_{65}$, $R_{\mathcal{B}}$ is no infimum of purely binary rejection functions. ◊

Lemma 89. *Consider any option sets* $A \coloneqq \{0, u_1, \dots, u_m\}$ *such that* $\max A = A$*, and* $A' \coloneqq A \cup \{v_1, \dots, v_n\}$*, with* $m \in \mathbb{N}$ *and* $n \in \mathbb{Z}_{\geq 0}$*. Then* $(\max A') \cap \{0, v_1, \dots, v_n\} \neq \emptyset$.

Proof. For notational convenience, let $A'_k \coloneqq A \cup \{v_1, \dots, v_k\}$ for all k in $\{0, \dots, n\}$. Then we need to prove that $(\max A'_n) \cap \{0, v_1, \dots, v_n\} \neq \emptyset$. We will use induction on n. For the base case $n = 0$, infer that $A'_0 = A$, and therefore $0 \in \max A'_0 = A'_0$, whence indeed $(\max A'_0) \cap \{0\} \neq \emptyset$. Consider now the case $n > 0$. Then by the induction hypothesis, we may assume that $(\max A'_{n-1}) \cap \{0, v_1, \dots, v_{n-1}\} \neq \emptyset$, and therefore $\max A'_{n-1} \nsubseteq \{u_1, \dots, u_m\}$, meaning that (i) $0 \in \max A'_{n-1}$ or (ii) $v_i \in \max A'_{n-1}$ for some i in $\{1, \dots, n-1\}$. Assume now *ex absurdo* that $\max A'_n \subseteq \{u_1, \dots, u_m\}$.

If (i) $0 \in \max A'_{n-1}$, then $(\forall w \in A'_{n-1}) 0 \nprec w$, and therefore, since $0 \notin \max A'_n$, necessarily $0 \prec v_n$. But, since $\prec$ is transitive, then $(\forall w \in A'_{n-1}) v_n \nprec w$, and, since $\prec$ is irreflexive, we can extend this to $(\forall w \in A'_n) v_n \nprec w$. Therefore $v_n \in \max A'_n$, a contradiction. So, if (i) holds, we infer that $\max A'_n \nsubseteq \{u_1, \dots, u_m\}$.

If (ii) $v_i \in \max A'_{n-1}$ for some i in $\{1, \dots, n-1\}$, then $(\forall w \in A'_{n-1}) v_i \nprec w$, and therefore, since $v_i \notin \max A'_n$, necessarily $v_i \prec v_n$. But, since $\prec$ is transitive, then $(\forall w \in A'_{n-1}) v_n \nprec w$, and, since $\prec$ is irreflexive, we can extend this to $(\forall w \in A'_n) v_n \nprec w$. Therefore $v_n \in \max A'_n$, a contradiction. So, also if (ii) holds, we infer that $\max A'_n \nsubseteq \{u_1, \dots, u_m\}$.

So we have showed that indeed $(\max A'_n) \cap \{0, v_1, \dots, v_n\} \neq \emptyset$. □

To gain a feel for what Proposition 87104 means, we will consider two simple assessments $\mathcal{B}$, and find the outer approximation $D_{\mathcal{B}}$ of $D_{R_{\mathcal{B}}}$, mentioned in Proposition 87104.

Example 17. We will work with the special vector space of gambles $\mathcal{V} = \mathcal{L}$ on a binary possibility space $\mathcal{X} = \{\mathrm{H}, \mathrm{T}\}$, ordered by the standard point-wise ordering $\leq$.

The first example is purely qualitative. Let

$$\mathcal{B} \coloneqq \{\{0, f_1, f_2\}, \{0, g_1, g_2, g_3\}\} \subseteq \mathcal{Q}_0,$$

as indicated in Figure 3.8↷. The set $\mathcal{A}_{\mathcal{B}}$ is given by

$$\mathcal{A}_{\mathcal{B}} = \{\{f_1, g_1\}, \{f_1, g_2\}, \{f_1, g_3\}, \{f_2, g_1\}, \{f_2, g_2\}, \{f_2, g_3\}\},$$

and its subset that avoids non-positivity is

$$\{\{f_1, g_1\}, \{f_1, g_2\}, \{f_2, g_1\}, \{f_2, g_2\}, \{f_2, g_3\}\}.$$

Each of these desirability assessments leads to a coherent sets of desirable gambles, as indicated in Figure 3.9115. Therefore, since $D_{\mathcal{B}}$ is the intersection of all these sets of desirable gambles, we see that $D_{\mathcal{B}} = \mathcal{L}_{>0}$ is the vacuous set of desirable gambles, shown in Figure 3.10116. Then also $D_{R_{\mathcal{B}}} = D_{\mathcal{B}} = \mathcal{L}_{>0}$ is the vacuous set of desirable gambles: the behaviour of $R_{\mathcal{B}}$ on pairwise option sets is vacuous! This does not imply that $R_{\mathcal{B}}$ is vacuous: indeed, for instance $0 \in R_{\mathcal{B}}(\{0, f_1, f_2\})$.

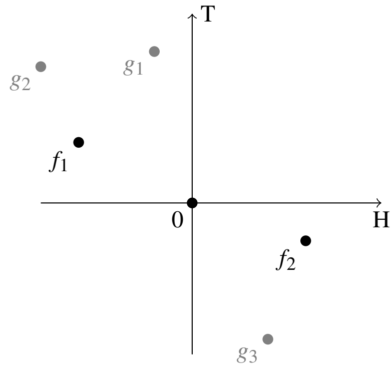


Figure 3.8: Illustration of the gambles in $\bigcup\mathcal{B}$

Without taking care, we might (incorrectly) conclude from this example that $\inf\{R_{\mathrm{posi}(\mathcal{V}_{>0}\cup A)} : A \in \mathcal{A}_{\mathcal{B}}$ and A avoids non-positivity$\}$ is the vacuous rejection function, and hence, Proposition 87_{104} would (incorrectly) imply that $R_{\mathcal{B}}$ is the vacuous rejection function. However, this is not true: indeed, consider for instance the option set $A' \coloneqq \{0, f_1, f_2\}$. Then $A' \cap D \neq \emptyset$ for all of the sets of desirable gambles D in Figure 3.9, whence $0 \in (\inf\{R_{\mathrm{posi}(\mathcal{V}_{>0}\cup A)} : A \in \mathcal{A}_{\mathcal{B}}$ and A avoids non-positivity$\})(A')$.

As a second, more general, example of Proposition 87_{104}, let $\mathcal{B}' \coloneqq \{\{0, f, g\}\}$, where f and g are any fixed gambles such that $f(\mathrm{H}) < 0 < f(\mathrm{T})$ and $g(\mathrm{T}) < 0 < g(\mathrm{H})$, of which a particular instance is indicated in Figure 3.11_{116}. By Corollary 88_{106} $\mathcal{B}'$ avoids complete rejection: indeed, let $D \coloneqq \mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})$, then $D \cap B' = \{f\} \neq \emptyset$ for the only—and hence every—$B' = \{0, f, g\}$ in $\mathcal{B}'$.

We wonder what the binary behaviour $D_{R_{\mathcal{B}}}$ of $R_{\mathcal{B}}$ is. To this end, we will use Proposition 87_{104} to find an outer approximation for it. Note that $\mathcal{A}_{\mathcal{B}'} = \{\{f\}, \{g\}\}$, and each of these desirability assessments avoid non-positivity, so we need to find the infimum of $\mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})$ and $\mathrm{posi}(\mathcal{L}_{>0} \cup \{g\})$, whose illustration is drawn in Figure 3.12_{117}. This infimum is $D_{\mathcal{B}'} = \mathcal{L}_{>0}$, and therefore the only possibility is $D_{R_{\mathcal{B}'}} = \mathcal{L}_{>0}$. So we see that the natural extension of a single assessment consisting of 0 and a gamble in each of the non-trivial quadrants, has vacuous pairwise behaviour! This generalises to bigger possibility spaces. $\Diamond$

This example clearly shows how Proposition 87_{104} comes in very handy to find an outer approximation of the implications of a given assessment. The approximated binary behaviour was found to be exact here.

In the next example, we revisit Example 16_{109} in the light of Proposition 87_{104}.

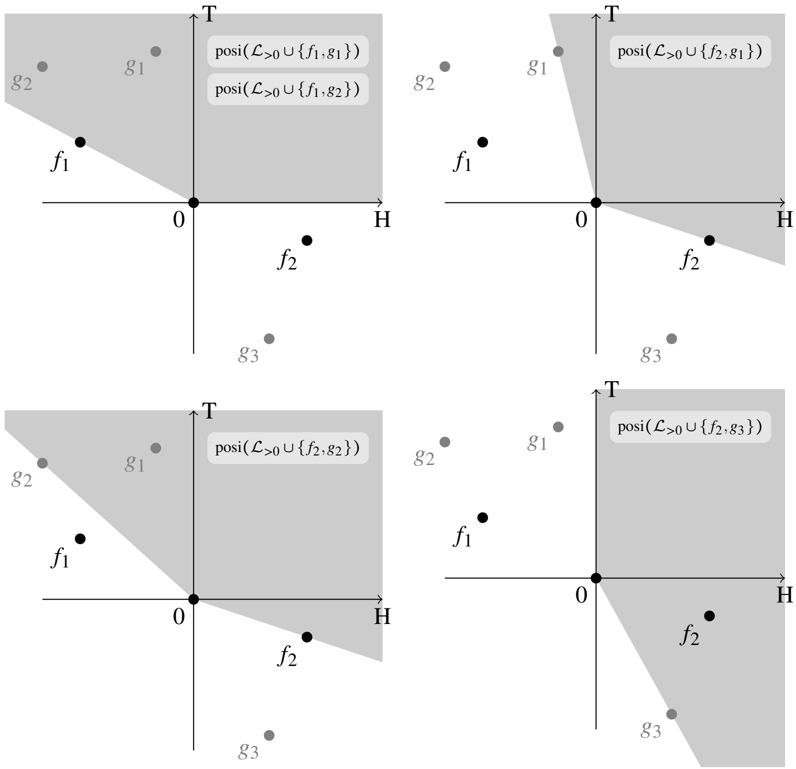


Figure 3.9: Illustration of the sets of desirable gambles corresponding to the elements of $\{A \in \mathcal{A}_{\mathcal{B}} : A \text{ avoids non-positivity}\}$

Example 18 (Binary behaviour of a non-binary assessment)**.** We consider the same assessment $\mathcal{B} := \{B\}$, where $B := \{0, f, \lambda f\}$ with f a fixed gamble on $\mathcal{X} = \{\mathrm{H}, \mathrm{T}\}$ such that $f(\mathrm{H}) < 0 < f(\mathrm{T})$ and $\lambda > 1$. We showed that $R_{\mathcal{B}}$ is no infimum of binary rejection functions, and that its behaviour is intrinsically non-binary: we found that $0 \notin R_{\mathcal{B}}(\{0, f\})$ but $0 \in R_{\mathcal{B}}(\{0, f, \lambda f\})$. In this example, we are looking for the *binary* implications of this assessment, so we wonder what $D_{R_{\mathcal{B}}}$ is. We showed that $\mathcal{B}$ avoids complete rejection, so by Theorem 81_{97} $R_{\mathcal{B}}$ is a coherent rejection function, and using Proposition 53_{61} we therefore already know that $D_{R_{\mathcal{B}}} := \{g \in \mathcal{L} : 0 \in R_{\mathcal{B}}(\{0, g\})\}$ is a coherent set of desirable gambles.

Recall also that $0 \notin R_{\mathcal{B}}(\{0, f\})$, so, quite surprisingly, $f \notin D_{R_{\mathcal{B}}}$ and, using Axiom $\mathrm{D3}_{57}$ therefore $D_{R_{\mathcal{B}}} \cap \mathrm{posi}\{f\} = \emptyset$, even though the assessment $\mathcal{B} = \{\{0, f, \lambda f\}\}$ states that 0 is rejected from 0 and two different gambles on the ray $\mathrm{posi}\{f\}$. Considering the *purely binary* assessments $\{\{0, f\}\}$, $\{\{0, \lambda f\}\}$ and $\{\{0, f\}, \{0, \lambda f\}\}$ that avoid complete rejection, note that by Theorem 86_{100} the respective sets of desirable gambles based on their natural

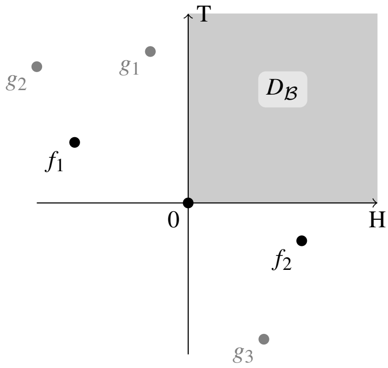


Figure 3.10: Illustration of the gambles in $D_{\mathcal{B}}$

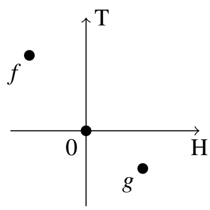


Figure 3.11: Illustration of the gambles in $\bigcup\mathcal{B}'$

extensions are given by $\operatorname{posi}(\{f\}\cup\mathcal{L}_{>0})$, $\operatorname{posi}(\{\lambda f\}\cup\mathcal{L}_{>0})$ and $\operatorname{posi}(\{f,\lambda f\}\cup\mathcal{L}_{>0})$ respectively. But these three set of desirable gambles are all equal to each other, and they all lead to the desirability of f, and therefore differ from $D_{R_{\mathcal{B}}}$. So a natural question is: 'What is $D_{R_{\mathcal{B}}}$?' We will show that $D_{R_{\mathcal{B}}}=\mathcal{L}_{>0}+\operatorname{posi}\{0,f\}$.

To show that $D_{R_{\mathcal{B}}}\subseteq\mathcal{L}_{>0}+\operatorname{posi}\{0,f\}$, first we will show the weaker statement that $D_{R_{\mathcal{B}}}\subseteq\operatorname{posi}(\mathcal{L}_{>0}\cup\{f\})$; see Figure 3.13 for an illustration of $\mathcal{L}_{>0}+\operatorname{posi}\{0,f\}$ and related sets of gambles, where full lines and grey points are included in the set, and dotted lines and white points excluded. To this end, use Proposition 87_{104} to infer that $D_{R_{\mathcal{B}}}\subseteq D_{\mathcal{B}}$, where

$$D_{\mathcal{B}}=\bigcap_{\substack{A\in\mathcal{A}_{\mathcal{B}}\\ \operatorname{posi}(A)\cap\mathcal{V}_{\leq 0}=\varnothing}}\operatorname{posi}(\mathcal{V}_{>0}\cup A)$$

and $\mathcal{A}_{\mathcal{B}}=\{\{f\},\{\lambda f\}\}$. Therefore $D_{\mathcal{B}}=\operatorname{posi}(\mathcal{L}_{>0}\cup\{f\})\cap\operatorname{posi}(\mathcal{L}_{>0}\cup\{\lambda f\})=\operatorname{posi}(\mathcal{L}_{>0}\cup\{f\})$, whence indeed $D_{R_{\mathcal{B}}}\subseteq\operatorname{posi}(\mathcal{L}_{>0}\cup\{f\})$, or, in other words, using Lemma 1_{11}, $D_{R_{\mathcal{B}}}\subseteq\mathcal{L}_{>0}\cup(\operatorname{posi}\{f\})\cup(\mathcal{L}_{>0}+\operatorname{posi}\{f\})$. Since we already know that $D_{R_{\mathcal{B}}}\cap\operatorname{posi}\{f\}=\varnothing$, therefore indeed $D_{R_{\mathcal{B}}}\subseteq\mathcal{L}_{>0}\cup(\mathcal{L}_{>0}+\operatorname{posi}\{f\})=\mathcal{L}_{>0}+\operatorname{posi}\{0,f\}$.

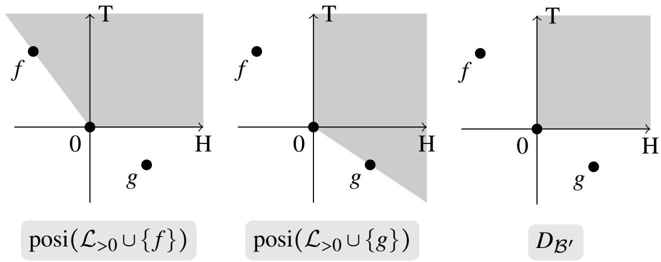


Figure 3.12: Illustration of $\mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})$, $\mathrm{posi}(\mathcal{L}_{>0} \cup \{g\})$ and its intersection $D_{\mathcal{B}'}$

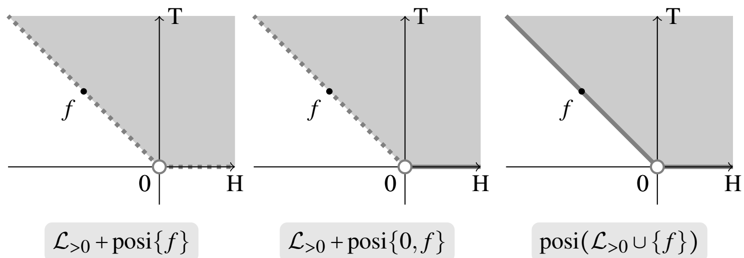


Figure 3.13: Illustration of $\mathcal{L}_{>0} + \mathrm{posi}\{f\} \subset \mathcal{L}_{>0} + \mathrm{posi}\{0, f\} \subset \mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})$

Note that this also shows that the inequalities in Proposition 87_{104} can be strict. Indeed, $D_{R_{\mathcal{B}}} \subseteq \mathcal{L}_{>0} + \mathrm{posi}\{0, f\} \subset \mathrm{posi}(\mathcal{L}_{>0} \cup f) = D_{\mathcal{B}}$. Also, as we have seen, both of the desirability assessments A in $\mathcal{A}_{\mathcal{B}}$ lead to the same set of desirable gambles $\mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})$, therefore $\inf\{R_{\mathrm{posi}(\mathcal{V}_{>0} \cup A)} : A \in \mathcal{A}_{\mathcal{B}} \text{ and } A \text{ avoids non-positivity}\} = R_{\mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})}$. We have shown in Example 16_{109} that $0 \notin R_{\mathcal{B}}(\{0, f\})$, while $0 \in R_{\mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})}(\{0, f\})$, and therefore indeed $R_{\mathcal{B}} \sqsubset \inf\{R_{\mathrm{posi}(\mathcal{L}_{>0} \cup A)} : A \in \mathcal{A}_{\mathcal{B}} \text{ and } A \text{ avoids non-positivity}\}$.

To show that $\mathcal{L}_{>0} + \mathrm{posi}\{0, f\} \subseteq D_{R_{\mathcal{B}}}$, consider any g in $\mathcal{L}_{>0} + \mathrm{posi}\{0, f\}$. We will show that then $g \in D_{R_{\mathcal{B}}}$, or, equivalently, that $0 \in R_{\mathcal{B}}(A)$, where $A \coloneqq \{0, g\}$. If g belongs to $\mathcal{L}_{>0}$, by Axiom $\mathrm{R1}_{20}$ then $0 \in R_{\mathcal{B}}(\{0, g\})$, so assume that $g \notin \mathcal{L}_{>0}$. Then $g(\mathrm{H}) < 0 < g(\mathrm{T})$. As a result, in particular $g \in \mathcal{L}_{>0} + \mathrm{posi}\{f\}$. To show that $0 \in R_{\mathcal{B}}(A)$, we need to find some $A' \supseteq A$ in $\mathcal{Q}$ such that

$$(\forall h \in \{0\} \cup (A' \setminus A))\big((A' - \{h\}) \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\exists \mu \in \mathbb{R}_{>0})\{h\} + \mu B \lessdot A'\big). \tag{3.13}$$

We state that

$$A' \coloneqq A \cup \left\{\mu' \beta_k f : k \in \mathbb{N} \text{ and } \beta_{k-1} < \frac{g(\mathrm{H})}{\mu' f(\mathrm{H})}\right\}$$

where we let $\mu' \coloneqq \frac{g(\mathrm{T})}{\lambda f(\mathrm{T})} > 0$ and $\beta\colon \mathbb{N} \to \mathbb{R}\colon k \mapsto \beta_k \coloneqq \sum_{i=1}^{k} (\frac{\lambda-1}{\lambda})^{i-1}$, and $\beta_0 \coloneqq 0$, satisfies Equation $(3.13)_{\curvearrowleft}$. Since $\lambda > 1$, therefore $\frac{\lambda-1}{\lambda} \in (0,1)$, and, since β forms a geometric progression, we infer that β satisfies the following properties: $\beta_k = \lambda - \lambda(\frac{\lambda-1}{\lambda})^k$ for all k in $\mathbb{N}$, so β is increasing and $\lim_{k\to+\infty} \beta_k = \lambda$.

Before we show that A' satisfies Equation $(3.13)_{\curvearrowleft}$, we first show that it belongs to $\mathcal{Q}$—that it is finite. We need to show that $\beta_{k-1} < \frac{g(\mathrm{H})}{\mu' f(\mathrm{H})}$ fails to hold for some k in $\mathbb{N}$: we will show that $\beta_{k-1} \geq \frac{g(\mathrm{H})}{\mu' f(\mathrm{H})}$ for some k in $\mathbb{N}$. Since $\lim_{k\to+\infty} \beta_k = \lambda$ this happens exactly when $\frac{g(\mathrm{H})}{\mu' f(\mathrm{H})} < \lambda$.[18] Infer, using that $f(\mathrm{H}) < 0 < f(\mathrm{T})$ and $g(\mathrm{H}) < 0 < g(\mathrm{T})$, that

$$\frac{g(\mathrm{H})}{\mu' f(\mathrm{H})} = \frac{\lambda f(\mathrm{T}) g(\mathrm{H})}{f(\mathrm{H}) g(\mathrm{T})} < \lambda \Leftrightarrow \frac{f(\mathrm{T})}{f(\mathrm{H})} > \frac{g(\mathrm{T})}{g(\mathrm{H})} \Leftrightarrow \frac{f(\mathrm{T})}{|f(\mathrm{H})|} < \frac{g(\mathrm{T})}{|g(\mathrm{H})|}.$$

Since $g \in \mathcal{L}_{>0} + \operatorname{posi}\{g\}$, therefore $g > \mu f$ for some μ in $\mathbb{R}_{>0}$. Therefore $g(\mathrm{H}) \geq \mu f(\mathrm{H})$—because $g(\mathrm{H}) < 0$ and $f(\mathrm{H}) < 0$, equivalently $|g(\mathrm{H})| \leq |\mu f(\mathrm{H})|$—and $g(\mathrm{T}) \geq \mu f(\mathrm{T})$, with one of the two inequalities strict. Then indeed $\frac{f(\mathrm{T})}{|f(\mathrm{H})|} = \frac{\mu f(\mathrm{T})}{|\mu f(\mathrm{H})|} < \frac{g(\mathrm{T})}{|g(\mathrm{H})|}$, so we have showed that $\frac{g(\mathrm{H})}{\mu' f(\mathrm{H})} < \lambda$ and therefore indeed that A' belongs to $\mathcal{Q}$. For notational convenience, we let $\bar{k}$ be the biggest k in $\mathbb{N}$ such that $\beta_{k-1} < \frac{g(\mathrm{H})}{\mu' f(\mathrm{H})}$, and $h_k \coloneqq \mu' \beta_k f$ for every k in $\{1,\ldots,\bar{k}\}$. Figure 3.14 shows an illustration of A', which in this example is $\{0, g, h_1, h_2, h_3, h_4\}$, since 4 is the first index k for which $h_k(\mathrm{H})$ is smaller than $g(\mathrm{H})$; therefore $\bar{k} = 4$.

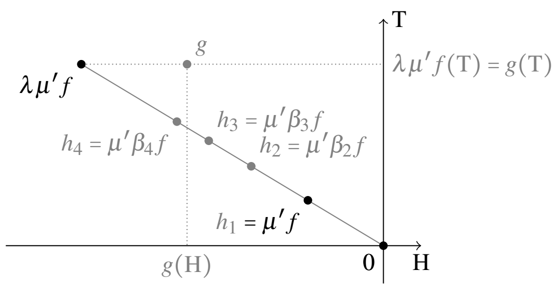


Figure 3.14: Illustration of $A' = \{0, g, h_1, h_2, h_3, h_4\}$

We are now ready to show that A' satisfies Equation $(3.13)_{\curvearrowleft}$. Note that $\{0\} \cup (A' \setminus A) = \{0, h_1, \ldots, h_{\bar{k}}\}$. So we need to show that $A' \cap \mathcal{L}_{>0} \neq \emptyset$ or $\mu B \lessdot A'$

[18] To see this, observe that $\beta_{k-1} \geq \frac{g(\mathrm{H})}{\mu' f(\mathrm{H})}$ if $k \geq 1 + \frac{\ln\left(1 - \frac{g(\mathrm{H})}{\lambda\mu' f(\mathrm{H})}\right)}{\ln(\frac{\lambda-1}{\lambda})}$, being a positive real number if and only if $\frac{g(\mathrm{H})}{\lambda\mu' f(\mathrm{H})} < 1$.

for some $\mu \in \mathbb{R}_{>0}$, and, for every k in $\{1,\dots,\bar{k}\}$, that $(A' - \{h_k\}) \cap \mathcal{L}_{>0} \neq \emptyset$ or $\{h_k\} + \mu_k B \lessdot A'$ for some μ_k in $\mathbb{R}_{>0}$. We start with $h = 0$. We will show that $\mu B \lessdot A'$ for some $\mu \in \mathbb{R}_{>0}$. Let $\mu = \mu'$, then $\mu B = \{0, \mu' f, \lambda\mu' f\} = \{0, h_1, \lambda\mu' f\}$. Since $\mu' = \frac{g(\mathrm{T})}{\lambda f(\mathrm{T})}$, therefore $\lambda\mu' f(\mathrm{T}) = g(\mathrm{T})$, and since g belongs to $\mathcal{L}_{>0} + \mathrm{posi}\{f\}$, therefore also $\lambda\mu' f(\mathrm{H}) < g(\mathrm{H})$, so $\lambda\mu' f < g$, and hence $\mu B \lessdot \{0, h_1, g\}$. But $\{0, h_1, g\} \subseteq A'$, whence by Proposition 33_{43}(i)&(ii) indeed $\mu B \lessdot A'$. Next, we will show that $(A' - \{h_{\bar{k}}\}) \cap \mathcal{L}_{>0} \neq \emptyset$. We state that $h_{\bar{k}} < g$. To see this, since $\bar{k}$ is the biggest k in $\mathbb{N}$ such that $\beta_{k-1} < \frac{g(\mathrm{H})}{\mu' f(\mathrm{H})}$, we have that $\beta_{\bar{k}} \geq \frac{g(\mathrm{H})}{\mu' f(\mathrm{H})}$, or, in other words, that $h_{\bar{k}}(\mathrm{H}) \leq g(\mathrm{H})$. Furthermore, since we have already shown that $\beta_k < \lambda$ for all k in $\mathbb{N}$, and that $\lambda\mu' f(\mathrm{T}) = g(\mathrm{T})$, we have that $h_k(\mathrm{T}) = \mu'\beta_k f(\mathrm{T}) < \lambda\mu' f(\mathrm{T}) = g(\mathrm{T})$, so indeed $h_{\bar{k}} < g$. Since g belongs to A', this proves that indeed $(A' - \{h_{\bar{k}}\}) \cap \mathcal{L}_{>0} \neq \emptyset$. Finally, we will prove, for all k in $\{1,\dots,\bar{k}-1\}$, that there is some μ_k in $\mathbb{R}_{>0}$ such that $\{h_k\} + \mu_k B \lessdot A'$. Consider any k in $\{1,\dots,\bar{k}-1\}$ and let $\mu_k = \frac{\mu'}{\lambda}(\lambda - \beta_k) > 0$. Note that $\mu_k = \frac{\mu'}{\lambda}(\lambda - \beta_k) = \mu'(1 - \frac{\beta_k}{\lambda}) = \mu'(1 - (1 - (\frac{\lambda-1}{\lambda})^k)) = \mu'(\frac{\lambda-1}{\lambda})^k = \mu'(\beta_{k+1} - \beta_k)$. Then

$$\begin{aligned}
\{h_k\} + \mu B &= \{h_k, h_k + \mu_k f, h_k + \mu_k \lambda f\} \\
&= \left\{h_k, \mu'\beta_k f + \mu'(\beta_{k+1} - \beta_k) f, \mu'\beta_k f + \frac{\mu'}{\lambda}(\lambda - \beta_k)\lambda f\right\} \\
&= \{h_k, \mu'\beta_{k+1} f, \mu'\lambda f\} = \{h_k, h_{k+1}, \mu'\lambda f\}.
\end{aligned}$$

We already know that $\mu'\lambda f$ is dominated by g, whence $\{h_k\} + \mu_k B \lessdot \{h_k, h_{k+1}, g\}$. But $\{h_k, h_{k+1}, g\} \subseteq A'$, whence by Proposition 33_{43}(i)&(ii) indeed $\{h_k\} + \mu_k B \lessdot A'$. So we have shown that Equation $(3.13)_{117}$ holds, whence $0 \in R_{\mathcal{B}}(\{0, g\})$, and therefore indeed $\mathcal{L}_{>0} + \mathrm{posi}\{0, f\} \subseteq D_{R_{\mathcal{B}}}$. Since we already know that $D_{R_{\mathcal{B}}} \subseteq \mathcal{L}_{>0} + \mathrm{posi}\{0, f\}$, this means that indeed $D_{R_{\mathcal{B}}} = \mathcal{L}_{>0} + \mathrm{posi}\{0, f\}$: this is the binary behaviour that is incorporated in $R_{\mathcal{B}}$.

Compare this with the purely binary assessments $\mathcal{B}' := \{\{0, f\}\} \subseteq \mathcal{Q}_0$ and $\mathcal{B}'' := \{\{0, f + \varepsilon\} : \varepsilon \in \mathbb{R}_{>0}\} \subseteq \mathcal{Q}_0$, where f is the same fixed gamble as before. Both $\mathcal{B}'$ and $\mathcal{B}''$ avoid complete rejection, since, with $D := \mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})$ we have $((\forall B' \in \mathcal{B}') B' \cap D \neq \emptyset) \Leftrightarrow \{0, f\} \cap D \neq \emptyset \Leftrightarrow \{f\} \neq \emptyset$ and

$$\begin{aligned}
((\forall B'' \in \mathcal{B}'') B'' \cap D \neq \emptyset) &\Leftrightarrow ((\forall \varepsilon \in \mathbb{R}_{>0}) \{0, f + \varepsilon\} \cap D \neq \emptyset) \\
&\Leftrightarrow ((\forall \varepsilon \in \mathbb{R}_{>0}) \{f + \varepsilon\} \neq \emptyset),
\end{aligned}$$

which are both true. Therefore, by Corollary 88_{106}, indeed $\mathcal{B}'$ and $\mathcal{B}''$ avoid complete rejection, and by Theorem 81_{97}, their respective natural extensions $R_{\mathcal{B}'}$ and $R_{\mathcal{B}''}$ are coherent. Since both the assessments $\mathcal{B}'$ and $\mathcal{B}''$ are binary assessments, by Theorem 86_{100}, therefore $R_{\mathcal{B}'}$ and $R_{\mathcal{B}''}$ are binary rejection functions, derived from $\mathrm{posi}(\mathcal{L}_{>0} \cup B')$ and $\mathrm{posi}(\mathcal{L}_{>0} \cup B'')$ respectively, where $B' := \bigcup(\mathcal{B}' \setminus \{0\}) = \{f\}$ and $B'' := \bigcup(\mathcal{B}'' \setminus \{0\}) = \{f + \varepsilon : \varepsilon \in \mathbb{R}_{>0}\}$. Therefore $R_{\mathcal{B}'} = R_{\mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})}$ and, using Lemma 90_{121}, $R_{\mathcal{B}''} = R_{\mathcal{L}_{>0} + \mathrm{posi}\{0, f\}}$.

We will show that $R_{\mathcal{B}}$ is strictly bounded by $R_{\mathcal{B}''}$ and $R_{\mathcal{B}'}$, in the sense that $R_{\mathcal{B}''} \sqsubset R_{\mathcal{B}} \sqsubset R_{\mathcal{B}'}$. First, we will show that $R_{\mathcal{B}''} \neq R_{\mathcal{B}} \neq R_{\mathcal{B}'}$. Note that $R_{\mathcal{B}}$ does not satisfy Axiom R6$_{25}$, as we have shown in Example 16$_{109}$, while, using Proposition 56$_{65}$ and since they are derived from a set of desirable gambles, both $R_{\mathcal{B}''}$ and $R_{\mathcal{B}'}$ do satisfy Axiom R6$_{25}$. Therefore indeed $R_{\mathcal{B}''} \neq R_{\mathcal{B}} \neq R_{\mathcal{B}'}$. So it suffices to prove that $R_{\mathcal{B}''} \sqsubseteq R_{\mathcal{B}} \sqsubseteq R_{\mathcal{B}'}$. Since all three rejection functions involved are coherent, it suffices to show that $0 \in R_{\mathcal{B}''}(A) \Rightarrow 0 \in R_{\mathcal{B}}(A)$ and $0 \in R_{\mathcal{B}}(A) \Rightarrow 0 \in R_{\mathcal{B}'}(A)$ for every A in $\mathcal{Q}$. To show that $R_{\mathcal{B}''} \sqsubseteq R_{\mathcal{B}}$, consider any A in $\mathcal{Q}$ and assume that $0 \in R_{\mathcal{B}''}(A)$. Therefore, since $R_{\mathcal{B}''} = R_{\mathcal{L}_{>0}+\mathrm{posi}\{0,f\}}$, equivalently $0 \in A$ and $A \cap (\mathcal{L}_{>0} + \mathrm{posi}\{0,f\}) \neq \emptyset$, whence there is some h in A such that h belongs to $\mathcal{L}_{>0} + \mathrm{posi}\{0,f\}$. Since we have shown earlier in this example that $D_{R_{\mathcal{B}}} = \mathcal{L}_{>0} + \mathrm{posi}\{0,f\}$, therefore $0 \in R_{\mathcal{B}}(\{0,h\})$. Using Axiom R3a$_{20}$ [with $\tilde{A} \coloneqq A$, $\tilde{A}_1 \coloneqq \{0\}$ and $\tilde{A}_2 \coloneqq \{0,h\}$] then indeed $0 \in R_{\mathcal{B}}(A)$. Since the choice of A was arbitrary, we have shown that $R_{\mathcal{B}''} \sqsubseteq R_{\mathcal{B}}$, and since they are not equal, therefore indeed $R_{\mathcal{B}''} \sqsubset R_{\mathcal{B}}$. To show that $R_{\mathcal{B}} \sqsubseteq R_{\mathcal{B}'}$, note that the assessment $\mathcal{B}'$ is at least as strong as $\mathcal{B}$: indeed, $\{0,f\} \lessdot \{0,f,\lambda f\}$, so by Definition 30$_{91}$ $\mathcal{B}'$ is indeed at least as strong as $\mathcal{B}$. By Proposition 76$_{92}$, then $R_{\mathcal{B}} \sqsubseteq R_{\mathcal{B}'}$ and since they are not equal, therefore indeed $R_{\mathcal{B}} \sqsubset R_{\mathcal{B}'}$. So we conclude that $R_{\mathcal{B}''} \sqsubset R_{\mathcal{B}} \sqsubset R_{\mathcal{B}'}$: $R_{\mathcal{B}}$ is strictly bounded by $R_{\mathcal{B}''}$ and $R_{\mathcal{B}'}$.

We summarise our comparison in the next table, where A is any element of $\mathcal{Q}$ such that $0 \in A$.

	R6?	Binary behaviour	0 rejected from A
$R_{\mathcal{B}''}$	yes	$D_{R_{\mathcal{B}''}} = \mathcal{L}_{>0} + \mathrm{posi}\{0,f\}$	$\Leftrightarrow A \cap (\mathcal{L}_{>0} + \mathrm{posi}\{0,f\}) \neq \emptyset$
$R_{\mathcal{B}}$	no	$D_{R_{\mathcal{B}}} = \mathcal{L}_{>0} + \mathrm{posi}\{0,f\}$	$\Rightarrow A \cap \mathrm{posi}(\mathcal{L}_{>0} \cup \{f\}) \neq \emptyset$ $\Leftarrow A \cap (\mathcal{L}_{>0} + \mathrm{posi}\{0,f\}) \neq \emptyset$
$R_{\mathcal{B}'}$	yes	$D_{R_{\mathcal{B}'}} = \mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})$	$\Leftrightarrow A \cap \mathrm{posi}(\mathcal{L}_{>0} \cup \{f\}) \neq \emptyset$

For $R_{\mathcal{B}'} = R_{\mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})}$ and $R_{\mathcal{B}''} = R_{\mathcal{L}_{>0}+\mathrm{posi}\{0,f\}}$, the rightmost column follows from Proposition 55$_{64}$, and for $R_{\mathcal{B}}$, this is a consequence from the fact that $R_{\mathcal{B}''} \sqsubset R_{\mathcal{B}} \sqsubset R_{\mathcal{B}'}$.

I want to highlight that the difference between $R_{\mathcal{B}}$, $R_{\mathcal{B}'}$ and $R_{\mathcal{B}''}$ shows itself in option sets that have parts in common with $\mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})$ but not with $\mathcal{L}_{>0} + \mathrm{posi}\{0,f\}$, so in option sets A such that

$$A \cap (\mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})) \neq \emptyset \text{ and } A \subseteq (\mathcal{L}_{>0} + \mathrm{posi}\{0,f\})^c.$$

Indeed, for such option sets, $0 \notin R_{\mathcal{B}''}(A)$, $0 \in R_{\mathcal{B}'}(A)$, and it is undetermined whether 0 belongs to $R_{\mathcal{B}}(A)$. Let us investigate what this means. Since $A \cap (\mathrm{posi}(\mathcal{L}_{>0} \cup \{f\})) \neq \emptyset$, by Lemma 1$_{11}$,

$$A \cap \mathrm{posi}(f) \neq \emptyset \text{ or } A \cap (\mathcal{L}_{>0} + \mathrm{posi}\{0,f\}) \neq \emptyset,$$

but since $A \subseteq (\mathcal{L}_{>0} + \mathrm{posi}\{0,f\})^c$, therefore

$$A \cap \mathrm{posi}(f) \neq \emptyset,$$

so the border $\operatorname{posi}\{f\}$ is the important part of $A \subseteq (\mathcal{L}_{>0} + \operatorname{posi}\{0,f\})^c$ to determine whether $0 \in R_{\mathcal{B}'}(A)$ and $0 \in R_{\mathcal{B}}(A)$. This shows that $R_{\mathcal{B}}$ has complex border behaviour—more complex behaviour than desirability can model. We see that, besides their generalisation towards non-binary choice, choice models are capable of modelling even more complex border behaviour than desirability.

As a final remark, I would like to draw attention to some subtlety, by comparing with Example 14_{106}: in that setting, there we found that $f \in R_{\mathcal{B}}(\{0,f,g\})$, where f and g are two specific gambles such that $f(\mathrm{H}) < 0 < f(\mathrm{T})$ and $g(\mathrm{H}) < 0 < g(\mathrm{T})$—just as in this example, but they do not lie on the same ray through 0. We found that $f \in R_{\mathcal{B}}(\{0,f,g\})$, which, taking into account that $0 \in R_{\mathcal{B}}(\{0,f,g\})$, by Axiom R3b$_{20}$ means that $0 \in R_{\mathcal{B}}(\{0,g\})$, and therefore $D_{R_{\mathcal{B}}} \supseteq \operatorname{posi}(\mathcal{L}_{>0} \cup \{g\})$. By combining this with Proposition 87_{104}, we find that $D_{R_{\mathcal{B}}} = \operatorname{posi}(\mathcal{L}_{>0} \cup \{g\})$. So in Example 14_{106}, the outer approximation of the binary behaviour given in Proposition 87_{104} is exact, while in Example 16_{109}, it is not. This observation hints at the reason why it is difficult to find an exact connection between $\mathcal{B}$ and the binary behaviour $D_{R_{\mathcal{B}}}$ of $R_{\mathcal{B}}$, other than the outer approximation of Proposition 87_{104} or by calculating $R_{\mathcal{B}}$ explicitly: even in these easy examples where $|\mathcal{X}| = 2$ and the assessments are relatively small, it is hard to predict when Proposition 87_{104} gives $D_{R_{\mathcal{B}}}$ exactly, and when it gives an outer approximation for it—and it is even harder to see the reason for this. $\lozenge$

Lemma 90. *Consider the possibility space $\mathcal{X} := \{\mathrm{H},\mathrm{T}\}$ and the linear space $\mathcal{L}$ of gambles on $\mathcal{X}$. Consider any h in $(\mathcal{L}_{>0} \cup \mathcal{L}_{<0})^c$, and let $A := \{h + \varepsilon : \varepsilon \in \mathbb{R}_{>0}\} \subseteq \mathcal{L}$. Then* $\operatorname{posi}(\mathcal{L}_{>0} \cup A) = \mathcal{L}_{>0} + \operatorname{posi}\{0,h\}$.

Proof. We first show that $\operatorname{posi}(\mathcal{L}_{>0} \cup A) \subseteq \mathcal{L}_{>0} + \operatorname{posi}\{0,h\}$. Consider any f in $\operatorname{posi}(\mathcal{L}_{>0} \cup A)$. By Lemma 1_{11} then $f \in \mathcal{L}_{>0} \cup \operatorname{posi}(A) \cup (\mathcal{L}_{>0} + \operatorname{posi}(A))$. If f belongs to $\mathcal{L}_{>0}$, then it belongs to $\mathcal{L}_{>0} + \operatorname{posi}\{0,h\}$ since $0 \in \operatorname{posi}\{0,h\}$. If f belongs to $\operatorname{posi}(A) \cup (\mathcal{L}_{>0} + \operatorname{posi}(A)) = \mathcal{L}_{\geq 0} + \operatorname{posi}(A)$ then $f \geq \sum_{k=1}^{n} \lambda_k h_k$, where n is an element of $\mathbb{N}$, $\lambda_1, \ldots, \lambda_n$ are elements of $\mathbb{R}_{>0}$ and $h_1, \ldots, h_n$ belong to A. For every k in $\{1,\ldots,m\}$ therefore $h_k = h + \varepsilon_k$ for some ε_k in $\mathbb{R}_{>0}$, so $f \geq \sum_{k=1}^{n} (\lambda_k h + \lambda_k \varepsilon_k) = (\sum_{k=1}^{n} \lambda_k h) + \sum_{k=1}^{n} \lambda_k \varepsilon_k$, and because $\sum_{k=1}^{n} \lambda_k \varepsilon_k > 0$, then $f > \sum_{k=1}^{n} \lambda_k h$. Therefore $f \in \mathcal{L}_{>0} + \operatorname{posi}\{h\}$, so in particular indeed $f \in \mathcal{L}_{>0} + \operatorname{posi}\{0,h\}$.

Conversely, to show that $\mathcal{L}_{>0} + \operatorname{posi}\{0,h\} \subseteq \operatorname{posi}(\mathcal{L}_{>0} \cup A)$, consider any f in $\mathcal{L}_{>0} + \operatorname{posi}\{0,h\}$. By Lemma 5_{12} then $f \in \mathcal{L}_{>0}$ or $\mu h < f$ for some μ in $\mathbb{R}_{>0}$. If $f \in \mathcal{L}_{>0}$ then $f \in \operatorname{posi}(\mathcal{L}_{>0} \cup A)$ which proves the desired statement, so assume that $\mu h < f$ for some μ in $\mathbb{R}_{>0}$. If $h = 0$, then $f \in \mathcal{L}_{>0}$ and the proof is done, so assume that $h \neq 0$. We therefore may assume that f belongs to the same quadrant as h—if this is not the case, then necessarily $f \in \mathcal{L}_{>0}$, for which we have just shown that then the desired statement holds. Note that then either (i) $f(\mathrm{H}) < 0 < f(\mathrm{T})$ and $h(\mathrm{H}) < 0 < h(\mathrm{T})$, or (ii) $f(\mathrm{H}) > 0 > f(\mathrm{T})$ and $h(\mathrm{H}) > 0 > h(\mathrm{T})$.

We will show that f belongs to $\operatorname{posi}(\mathcal{L}_{>0} \cup A)$. To do so, we will show that $f = \lambda\mu h + \varepsilon$ for some λ and ε in $\mathbb{R}_{>0}$, whence $f \in \operatorname{posi}\left\{h + \frac{\varepsilon}{\lambda\mu}\right\} \subseteq \operatorname{posi}(A) \subseteq \operatorname{posi}(\mathcal{L}_{>0} \cup A)$. We state that $\lambda := \frac{f(\mathrm{H}) - f(\mathrm{T})}{\mu(h(\mathrm{H}) - h(\mathrm{T}))}$ and $\varepsilon := \frac{f(\mathrm{T})h(\mathrm{H}) - f(\mathrm{H})h(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})}$ both belong to $\mathbb{R}_{>0}$ and satisfy $f = \lambda\mu h + \varepsilon$. Since f and h belong to the same quadrant, it is clear that λ belongs to $\mathbb{R}_{>0}$. To show that also ε belongs to $\mathbb{R}_{>0}$, infer from $f > \mu h$ that $f(\mathrm{H}) \geq \mu h(\mathrm{H})$ and $f(\mathrm{T}) \geq \mu h(\mathrm{T})$, where at least one of the inequalities is strict.

If (i) $f(\mathrm{H}) < 0 < f(\mathrm{T})$ and $h(\mathrm{H}) < 0 < h(\mathrm{T})$, then $|f(\mathrm{H})| \leq |\mu h(\mathrm{H})|$ and $f(\mathrm{T}) \geq \mu h(\mathrm{T})$, where at least one of the inequalities is strict, and every term involved is non-negative. Therefore $f(\mathrm{T})|\mu h(\mathrm{H})| > |f(\mathrm{H})|\mu h(\mathrm{T})$, whence $f(\mathrm{T})|h(\mathrm{H})| > |f(\mathrm{H})|h(\mathrm{T})$, so $f(\mathrm{T})h(\mathrm{H}) < f(\mathrm{H})h(\mathrm{T})$. Since $h(\mathrm{H}) < h(\mathrm{T})$, this implies that indeed $\varepsilon = \frac{f(\mathrm{T})h(\mathrm{H}) - f(\mathrm{H})h(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})}$ belongs to $\mathbb{R}_{>0}$. On the other hand, if (ii) $f(\mathrm{H}) > 0 > f(\mathrm{T})$ and $h(\mathrm{H}) > 0 > h(\mathrm{T})$, then $f(\mathrm{H}) \geq \mu h(\mathrm{H})$ and $|f(\mathrm{T})| \leq |\mu h(\mathrm{T})|$, where at least one of the inequalities is strict, and every term involved is non-negative. Therefore $|f(\mathrm{T})|\mu h(\mathrm{H}) < f(\mathrm{H})|\mu h(\mathrm{T})|$, whence $|f(\mathrm{T})|h(\mathrm{H}) < f(\mathrm{H})|h(\mathrm{T})|$, so $f(\mathrm{T})h(\mathrm{H}) > f(\mathrm{H})h(\mathrm{T})$. Since $h(\mathrm{H}) > h(\mathrm{T})$, this implies that indeed $\varepsilon = \frac{f(\mathrm{T})h(\mathrm{H}) - f(\mathrm{H})h(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})}$ belongs to $\mathbb{R}_{>0}$.

The proof is complete if we prove that $f = \lambda\mu h + \varepsilon$. To show this, note that indeed

$$\begin{aligned}
\lambda\mu h(\mathrm{H}) + \varepsilon &= \frac{f(\mathrm{H}) - f(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})} h(\mathrm{H}) + \frac{f(\mathrm{T})h(\mathrm{H}) - f(\mathrm{H})h(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})} \\
&= \frac{f(\mathrm{H})h(\mathrm{H}) - f(\mathrm{T})h(\mathrm{H}) + f(\mathrm{T})h(\mathrm{H}) - f(\mathrm{H})h(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})} \\
&= \frac{f(\mathrm{H})h(\mathrm{H}) - f(\mathrm{H})h(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})} = f(\mathrm{H}) \frac{h(\mathrm{H}) - h(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})} = f(\mathrm{H})
\end{aligned}$$

and

$$\begin{aligned}
\lambda\mu h(\mathrm{T}) + \varepsilon &= \frac{f(\mathrm{H}) - f(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})} h(\mathrm{T}) + \frac{f(\mathrm{T})h(\mathrm{H}) - f(\mathrm{H})h(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})} \\
&= \frac{f(\mathrm{H})h(\mathrm{T}) - f(\mathrm{T})h(\mathrm{T}) + f(\mathrm{T})h(\mathrm{H}) - f(\mathrm{H})h(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})} \\
&= \frac{f(\mathrm{T})h(\mathrm{H}) - f(\mathrm{T})h(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})} = f(\mathrm{T}) \frac{h(\mathrm{H}) - h(\mathrm{T})}{h(\mathrm{H}) - h(\mathrm{T})} = f(\mathrm{T}). \qquad \square
\end{aligned}$$

3.7 DISCUSSION

In this chapter, we have investigated the natural extension of choice functions, found an expression for it, and characterised the assessments that have coherent extensions. We made the connection with binary choice, and showed how the well-known natural extension for desirability follows from our natural extension.

One of the open problems is whether the condition in Corollary 88_{106} is also necessary for an assessment to avoid complete rejection. This is important, because it would imply that Proposition 87_{104} always obtains a non-trivial outer approximation of $R_{\mathcal{B}}$. Furthermore, it would imply the following useful property of any coherent rejection function R:[19]

$$(\exists D \in \overline{\mathbf{D}})(\forall A \in \mathcal{Q})(0 \in R(A) \Rightarrow A \cap D \neq \emptyset),$$

[19]If the converse statement of Corollary 88_{106} would hold, then we could derive this property

which I feel is a crucial property to find a representation of coherent choice in terms of desirability.

At this point, it is unknown whether this converse statement of Corollary 88_{106} holds. What is hopeful, is that some 'obvious' counterexamples fail. Indeed, taking Example 16_{109} in mind, we could focus our hope on the assessment $\mathcal{B} = \{\{0, f, \lambda f\}, \{0, -f\}\}$. Since $D \cap \{0, f, \lambda f\} = \emptyset$ or $D \cap \{0, -f\} = \emptyset$ for all D in $\overline{\mathbf{D}}$,[20] this might serve as a counterexample. The only thing we need to show for it to be a valid counterexample, is that $R_{\mathcal{B}}$ satisfies Axiom R1_{20}. Since $R_{\mathcal{B}}$ satisfies Axioms R3b_{20} and R4b_{20} [by Lemma 77_{92}], by Corollary 26_{39} it suffices to check whether or not $0 \in R_{\mathcal{B}}(\{0\})$, or equivalently, using Equation $(3.1)_{92}$, that there is some A' in $\mathcal{Q}_0$ such that

$$(\forall g \in A')\big(A' \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{g\} + \mu B \lessdot A'\big).$$

The option set $A' \coloneqq \{0, f, \lambda f\}$ satisfies this equation: for $g = 0$, take $B = \{0, f, \lambda f\}$ and $\mu = 1$, then $\{g\} + \mu B = \{0, f, \lambda f\} \lessdot \{0, f, \lambda f\} = A'$; for $g = f$, take $B = \{0, -f\}$ and $\mu = 1$, then $\{g\} + \mu B = \{f, 0\} \lessdot \{0, f, \lambda f\} = A'$; and finally, for $g = \lambda f$, take $B = \{0, -f\}$ and $\mu = \lambda$, then $\{g\} + \mu B = \{\lambda f, 0\} \lessdot \{0, f, \lambda f\} = A'$. So $R_{\mathcal{B}}$ does not avoid complete rejection, and therefore this assessment $\mathcal{B}$ cannot serve as a counterexample. Furthermore, this reasoning also rules assessments like $\mathcal{B}' \coloneqq \{\{0, f\}, \{0, -f\}\}$ out as valid counterexamples: since the assessment $\mathcal{B}'$ is at least as strong as $\mathcal{B}$, by the contraposition of Corollary 82_{98} also $\mathcal{B}'$ does not avoid complete rejection.

simply by considering the assessment $\mathcal{B}_R \coloneqq \{A \in \mathcal{Q} : 0 \in R(A)\} \subseteq \mathcal{Q}_0$. Since R is the smallest coherent extension of itself, Theorem 81_{97} then implies that $R_{\mathcal{B}_R} = R$, so $\mathcal{B}_R$ avoids complete rejection.

[20] To see this, if *ex absurdo* $D \cap \{0, f, \lambda f\} \neq \emptyset$ and $D \cap \{0, -f\} \neq \emptyset$ for some D in $\overline{\mathbf{D}}$, then $\{f, -f\} \subseteq D$ or $\{\lambda f, -f\} \subseteq D$. By Axioms D$3_{57}$ and D4_{57} therefore $0 \in D$, a contradiction with Axiom D1_{57}.

4

REPRESENTATION

In Chapter 2_9 we have seen that the coherent choice functions, or coherent rejection functions or coherent choice relations for that matter, form a belief structure (see Proposition 63_{70}). However, many of the imprecise probability models (see Section $2.8.6_{71}$) extant in the literature satisfy the additional property of being dually atomic (see Proposition 52_{59} for sets of desirable options), making them *strong* belief structures [23].

The relevance of strong belief structures is discussed at length by De Cooman in Reference [23]. They guarantee that every coherent model is the infimum of its dominating *maximal* models. Often, the maximal models are easy to work with, and have nice properties. For instance, as shown in Proposition 51_{59}, every maximal set of desirable options $\hat{D}$ has the very useful property that either u or $-u$ belongs to $\hat{D}$, for every u in $\mathcal{V} \setminus \{0\}$. This is of crucial importance for many results—like the result in Proposition 62_{70} showing that $C_{\hat{D}}$ is a maximal choice function whenever $\hat{D}$ is a maximal set of desirable options.

The coherent choice functions considered by Seidenfeld et al. [67]—whose rationality axioms and their relation with our notion of coherence we discussed in Section 2.4_{28}—do form a strong belief structure [67, Theorem 4]. Their maximal choice functions are exactly those that are representable by a probability–utility pair, and every coherent (in their notion) choice function is an infimum of such choice functions.

Up to now, for our notion of coherence, we have ignored the question of whether they can be represented in terms of maximal (or 'easy') choice functions. To answer this question, we need to take a closer look at two related questions. First, we need to know which are the maximal choice functions, and second, we need to find out whether every coherent choice function is indeed represented in terms of those maximal choice functions—is an infimum of its dominating maximal choice functions. With respect to the first question, in Proposition 62_{70} we have already found a subset of the maximal choice functions: we have shown that $\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\} \subseteq \hat{\mathbf{C}}$. Furthermore, its proof relies on the useful property that $C_{\hat{D}}$ identifies exactly one option to be chosen, from within any option set: $(\forall A \in \mathcal{Q})|C_{\hat{D}}(A)| = 1$. Ideally, we would like to have that these $C_{\hat{D}}$ constitute *all* the maximal choice functions, since this would guarantee that all of them satisfy this nice property.

However, in Example 16_{109}, we have essentially already shown that in general we cannot expect to have representation in terms of maximal choice functions that represent only binary choice:

Example 19. Consider the rejection function $R_{\mathcal{B}}$ that is the natural extension of the assessment $\mathcal{B} = \{0, f, \lambda f\}$ where $0 < \lambda \neq 1$. In Example 16_{109}, we have shown that $R_{\mathcal{B}}$ is no infimum of purely binary rejection functions. *A fortiori*, it therefore is no infimum of $R_{\hat{D}}$, with $\hat{D}$ in $\hat{\mathbf{D}}$. This shows that, if $\hat{\mathbf{R}} = \{R_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\}$, then not every coherent rejection function R is an infimum of its dominating rejection functions in $\hat{\mathbf{R}}$: indeed, consider for instance $R_{\mathcal{B}}$. Furthermore, it shows that, if every coherent rejection function is an infimum of its dominating rejection functions in $\hat{\mathbf{R}}$, then $\hat{\mathbf{R}} \supset \{R_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\}$: indeed, if $R_{\mathcal{B}} = \inf\{\hat{R} \in \hat{\mathbf{R}} : R_{\mathcal{B}} \sqsubseteq \hat{R}\}$, then, since $R_{\mathcal{B}}$ is no infimum of R_D, with D in $\mathcal{D}$, necessarily there are R in $\hat{\mathbf{R}}$ that are not purely binary. $\Diamond$

This clearly shows that coherence is not sufficient to obtain a representation for our choice functions in terms of maximal elements that represent binary choice. We will therefore need to add additional properties (or axioms) in order to try and guarantee such a representation. Since Seidenfeld et al.'s [67] choice functions do have this nice representation, we seek inspiration in their two additional axioms (Archimedeanity and the convexity Property C5_{25}). Because Archimedeanity is hard to reconcile with desirability (see Reference [86] and Section $2.8.7_{74}$), we will focus initially on their convexity axiom only.

In the first part of this chapter—Sections 4.1–4.3_{143}—, we investigate in detail the implications of Seidenfeld et al.'s [67] convexity axiom in our context. We will prove that, perhaps somewhat surprisingly, for purely binary choice functions, convexity is equivalent to being representable by means of a *lexicographic probability measure*. This is done by first establishing the implications of convexity in terms of the binary comparisons associated with a choice function, giving rise to what we will call *lexicographic sets of desirable gambles*. These sets include as particular cases the maximal (see Section $2.8.3_{58}$) and the strictly desirable (see Section $2.8.6_{71}$) sets of desirable gambles. Although in the particular case of binary possibility spaces these are the only two possibilities, for more general spaces lexicographic sets of gambles allow for a greater level of generality, as one would expect considering the above-mentioned equivalence.

A consequence of our equivalence result is that we can consider infima of choice functions associated with lexicographic probability measures, and in this manner generalise, or subsume as special cases, the examples of E-admissibility and M-admissibility discussed in Section 2.10_{81}. It will follow from the discussion that these infima also satisfy the convexity axiom. As one particularly relevant application of these ideas, we prove that the most conservative convex choice function associated with a binary preference relation can be obtained as the infimum of its dominating lexicographic choice functions.

In the second part of this chapter, we will prove the negative result that the coherent choice functions that satisfy the convexity property $\mathrm{C5}_{25}$ are *not* representable by $\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\}$, and not even by lexicographic choice functions. We will present a counterexample in the case of a binary possibility space.

4.1 PURELY BINARY CHOICE FUNCTIONS AND PROPERTY C5

We have seen in Example 3_{64} that not every purely binary choice function satisfies Property $\mathrm{C5}_{25}$: for the specific coherent set of desirable options D considered there, the corresponding choice function C_D fails to satisfy this property. However, there are other sets of desirable options D for which C_D does satisfy the convexity axiom. They are identified in the next proposition.

Proposition 91. *Consider any coherent set of desirable options D. Then the corresponding coherent choice function C_D satisfies Property* $\mathrm{C5}_{25}$ *if and only if D^c is a convex cone, or in other words, if and only if* $\operatorname{posi}(D^c) = D^c$*, or equivalently,* $\operatorname{posi}(D^c) \cap D = \varnothing$.

Proof. Proposition 54_{62} guarantees that C_D is a coherent choice function.

For necessity, assume that $\operatorname{posi}(D^c) \neq D^c$, or equivalently, that $\operatorname{posi}(D^c) \cap D \neq \varnothing$. Then there is some option u in D such that $u \in \operatorname{posi}(D^c)$, meaning that there are n in $\mathbb{N}$, $\lambda_1, \dots, \lambda_n$ in $\mathbb{R}_{>0}$ and $u_1, \dots, u_n$ in D^c such that $u = \sum_{k=1}^{n} \lambda_k u_k$. Let $A := \{0, u_1, \dots, u_n\}$ and $A_1 := A \cup \{u\}$. Due to the coherence of D [more precisely Axiom $\mathrm{D3}_{57}$], we can rescale $u \in D$ while keeping the u_k fixed, in such a way that we achieve that $\sum_{k=1}^{n} \lambda_k = 1$, whence $A \subseteq A_1 \subseteq \operatorname{conv}(A)$. We find that $0 \in C_D(A)$ by Proposition 55_{64}, because $A \cap D = \varnothing$, but $0 \notin C_D(A_1)$ because $u \in D$, so $A_1 \cap D \neq \varnothing$. This tells us that C_D does not satisfy Property $\mathrm{C5}_{25}$, because clearly $C_D(A) \nsubseteq C_D(A_1)$.

For sufficiency, assume that C_D does not satisfy Property $\mathrm{C5}_{25}$. Therefore, there are A and A_1 in $\mathcal{Q}$ such that $A \subseteq A_1 \subseteq \operatorname{conv}(A)$ and $C_D(A) \nsubseteq C_D(A_1)$, or, in other words, such that $u \in C_D(A)$ and $u \notin C_D(A_1)$ for some u in A. Consider such A and A_1 in $\mathcal{Q}$, and u in A. Due to Axiom $\mathrm{C4b}_{20}$, we find that $0 \in C_D(A - \{u\})$ and $0 \notin C_D(A_1 - \{u\})$, or equivalently, by Proposition 55_{64}, that $A - \{u\} \subseteq D^c$ and $A_1 - \{u\} \cap D \neq \varnothing$. But $A_1 - \{u\} \subseteq \operatorname{conv}(A) - \{u\} = \operatorname{conv}(A - \{u\}) \subseteq \operatorname{posi}(A - \{u\}) \subseteq \operatorname{posi}(D^c)$, so $\operatorname{posi}(D^c) \cap D \neq \varnothing$. □

This proposition seems to indicate that there is something special about coherent sets of desirable options whose complement is a convex cone too. We give them a special name that will be motivated and explained in the next section.

Definition 33. *A coherent set of desirable options D is called* lexicographic *if*

$$\operatorname{posi}(D^c) = D^c, \textit{ or, equivalently, if } \operatorname{posi}(D^c) \cap D = \varnothing.$$

We collect all the lexicographic coherent sets of desirable options in $\overline{\mathbf{D}}_{\mathrm{L}}$.

Any maximal coherent set of desirable options is also a lexicographic one:[1]

Proposition 92. *We have that $\hat{\mathbf{D}} \subseteq \overline{\mathbf{D}}_{\mathrm{L}}$.*

Proof. Consider any maximal set of desirable gambles D, and arbitrary n in $\mathbb{N}$, u_1, …, u_n in D^c and $\lambda_1, \dots, \lambda_n$ in $\mathbb{R}_{>0}$. Then since all $-u_k \in D \cup \{0\}$ by Proposition 51$_{59}$, we infer that $-\sum_{k=1}^n \lambda_k u_k \in D \cup \{0\}$, because the coherent D is in particular a convex cone. If $\sum_{k=1}^n \lambda_k u_k = 0$, then $\sum_{k=1}^n \lambda_k u_k \in D^c$ by Axiom D1$_{57}$. If $\sum_{k=1}^n \lambda_k u_k \neq 0$, then $-\sum_{k=1}^n \lambda_k u_k \in D$, and since coherence [more specifically, a combination of Axioms D1$_{57}$ and D4$_{57}$] implies that a coherent set of gambles cannot include both a gamble and its opposite, we conclude that, here too, $\sum_{k=1}^n \lambda_k u_k \in D^c$. Therefore, D^c is indeed a convex cone, so D belongs to $\overline{\mathbf{D}}_{\mathrm{L}}$. □

4.2 LEXICOGRAPHIC PROBABILITY SYSTEMS AND DESIRABILITY

In this section, we embark on a more detailed study of lexicographic sets of desirable options, and amongst other things, explain where their name comes from. We will restrict ourselves here to the special case where $\mathcal{V}$ is the linear space $\mathcal{L}(\mathcal{X})$ of all gambles on a *finite* possibility space $\mathcal{X}$, provided with the component-wise order $\leq$ as its vector ordering.

We first show that the lower expectation functional associated with a lexicographic D is actually a linear prevision (we refer to Section 2.8.6$_{71}$ and References [51, 72, 82] for more information about lower and linear previsions):

Proposition 93. *For any D in $\overline{\mathbf{D}}_{\mathrm{L}}$, the coherent lower prevision $\underline{P}_D$ on $\mathcal{L}(\mathcal{X})$ defined by*

$$\underline{P}_D(f) := \sup\{\mu \in \mathbb{R} : f - \mu \in D\} \textit{ for all } f \textit{ in } \mathcal{L}(\mathcal{X})$$

is a linear prevision.

Proof. Consider any f in $\mathcal{L}$ and ε in $\mathbb{R}_{>0}$, then we first prove that $f \in D$ or $\varepsilon - f \in D$. Assume *ex absurdo* that $f \notin D$ and $\varepsilon - f \notin D$. Then, because by assumption $\operatorname{posi}(D^c) = D^c$ is a convex cone, we also have that $f + \varepsilon - f = \varepsilon \notin D$, which contradicts Axiom D2$_{57}$. Now, Walley [82, Theorem 3.8.3] guarantees that for any such D, the corresponding functional $\underline{P}_D$ is indeed a linear prevision. □

[1] This can actually be obtained as a corollary to a result by Hammer [40, Theorem 2], by taking into account that maximal sets of desirable gambles are *semispaces* and that lexicographic sets of desirable gambles correspond to *hemispaces*. To make this thesis more self-contained, we give a proof using the coherence axioms of sets of desirable gambles we are employing in this paper.

4.2.1 *Binary possibility spaces*

To get some feeling for what these lexicographic models represent, we first look at the special case of binary possibility spaces $\{\mathrm{H},\mathrm{T}\}$, leading to a two-dimensional option space $\mathcal{V}=\mathcal{L}(\{\mathrm{H},\mathrm{T}\})$ provided with the point-wise order. It turns out that lexicographic sets of desirable options (gambles) are easy to characterise there, so we have a simple expression for $\overline{\mathbf{D}}_{\mathrm{L}}$.

Proposition 94. *All lexicographic coherent sets of desirable gambles on the binary possibility space* $\{\mathrm{H},\mathrm{T}\}$ *are given by (see also Figure 4.1*$_{\curvearrowright}$*):*

$$\overline{\mathbf{D}}_{\mathrm{L}} \coloneqq \{D_\rho, D_\rho^{\mathrm{H}}, D_\rho^{\mathrm{T}} : \rho \in (0,1)\} \cup \{D_0, D_1\} = \{D_\rho : \rho \in (0,1)\} \cup \hat{\mathbf{D}},$$

where

$$\begin{aligned}
D_\rho &\coloneqq \{\lambda(\rho - \mathbb{I}_{\{\mathrm{H}\}}) : \lambda \in \mathbb{R}\} + \mathcal{L}_{>0} = \operatorname{span}(\{\rho - \mathbb{I}_{\{\mathrm{H}\}}\}) + \mathcal{L}_{>0}\\
D_\rho^{\mathrm{H}} &\coloneqq D_\rho \cup \{\lambda(\rho - \mathbb{I}_{\{\mathrm{H}\}}) : \lambda \in \mathbb{R}_{<0}\} = D_\rho \cup \operatorname{posi}(\{\mathbb{I}_{\{\mathrm{H}\}} - \rho\})\\
D_\rho^{\mathrm{T}} &\coloneqq D_\rho \cup \{\lambda(\rho - \mathbb{I}_{\{\mathrm{H}\}}) : \lambda \in \mathbb{R}_{>0}\} = D_\rho \cup \operatorname{posi}(\{\rho - \mathbb{I}_{\{\mathrm{H}\}}\})\\
D_0 &\coloneqq \{f \in \mathcal{L} : f(\mathrm{T}) > 0\} \cup \mathcal{L}_{>0}\\
D_1 &\coloneqq \{f \in \mathcal{L} : f(\mathrm{H}) > 0\} \cup \mathcal{L}_{>0}
\end{aligned}$$

for all ρ *in* $(0,1)$.

Proof. We first observe that every set of desirable options in $\{D_\rho, D_\rho^{\mathrm{H}}, D_\rho^{\mathrm{T}} : \rho \in (0,1)\} \cup \{D_0, D_1\}$ is coherent. Indeed, for any ρ in $(0,1)$, D_ρ is the smallest coherent set of desirable gambles corresponding to the linear prevision E_p, with $p \coloneqq (\rho, 1-\rho)$, while D_ρ^{H} and D_ρ^{T} are maximal coherent sets of desirable gambles corresponding to the same linear prevision E_p. Finally, D_0 is the maximal (and only) coherent set of desirable gambles corresponding to E_p with $p \coloneqq (0,1)$, while D_1 is the maximal (and only) coherent set of desirable gambles corresponding to E_p with $p \coloneqq (1,0)$.

We now prove that we recover all lexicographic coherent sets of desirable gambles in this way. Consider any lexicographic coherent set of desirable gambles D. Then $\underline{P}_D$ is a linear prevision, by Proposition 93, so $\underline{P}_D$ is characterised (i) by the mass function $(1,0)$, (ii) by the mass function $(0,1)$, or (iii) by the mass function $(\rho, 1-\rho)$ for some ρ in $(0,1)$. If (i), the only coherent set of desirable gambles that induces the linear prevision with mass function $(1,0)$ is $D_1 \in \overline{\mathbf{D}}_{\mathrm{L}}$. If (ii), the only coherent set of desirable gambles that induces the linear prevision with mass function $(0,1)$ is $D_0 \in \overline{\mathbf{D}}_{\mathrm{L}}$. If (iii), there are only three coherent sets of desirable gambles that induce the linear prevision with mass function $(\rho, 1-\rho)$: D_ρ, D_ρ^{H} and D_ρ^{T}, and all are elements of $\overline{\mathbf{D}}_{\mathrm{L}}$. □

In the language of sets of desirable gambles (see for instance Section 2.8$_{55}$ or Reference [57]), this means that in the binary case lexicographic sets of desirable gambles are either *maximal* (see Section 2.8.3$_{58}$) or *strictly desirable* (see Section 2.8.6$_{71}$) with respect to a linear prevision.

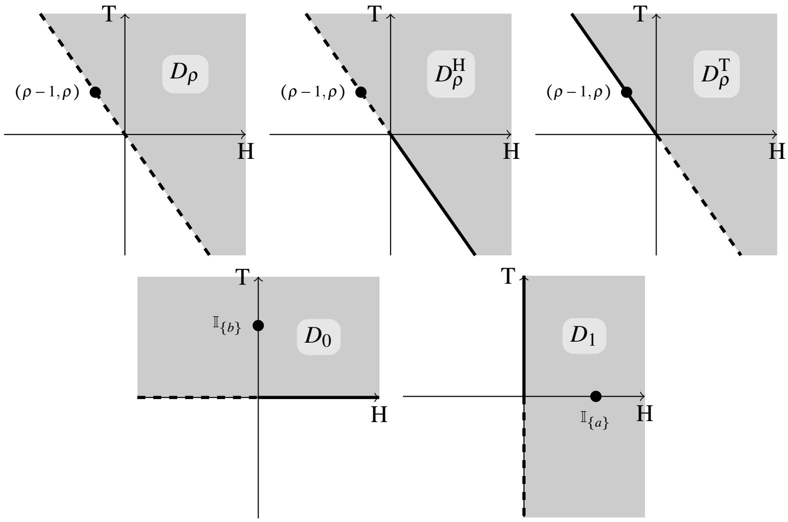


Figure 4.1: The lexicographic coherent sets of desirable gambles on the binary possibility space $\{\mathrm{H},\mathrm{T}\}$, with $\rho \in (0,1)$.

4.2.2 *Finite possibility spaces*

We now turn to the more general finite-dimensional case. We assume that the number of different outcomes—the cardinality of $\mathcal{X}$—is n in $\mathbb{N}$, throughout this section.

Recall that a *lexicographic order* $<_{\mathrm{L}}$ on a vector space $\mathcal{V}$ of finite dimension ℓ is defined by

$$u <_{\mathrm{L}} v \Leftrightarrow (\exists k \in \{1,\dots,\ell\})(u_k < v_k \text{ and } (\forall j \in \{1,\dots,k-1\})u_j = v_j),$$

and denote, as usual, its reflexive version $\leq_{\mathrm{L}}$ as $u \leq_{\mathrm{L}} v \Leftrightarrow (u <_{\mathrm{L}} v \text{ or } u = v)$ for any two vectors $u = (u_1,\dots,u_\ell)$ and $v = (v_1,\dots,v_\ell)$ in $\mathcal{V}$.

Definition 34 (Lexicographic probability system)**.** *A* lexicographic probability system *is an ℓ-tuple $p \coloneqq (p_1,\dots,p_\ell)$ of probability mass functions on a possibility space $\mathcal{X}$. We associate with this tuple p an expectation operator $E_p \coloneqq (E_{p_1},\dots,E_{p_\ell})$, and a (strict) preference relation $\prec_p$ on $\mathcal{L}(\mathcal{X})$, defined by:*

$$f \prec_p g \Leftrightarrow E_p(f) <_{\mathrm{L}} E_p(g), \text{ for all } f \text{ and } g \text{ in } \mathcal{L}(\mathcal{X}), \tag{4.1}$$

where, for every h in $\mathcal{L}(\mathcal{X})$, we let $E_p(h) \coloneqq (E_{p_1}(h),\dots,E_{p_\ell}(h))$, an element of an ℓ-dimensional vector space. We call ℓ the number of layers *of the lexicographic probability system.*

We refer to work by Blume et al. [7], Fishburn [37] and Seidenfeld et al. [64] for more details on generic lexicographic probability systems. The connection between lexicographic probability systems and sets of desirable gambles has also been studied by Cozman [16] and Benavoli et al. [6], and the connection with full conditional measures by Halpern [39] and Hammond [41]. Below, we first recall a number of relevant basic properties of lexicographic orders in Propositions 95 and $97_{\curvearrowright}$. We then provide a characterisation of lexicographic sets of desirable gambles in terms of lexicographic orders in Theorem 101_{135}.

Remark that the reflexive version[2] $\preceq_p$ of $\prec_p$ defined by $f \preceq_p g \Leftrightarrow E_p(f) \leq_{\mathrm{L}} E_p(g)$ for all f and g in $\mathcal{L}$ is a total order on $\mathcal{L}$ (see Reference [7]).

An important feature of preference relations $\prec_p$ based on lexicographic probability systems is the *incomparability relation* $\parallel_p$, defined by: $f \parallel_p g$ if and only if $f \nprec_p g$ and $g \nprec_p f$, for all f and g in $\mathcal{L}$. Since $\preceq_p$ is a total order, it follows that

$$f \parallel_p g \Leftrightarrow E_p(f) = E_p(g) \Leftrightarrow (\forall k \in \{1,\ldots,\ell\}) E_{p_k}(f) = E_{p_k}(g), \qquad (4.2)$$

for all f and g in $\mathcal{L}$. Finally, it also follows that

$$f \nprec_p g \Leftrightarrow g \prec_p f \text{ or } g \parallel_p f \Leftrightarrow E_p(g) \leq_{\mathrm{L}} E_p(f), \text{ for all } f \text{ and } g \text{ in } \mathcal{L}. \qquad (4.3)$$

Proposition 95. *Consider any lexicographic probability system p with ℓ layers. Then $\prec_p$ is a* strict weak order*, meaning that $\prec_p$ is irreflexive, and both $\prec_p$ and $\parallel_p$ are transitive. As a consequence, the binary relation $\nprec_p$ is transitive as well.*

Proof. This is a consequence of Equations (4.1), (4.2) and (4.3), taking into account that $<_{\mathrm{L}}$ and $\leq_{\mathrm{L}}$ are transitive, and that $<_{\mathrm{L}}$ is irreflexive. □

In what follows, we will restrict our attention to lexicographic probability systems p that satisfy the following condition:[3]

$$(\forall x \in \mathcal{X})(\exists k \in \{1,\ldots,\ell\}) p_k(x) > 0. \qquad (4.4)$$

This condition requires that there should be no possible outcome in $\mathcal{X}$ that has zero probability in every layer. It is closely related to the notion of a *Savage-null event* [60, Section 2.7]:

Definition 35 (Savage-null events)**.** *An event $E \subseteq \mathcal{X}$ is called* Savage-null *if $(\forall f, g \in \mathcal{L}) \mathbb{I}_E f \preceq_p \mathbb{I}_E g$. The event $\varnothing$ is always Savage-null, and is called the* trivial Savage-null event.

[2] This is not the usual way of deriving an irreflexive relation from a reflexive one: usually, we define $f \preceq_p g \Leftrightarrow (f \prec_p g \text{ or } f = g)$ for all f and g in $\mathcal{L}$.

[3] This condition is weaker than the requirement in Reference [6] that the stochastic matrix identified with p has full rank.

Proposition 96. *Consider any lexicographic probability system p with ℓ layers. Then Condition* (4.4)↶ *holds if and only if there are no non-trivial Savage-null events.*

Proof. For the direct implication, assume that p satisfies Condition (4.4)↶, and consider any non-empty event $E \subseteq \mathcal{X}$. Consider any x in E, then $\mathbb{I}_E \geq \mathbb{I}_{\{x\}}$ so $E_{p_k}(\mathbb{I}_E) \geq E_{p_k}(\mathbb{I}_{\{x\}})$ for every $k \in \{1,\dots,\ell\}$. Also, $E_p(0) <_{\mathrm{L}} E_p(\mathbb{I}_{\{x\}})$ by Condition (4.4)↶, so $0\mathbb{I}_E \prec_p 1\mathbb{I}_E$ whence $1\mathbb{I}_E \not\preceq_p 0\mathbb{I}_E$ and hence, by Definition 35↶, E is indeed no Savage-null event.

For the converse implication, assume that p fails Condition (4.4)↶. Then there is some x^* in $\mathcal{X}$ such that $p_k(x^*) = 0$ for all k in $\{1,\dots,\ell\}$, and therefore $E_{p_k}(f\mathbb{I}_{\{x^*\}}) = 0 = E_{p_k}(g\mathbb{I}_{\{x^*\}})$ for all f and g in $\mathcal{L}$ and k in $\{1,\dots,\ell\}$, so $E_p(f\mathbb{I}_{\{x^*\}}) = E_p(g\mathbb{I}_{\{x^*\}})$ for all f and g in $\mathcal{L}$. This implies that $f\mathbb{I}_{\{x^*\}} \preceq_p g\mathbb{I}_{\{x^*\}}$ for all f and g in $\mathcal{L}$, so indeed there is a non-trivial Savage-null event $\{x^*\}$. □

We now link the lexicographic ordering $\prec_p$ with the preference relation $\lessdot_D$ based a set of desirable gambles D, as defined in Section 2.8↶55. We begin with an auxiliary result, showing that $\prec_p$ also—just like $\lessdot_D$—is a strict vector ordering compatible with the natural order $<$ on gambles.

Proposition 97. *Consider any lexicographic probability system p with ℓ layers. Then $\prec_p$ is a (strict) vector order compatible with $<$: it is irreflexive, transitive and*

(i) $f \prec_p g \Leftrightarrow f+h \prec_p g+h \Leftrightarrow \lambda f \prec_p \lambda g$;

(ii) *if there are no non-trivial Savage-null events, then $f < g \Rightarrow f \prec_p g$,*

for all f, g and h in $\mathcal{L}$ and λ in $\mathbb{R}_{>0}$.

Proof. It is clear from Proposition 95↶ that $\prec_p$ is irreflexive and transitive. Let us prove the remaining statements.

(i) This follows from the definition of $\prec_p$ and the linearity of expectation operators.

(ii) Assume that there are no non-trivial Savage-null events. Use Proposition 96 to infer that then Condition (4.4)↶ holds. Consider any f in $\mathcal{L}$ such that $0 < f$. Then $0 \leq f$—so $0 \leq E_{p_k}(f)$ for every k in $\{1,\dots,\ell\}$—and $0 < f(x^*)$ for some x^* in $\mathcal{X}$. Then $p_k(x^*) > 0$ for some k in $\{1,\dots,\ell\}$ by Condition (4.4)↶, so $0 <_{\mathrm{L}} E_p(\mathbb{I}_{\{x^*\}})$. Use $f(x^*)\mathbb{I}_{\{x^*\}} \leq f$ to infer that then also $0 <_{\mathrm{L}} E_p(f)$, whence indeed $0 \prec_p f$. Since we just have showed that (i) holds, this immediately implies the desired result. □

We will establish a link between lexicographic probability systems and preference relations associated with lexicographic sets of desirable gambles. We refer to papers by Cozman [16, Section 2.1] and Seidenfeld et al. [64] for other relevant discussion on the connection between lexicographic probabilities and partial preference relations. Our proof is somewhat reminiscent of the representation of conditional probabilities by Krauss [46], and will make repeated use of the following separation theorem [43], in the form stated in Reference [82, Appendix E1]:

Theorem 98 (Separating hyperplane theorem)**.** *Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two convex subsets of a finite-dimensional linear topological space $\mathcal{B}$. If $0 \in \mathcal{W}_1 \cap \mathcal{W}_2$ and $\operatorname{int}(\mathcal{W}_1) \cap \mathcal{W}_2 = \varnothing$, then there is a non-zero continuous linear functional Λ on $\mathcal{B}$ such that*

$$\Lambda(w) \geq 0 \textit{ for all } w \textit{ in } \mathcal{W}_1 \textit{ and } \Lambda(w') \leq 0 \textit{ for all } w' \textit{ in } \mathcal{W}_2.$$

If $\mathcal{W}_1$ and $\mathcal{W}_2$ are finite, $\mathcal{W}_1$ non-empty, and $\sum_{i=1}^{m} \lambda_i w_i - \sum_{k=1}^{n} \mu_k w'_k \neq 0$ for all m and n in $\mathbb{N}$, all $\lambda_1, \ldots, \lambda_m$ in $\mathbb{R}_{\geq 0}$ with $\lambda_i > 0$ for at least one i in $\{1, \ldots, m\}$, all $\mu_1, \ldots, \mu_n$ in $\mathbb{R}_{\geq 0}$, all $w_1, \ldots, w_m$ in $\mathcal{W}_1$, and all $w'_1, \ldots, w'_n$ in $\mathcal{W}_2$, then there is a non-zero continuous linear functional Λ on $\mathcal{B}$ such that

$$\Lambda(w) > 0 \textit{ for all } w \textit{ in } \mathcal{W}_1 \textit{ and } \Lambda(w') \leq 0 \textit{ for all } w' \textit{ in } \mathcal{W}_2.$$

Two clarifications here are (i) that we will apply the theorem to linear subspaces of $\mathcal{L}$, which is a linear topological space [82, Appendix D] that is finite-dimensional because $\mathcal{X}$ is finite, and (ii) that when the linear topological space is finite-dimensional, the assumption $\operatorname{int}(\mathcal{W}_1) \neq \varnothing$ that is mentioned in Reference [82, Appendix E1] is not necessary for the separating hyperplane theorem to hold, as shown in Reference [43, Theorem 4B].

Our proof will also make use of the following two lemmas.

Lemma 99. *Consider any coherent set D of desirable gambles on a finite possibility space $\mathcal{X}$, and consider any linear subspace $\Lambda \subseteq \mathcal{L}$. Then $\operatorname{int}(\operatorname{cl}(D \cap \Lambda)) \cap D^c = \varnothing$, where* int *is the topological interior and* cl *the topological closure.*

Proof. We first prove $\operatorname{int}(\operatorname{cl}(D)) \cap D^c = \varnothing$. To show this, we will use the fact that D, and therefore also $\operatorname{cl}(D)$, is a convex set. Since the interior of a convex set is always included in the relative interior ri of that convex set (see Reference [10, Section 1.3]), we find that $\operatorname{int}(\operatorname{cl}(D)) \subseteq \operatorname{ri}(\operatorname{cl}(D))$. A well-known result [10, Theorem 3.4(d)] states that $\operatorname{ri}(\operatorname{cl}(C)) = \operatorname{ri}(C)$ for any convex set C in a finite-dimensional vector space, whence $\operatorname{int}(\operatorname{cl}(D)) \subseteq \operatorname{ri}(D)$. But $\operatorname{ri}(D)$ is a subset of D, so $\operatorname{int}(\operatorname{cl}(D)) \subseteq D$, and hence indeed $\operatorname{int}(\operatorname{cl}(D)) \cap D^c = \varnothing$.

Now consider $D \cap \Lambda$, a subset of D. Since both cl and int respect set inclusion, we find that $\operatorname{int}(\operatorname{cl}(D \cap \Lambda)) \subseteq \operatorname{int}(\operatorname{cl}(D)) \subseteq D$, whence indeed $\operatorname{int}(\operatorname{cl}(D \cap \Lambda)) \cap D^c = \varnothing$. $\square$

Lemma 100. *Consider any non-zero real linear functional Λ_1 on the n-dimensional real vector space $\mathcal{L}$, and any sequence of non-zero real linear functionals Λ_k defined on the $n-k+1$-dimensional real vector space $\ker \Lambda_{k-1} := \{f \in \mathcal{L} : \Lambda_{k-1}(f) = 0\}$ for all k in $\{2, \ldots, \ell\}$, where $\ell \in \{2, \ldots, n\}$. Assume that all Λ_k are positive in the sense that $(\forall f \in \mathcal{L}_{\geq 0} \cap \operatorname{dom} \Lambda_k)(\Lambda_k(f) \geq 0)$,[4] for all $k \in \{1, \ldots, \ell\}$. Then for each k in $\{2, \ldots, \ell\}$ the real linear functional Λ_k on $\ker \Lambda_{k-1}$ can be extended to a real linear functional Γ_k on $\mathcal{L}$ with the following properties:*

[4] We let $\operatorname{dom} \Lambda_k$ be the domain of the functional Λ_k.

(i) *for all* f *in* $\mathcal{L}_{\geq 0}$*:* $\Gamma_k(f) \geq 0$*;*
(ii) $\Gamma_k(1) > 0$*;*
(iii) $\ker\Gamma_k \cap \ker\Lambda_{k-1} = \ker\Lambda_k$*;*
(iv) *for all* f *in* $\ker\Lambda_{k-1}$*:* $\Gamma_k(f) > 0 \Leftrightarrow \Lambda_k(f) > 0$.

Proof. Consider any k in $\{2,\dots,\ell\}$. Since the real functional Λ_k on the $n-k+1$-dimensional real vector space $\ker\Lambda_{k-1}$ is non-zero, there is some h_k in $\ker\Lambda_{k-1}$ such that $\Lambda_k(h_k) > 0$. We will consider the quotient space $\mathcal{V}/I$, a k-dimensional vector space whose elements $[f] = \{f\} + \ker\Lambda_k$ are the affine subspaces through f, parallel to the subspace $\ker\Lambda_k$, for every f in $\mathcal{L}(\mathcal{X})$. We first show that it follows from Theorem 98 that there is a non-zero linear functional $\tilde{\Gamma}_k$ on $\mathcal{V}/I$ such that

$$\tilde{\Gamma}_k(u) \leq 0 \text{ for all } u \text{ in } \mathcal{W}_k^2 := \left\{[-\mathbb{I}_{\{x\}}] : x \in \mathcal{X}_k\right\}, \text{ and}$$
$$\tilde{\Gamma}_k(u) > 0 \text{ for all } u \text{ in } \mathcal{W}_k^1 := \{[h_k]\} \cup \left\{[\mathbb{I}_{\{x\}}] : x \in \mathcal{X}_k\right\}, \quad (4.5)$$

where we let $\mathcal{X}_k := \{x \in \mathcal{X} : \mathbb{I}_{\{x\}} \notin \ker\Lambda_k\} \subseteq \mathcal{X}$. The set $\mathcal{X}_k$ is non-empty: since $\ker\Lambda_k$ is $n-k$-dimensional, at most $n-k$ of the linearly independent indicators $\mathbb{I}_{\{x\}}$, $x \in \mathcal{X}$ may lie in $\ker\Lambda_k$, so $|\mathcal{X}_k| \geq k$. To show that we can apply Theorem 98, we prove that the condition for it is satisfied: $\sum_{i=1}^n \lambda_i w_i^1 - \sum_{k=1}^m \mu_k w_k^2 \neq 0$ for all m and n in $\mathbb{N}$, all $\lambda_1, \dots, \lambda_m$ in $\mathbb{R}_{\geq 0}$ with $\lambda_i > 0$ for at least one i in $\{1,\dots,m\}$, all $\mu_1, \dots, \mu_n$ in $\mathbb{R}_{\geq 0}$, all $w_1^1, \dots, w_n^1$ in $\mathcal{W}_k^1$, and all $w_1^2, \dots, w_m^2$ in $\mathcal{W}_k^2$. Since $\mathcal{W}_k^1$ and $\mathcal{W}_k^2$ are finite, it is not difficult to see that it suffices to consider $\sum_{i=1}^n \lambda_i w_i^1 = \lambda[h_k] + \sum_{x\in\mathcal{X}_k} \lambda_x[\mathbb{I}_{\{x\}}]$ and $\sum_{j=1}^m \mu_j w_j^2 = -\sum_{x\in\mathcal{X}_k} \mu_x[\mathbb{I}_{\{x\}}]$. So assume *ex absurdo* that $\lambda[h_k] + \sum_{x\in\mathcal{X}_k}(\lambda_x + \mu_x)[\mathbb{I}_{\{x\}}] = 0$, or equivalently, that $\lambda h_k + \sum_{x\in\mathcal{X}_k}(\lambda_x + \mu_x)\mathbb{I}_{\{x\}} \in \ker\Lambda_k$ for some $\mu_x \geq 0$, $\lambda_x \geq 0$ and $\lambda \geq 0$ for all x in $\mathcal{X}_k$, where λ or at least one of $\{\lambda_x : x \in \mathcal{X}_k\}$ are positive. Let $\mathcal{X}_k' := \{x \in \mathcal{X}_k : \lambda_x + \mu_x > 0\}$ and $g := \sum_{x\in\mathcal{X}_k'}(\lambda_x + \mu_x)\mathbb{I}_{\{x\}}$, then we know that $\lambda h_k + g \in \ker\Lambda_k$.

There are now a number of possibilities. The first is that $\lambda = 0$, whence $\mathcal{X}_k' \neq \emptyset$ and therefore $g \in \ker\Lambda_k \subseteq \cdots \subseteq \ker\Lambda_1$. This implies that $0 = \Lambda_1(g) = \sum_{x\in\mathcal{X}_k'}(\lambda_x + \mu_x)\Lambda_1(\mathbb{I}_{\{x\}})$. Since $\mathbb{I}_{\{x\}} > 0$ and Λ_1 is positive, we find that $\mathbb{I}_{\{x\}} \in \ker\Lambda_1 = \operatorname{dom}\Lambda_2$ for all x in $\mathcal{X}_k'$. This in turn allows us to conclude that $0 = \Lambda_2(g) = \sum_{x\in\mathcal{X}_k'}(\lambda_x + \mu_x)\Lambda_2(\mathbb{I}_{\{x\}})$. Since $\mathbb{I}_{\{x\}} > 0$ and Λ_2 is positive, we find that $\mathbb{I}_{\{x\}} \in \ker\Lambda_2 = \operatorname{dom}\Lambda_3$ for all x in $\mathcal{X}_k'$. We can go on in this way until we eventually conclude that $0 = \Lambda_k(g) = \sum_{x\in\mathcal{X}_k'}(\lambda_x + \mu_x)\Lambda_k(\mathbb{I}_{\{x\}})$. Since $\mathbb{I}_{\{x\}} > 0$ and Λ_k is positive, we find that $\mathbb{I}_{\{x\}} \in \ker\Lambda_k$ for all x in $\mathcal{X}_k'$, a contradiction.

The second possibility is that $\lambda > 0$. If now $\mathcal{X}_k' = \emptyset$, we find that $\lambda h_k \in \ker\Lambda_k$, whence $\lambda\Lambda_k(h_k) = 0$, a contradiction. If $\mathcal{X}_k' \neq \emptyset$, we find that $\lambda h_k + g \in \ker\Lambda_k \subseteq \cdots \subseteq \ker\Lambda_1$. Since $h_k \in \ker\Lambda_{k-1} \subseteq \cdots \subseteq \ker\Lambda_1$, this implies that $g \in \ker\Lambda_{k-1} \subseteq \cdots \subseteq \ker\Lambda_1$ too. This implies that $0 = \Lambda_1(g) = \sum_{x\in\mathcal{X}_k'}(\lambda_x + \mu_x)\Lambda_1(\mathbb{I}_{\{x\}})$. Since $\mathbb{I}_{\{x\}} > 0$ and Λ_1 is positive, we find that $\mathbb{I}_{\{x\}} \in \ker\Lambda_1 = \operatorname{dom}\Lambda_2$ for all x in $\mathcal{X}_k'$. This in turn allows us to conclude that $0 = \Lambda_2(g) = \sum_{x\in\mathcal{X}_k'}(\lambda_x + \mu_x)\Lambda_2(\mathbb{I}_{\{x\}})$. Since $\mathbb{I}_{\{x\}} > 0$ and Λ_2 is positive, we find that $\mathbb{I}_{\{x\}} \in \ker\Lambda_2 = \operatorname{dom}\Lambda_3$ for all x in $\mathcal{X}_k'$. We can go on in this way until we eventually conclude that $0 = \Lambda_{k-1}(g) = \sum_{x\in\mathcal{X}_k'}(\lambda_x + \mu_x)\Lambda_{k-1}(\mathbb{I}_{\{x\}})$. Since $\mathbb{I}_{\{x\}} > 0$ and Λ_{k-1} is positive, we find that $\mathbb{I}_{\{x\}} \in \ker\Lambda_{k-1} = \operatorname{dom}\Lambda_k$ for all x in $\mathcal{X}_k'$. This now allows us to rewrite $\lambda h_k + g \in \ker\Lambda_k$ as $0 = \Lambda_k(\lambda h_k + g) = \lambda\Lambda_k(h_k) + \sum_{x\in\mathcal{X}_k'}(\lambda_x + \mu_x)\Lambda_k(\mathbb{I}_{\{x\}})$. Since $\mathbb{I}_{\{x\}} > 0$ and Λ_k is positive, this implies that $\lambda\Lambda_k(h_k) \leq 0$, a contradiction. We conclude that, indeed, there is a non-zero linear functional $\tilde{\Gamma}_k$ on $\mathcal{V}/I$ that satisfies Equation (4.5).

We now define the new real linear functional Γ_k on $\mathcal{L}(\mathcal{X})$ by letting

$$\Gamma_k(f) := \tilde{\Gamma}_k([f]) \text{ for all } f \text{ in } \mathcal{L}(\mathcal{X}).$$

Observe that, since $f = \sum_{x\in\mathcal{X}} f(x)\mathbb{I}_{\{x\}}$, this leads to

$$\Gamma_k(f) = \sum_{x\in\mathcal{X}} f(x)\tilde{\Gamma}_k\big([\mathbb{I}_{\{x\}}]\big) = \sum_{x\in\mathcal{X}_k} f(x)\tilde{\Gamma}_k\big([\mathbb{I}_{\{x\}}]\big),$$

where the second equality follows from $\mathbb{I}_{\{x\}} \in \ker\Lambda_k$, and therefore $[\mathbb{I}_{\{x\}}] = 0$, for all $x \in \mathcal{X} \setminus \mathcal{X}_k$. If we also take into account Equation (4.5), this proves in particular that (i) and (ii) hold.

For the rest of the proof, consider any f in $\ker\Lambda_{k-1}$ and $\lambda := \frac{\Lambda_k(f)}{\Lambda_k(h_k)}$, a well-defined real number because $\Lambda_k(h_k) > 0$. Then $0 = \Lambda_k(f) - \lambda\Lambda_k(h_k) = \Lambda_k(f - \lambda h_k)$, so $f - \lambda h_k \in \ker\Lambda_k$. As a result, $[f] = [\lambda h_k]$ and therefore $\Gamma_k(f) = \tilde{\Gamma}_k([f]) = \tilde{\Gamma}_k([\lambda h_k]) = \lambda\tilde{\Gamma}_k([h_k])$. Substituting back for λ, we get the equality:

$$\Gamma_k(f)\Lambda_k(h_k) = \tilde{\Gamma}_k([h_k])\Lambda_k(f).$$

Since both $\Lambda_k(h_k) > 0$ and $\tilde{\Gamma}_k([h_k]) > 0$ [by Equation (4.5)], we see that $\Gamma_k(f)$ and $\Lambda_k(f)$ are either both zero, both (strictly) positive, or both (strictly) negative. This proves (iii) and (iv). □

Theorem 101. *Consider any lexicographic probability system $p = (p_1,\dots,p_\ell)$ that has no non-trivial Savage-null events. Then the set of desirable gambles $D_p := \{f \in \mathcal{L} : 0 \prec_p f\}$ corresponding with the preference relation $\prec_p$, is an element of $\overline{\mathbf{D}}_{\mathrm{L}}$—a coherent and lexicographic set of desirable gambles. Conversely, consider any lexicographic set of desirable gambles D in $\overline{\mathbf{D}}_{\mathrm{L}}$. Then its corresponding preference relation $\prec_D$ is a preference relation based on some lexicographic probability system $p = (p_1,\dots,p_\ell)$ that has no non-trivial Savage-null events.*

Proof. We begin with the first statement. We first show that D_p is coherent. For Axiom D1$_{57}$, infer from $0 \not\prec_p 0$ by the irreflexivity of $\prec_p$ [see Proposition 95$_{131}$] that indeed $0 \notin D_p$. For Axiom D2$_{57}$, consider any f in $\mathcal{L}_{>0}$. Use Proposition 97$_{132}$ to infer that $0 \prec_p f$, whence indeed $f \in D_p$. For Axiom D3$_{57}$, consider any f in D_p and λ in $\mathbb{R}_{>0}$. Then $0 \prec_p f$, and hence $0 \prec_p \lambda f$ using Proposition 97$_{132}$. Therefore indeed $\lambda f \in D_p$. For Axiom D4$_{57}$, consider any f and g in D_p, whence $0 \prec_p f$ and $0 \prec_p g$. From $0 \prec_p g$ infer that $f \prec_p f + g$ by Proposition 97$_{132}$, and using $0 \prec_p f$, that $0 \prec_p f + g$ by the transitivity of $\prec_p$ [see Proposition 95$_{131}$]. Therefore indeed $f + g \in D_p$.

So it only remains to show that $\mathrm{posi}(D_p^c) = D_p^c$. Consider any f and g in D_p^c and any λ_1 and λ_2 in $\mathbb{R}_{>0}$, then we must prove that $\lambda_1 f + \lambda_2 g \in D_p^c$. Since by assumption $0 \not\prec_p f$ and $0 \not\prec_p g$, Equation (4.3)$_{131}$ guarantees that

$$E_p(f) \leq_{\mathrm{L}} E_p(0) = 0 \text{ and } E_p(g) \leq_{\mathrm{L}} E_p(0).$$

By the linearity of the expectation operators,

$$E_p(\lambda_1 f + \lambda_2 g) \leq_{\mathrm{L}} E_p(0) = 0,$$

whence $0 \not\prec_p \lambda_1 f + \lambda_2 g$. Therefore indeed $\lambda_1 f + \lambda_2 g \in D_p^c$.

For the second statement, we consider any D in $\overline{\mathbf{D}}_{\mathrm{L}}$, and we construct a lexicographic probability system p with no non-trivial Savage-null events and such that $\prec_p$ equals $\prec_D$. Define the real functional Λ_1 on $\mathcal{L}$ by letting $\Lambda_1(f) := \sup\{\alpha \in \mathbb{R} : f - \alpha \in D\}$ for all f in $\mathcal{L}$. Proposition 93_{128} guarantees that Λ_1 is a linear functional. Its kernel $\ker\Lambda_1$ is an $n-1$-dimensional linear space,[5] where, as usual in this section, n is the finite dimension of the real vector space $\mathcal{L}$—the cardinality of $\mathcal{X}$. Since both D^c and $\ker\Lambda_1$ are convex cones, so is their intersection $D^c \cap \ker\Lambda_1$, and it contains 0 because $0 \in D^c$ and $0 \in \ker\Lambda_1$. Using similar arguments, we see that $D \cap \ker\Lambda_1$ is either a convex cone or empty. When $D \cap \ker\Lambda_1 = \emptyset$, let $\ell := 1$, and stop. When $D \cap \ker\Lambda_1 \neq \emptyset$, it follows from Theorem 98_{133} that there is some non-zero (continuous) linear functional Λ_2 on $\ker\Lambda_1$ such that

$$\Lambda_2(f) \leq 0 \text{ for all } f \text{ in } D^c \cap \ker\Lambda_1 \text{ and } \Lambda_2(f) \geq 0 \text{ for all } f \text{ in } D \cap \ker\Lambda_1.$$

[Apply Theorem 98_{133} with $\mathcal{B} = \ker\Lambda_1$, $\mathcal{W}_2 = D^c \cap \ker\Lambda_1$ and $\mathcal{W}_1 = \mathrm{cl}(D \cap \ker\Lambda_1)$ (the topological closure of $D \cap \ker\Lambda_1$ in $\ker\Lambda_1$); then $\mathrm{int}(\mathcal{W}_1) \cap \mathcal{W}_2 = \emptyset$ by Lemma 99_{133}, and $0 \in \mathcal{W}_1 \cap \mathcal{W}_2$.] $\ker\Lambda_2$ is a $n-2$-dimensional linear space. Also, $D \cap \ker\Lambda_2$ is either empty or a non-empty convex cone. If it is empty, let $\ell := 2$; otherwise, we repeat the same procedure again: it follows from Theorem 98_{133} that there is some non-zero (continuous) linear functional Λ_3 on $\ker\Lambda_2$ such that

$$\Lambda_3(f) \leq 0 \text{ for all } f \text{ in } D^c \cap \ker\Lambda_2 \text{ and } \Lambda_3(f) \geq 0 \text{ for all } f \text{ in } D \cap \ker\Lambda_2.$$

[Apply Theorem 98_{133} with $\mathcal{B} = \ker\Lambda_2$, $\mathcal{W}_2 = D^c \cap \ker\Lambda_2$ and $\mathcal{W}_1 = \mathrm{cl}(D \cap \ker\Lambda_2)$ (the topological closure of $D \cap \ker\Lambda_2$ in $\ker\Lambda_2$); then $\mathrm{int}(\mathcal{W}_1) \cap \mathcal{W}_2 = \emptyset$ by Lemma 99_{133}, and $0 \in \mathcal{W}_1 \cap \mathcal{W}_2$.] $\ker\Lambda_3$ is a $n-3$-dimensional linear space. Also, $D \cap \ker\Lambda_3$ is either empty or a non-empty convex cone. If it is empty, let $\ell := 3$; if not, continue in the same vein. This leads to successive linear functionals Λ_k defined on the $n-k+1$-dimenional linear spaces $\ker\Lambda_{k-1}$ such that

$$\Lambda_k(f) \leq 0 \text{ for all } f \text{ in } D^c \cap \ker\Lambda_{k-1} \text{ and } \Lambda_k(f) \geq 0 \text{ for all } f \text{ in } D \cap \ker\Lambda_{k-1}. \tag{4.6}$$

This sequence stops as soon as $D \cap \ker\Lambda_k = \emptyset$, and we then let $\ell := k$. Because the finite dimensions of the successive $\ker\Lambda_k$ decrease with 1 at each step, we are guaranteed to stop after at most n repetitions: should $D \cap \ker\Lambda_k \neq \emptyset$ for all $k \in \{1, \dots, n-1\}$ then $\ker\Lambda_n$ will be the 0-dimensional linear space $\{0\}$, and then necessarily $D \cap \ker\Lambda_n = \emptyset$. For the last functional Λ_ℓ, we have moreover that

$$\Lambda_\ell(f) > 0 \text{ for all } f \text{ in } D \cap \ker\Lambda_{\ell-1}. \tag{4.7}$$

To see this, recall that by construction $\Lambda_\ell(f) \geq 0$ for all f in $D \cap \ker\Lambda_{\ell-1}$, and that $D \cap \ker\Lambda_\ell = \emptyset$.

In this fashion we obtain ℓ linear functionals $\Lambda_1, \dots, \Lambda_\ell$, each defined on the kernel of the previous functional—except for the domain $\mathcal{L}$ of Λ_1. We now show that we can

[5] To see that $\ker\Lambda_1$ is a linear space, note that by Proposition 93_{128}, $\Lambda_1 = \underline{P}_D$ is a linear prevision—so Λ_1 is a linear map from the n-dimensional linear space $\mathcal{L}$ to $\mathbb{R}$. Since Λ_1 is a linear map, its kernel is closed under addition and scalar multiplication, so it is a linear space, and by the rank-nullity theorem, its dimension is $(\dim\mathcal{L}) - \dim\mathbb{R} = n - 1$.

turn the $\Lambda_2, \dots, \Lambda_\ell$ into expectation operators: positive and normalised linear functionals on the linear space $\mathcal{L}$. Indeed, consider their respective extensions $\Gamma_2, \dots, \Gamma_\ell$ to $\mathcal{L}$ from Lemma 100_{133}, and let $\Gamma_1 \coloneqq \Lambda_1$. They satisfy $\Gamma_k(1) > 0$ for all $k \in \{1, \dots, \ell\}$; see Proposition 93_{128} and Lemma 100_{133}(ii). Now consider the real linear functionals on $\mathcal{L}$ defined by $E_1 \coloneqq \Gamma_1$, and $E_k(f) \coloneqq \frac{\Gamma_k(f)}{\Gamma_k(1)}$ for all k in $\{2, \dots, \ell\}$ and f in $\mathcal{L}$. It is obvious from Proposition 93_{128} and Lemma 100_{133}(i) that these linear functionals are normalised and positive, and therefore expectation operators on $\mathcal{L}$. Indeed each E_k is the expectation operator associated with the mass function p_k defined by $p_k(x) \coloneqq E_k(\mathbb{I}_{\{x\}})$ for all x in $\mathcal{X}$. In this way, $p \coloneqq (p_1, \dots, p_\ell)$ defines a lexicographic probability system.

We now prove that p has no non-trivial Savage-null events, using Proposition 96_{132}. Assume *ex absurdo* that there is some x^* in $\mathcal{X}$ such that $p_k(x^*) = E_k(\mathbb{I}_{\{x^*\}}) = 0$ for all k in $\{1, \dots, \ell\}$. Then $\mathbb{I}_{\{x^*\}} \in \ker \Gamma_1 = \ker \Lambda_1$ and $\mathbb{I}_{\{x^*\}} \in \ker \Gamma_k$ for all k in $\{2, \dots, \ell\}$. Invoke Lemma 100_{133}(iii) to find that $\mathbb{I}_{\{x^*\}} \in \ker \Lambda_1 \cap \ker \Gamma_2 = \ker \Lambda_2$. Repeated application of this same lemma eventually leads us to conclude that $\mathbb{I}_{\{x^*\}} \in \ker \Lambda_{\ell-1}$ and $\mathbb{I}_{\{x^*\}} \in \ker \Lambda_\ell$. Since also $\mathbb{I}_{\{x^*\}} \in D$ and hence $\mathbb{I}_{\{x^*\}} \in D \cap \ker \Lambda_{\ell-1}$ [Axiom D2_{57}], Equation (4.7) implies that $\Lambda_\ell(\mathbb{I}_{\{x^*\}}) > 0$, a contradiction.

It now only remains to prove that $\prec_D$ is the lexicographic ordering with respect to *this* lexicographic probability system, or in other words that

$$f \in D \Leftrightarrow 0 <_{\mathrm{L}} (E_1(f), \dots, E_\ell(f)), \text{ for all } f \text{ in } \mathcal{L}.$$

For necessity, assume that $f \in D$. Then $E_1(f) \geq 0$ by the definition of Λ_1. If $E_1(f) > 0$, then we are done. So assume that $E_1(f) = 0$. Then $f \in \ker \Lambda_1$ and $\Lambda_2(f) \geq 0$ by Equation (4.6). Again, if $\Lambda_2(f) > 0$, we can invoke Lemma 100_{133}(iv) to find that $\Gamma_2(f) > 0$ and hence $E_2(f) > 0$, and we are done. So assume that $\Lambda_2(f) = 0$. Then $f \in \ker \Lambda_2$ and $\Lambda_3(f) \geq 0$ by Equation (4.6). We can go on in this way, and we call k the largest number for which $E_j(f) = 0$ for all j in $\{1, \dots, k-1\}$, or in other words, the smallest number for which $E_k(f) > 0$. Then $k \leq \ell$ by construction—see Equation (4.7)—, whence indeed $0 <_{\mathrm{L}} (E_1(f), \dots, E_\ell(f))$.

For sufficiency, assume that $0 <_{\mathrm{L}} (E_1(f), \dots, E_\ell(f))$, meaning that there is some k in $\{1, \dots, \ell\}$ for which $E_j(f) = 0 = \Gamma_j(f)$ for all j in $\{1, \dots, k-1\}$ and $E_k(f) > 0$, whence also $\Gamma_k(f) > 0$. So $f \in \ker \Gamma_j$ for all $j \in \{1, \dots, k-1\}$ and therefore repeated application of Lemma 100_{133}(iii) tells us that $f \in \ker \Lambda_j$ for all $j \in \{1, \dots, k-1\}$. Since $\Gamma_k(f) > 0$, we infer from Lemma 100_{133}(iv) that also $\Lambda_k(f) > 0$, whence indeed $f \in D$ by Equation (4.6). $\square$

We conclude that the sets of desirable options in $\overline{\mathbf{D}}_{\mathrm{L}}$ are exactly the ones that are representable by a lexicographic probability system that has no non-trivial Savage-null events. This is, of course, the reason why we have called the coherent sets of desirable options in $\overline{\mathbf{D}}_{\mathrm{L}} \coloneqq \{D \in \overline{\mathbf{D}} : \operatorname{posi}(D^c) = D^c\}$ *lexicographic*.

This does not mean, however, that the correspondence between the two families is bijective:[6] Indeed, consider the binary possibility space $\mathcal{X} = \{\mathrm{H}, \mathrm{T}\}$

[6]That two different lexicographic systems (represented as stochastic matrices of full rank) may be associated with the same coherent set of desirable gambles can also be inferred from an example by Benavoli et al. [6, Example 1].

and the lexicographic systems $p = (p_1, p_2)$ and $p' = (p_1, p_2')$, associated with the respective mass functions $p_1 = (1/2, 1/2)$, $p_2 = (0,1)$ and $p_2' = (1/4, 3/4)$ on $\{\mathrm{H}, \mathrm{T}\}$. Then p and p' have no non-trivial Savage-null events. However, $D_p = D_{p'} = \{f \in \mathcal{L} : f(a) + f(b) > 0 \text{ or } f(b) = -f(a) > 0\}$, so we see that there are two different lexicographic probability systems that map to the same lexicographic set of desirable gambles. Note, however, that $<_p$ and $<_{p'}$ are both equal to $\prec_D$—and therefore equal to each other, so $<_p = <_{p'}$—, as is also guaranteed by the following corollary.

Corollary 102. *Consider any coherent lexicographic set of desirable gambles D in $\overline{\mathbf{D}}_{\mathrm{L}}$ and any lexicographic probability system p that has no non-trivial Savage-null events. Then $\prec_D = <_p \Leftrightarrow D = D_p$. As a consequence, given any coherent lexicographic set of desirable gambles D in $\overline{\mathbf{D}}_{\mathrm{L}}$, then $\prec_D = <_p$ for all lexicographic probability systems p that have no non-trivial Savage-null events such that $D_p = D$.*

Proof. For the first statement, infer the following chain of equivalences:

$$
\begin{aligned}
\prec_D = <_p &\Leftrightarrow (\forall f \in \mathcal{L})(0 \prec_D f \Leftrightarrow 0 <_p f) && \text{by Proposition } 97_{132}\text{(i)} \\
&\Leftrightarrow (\forall f \in \mathcal{L})(f \in D \Leftrightarrow f \in D_p) && \text{by the definitions of } \prec_D \text{ and } D_p \\
&\Leftrightarrow D = D_p.
\end{aligned}
$$

The second statement now follows immediately. □

This corollary is important, since it guarantees that $\prec_p$ and $\prec_{p'}$ differ if and only if D_p and $D_{p'}$ differ: it rules out that two different preference relations $\prec_p$ and $\prec_{p'}$ (based on two different lexicographic probability systems p and p' that have no non-trivial Savage-null events) map to the same coherent lexicographic set of desirable gambles $D_p = D_{p'}$. More information about this relation—and also taking updating into account—can be found in work by Benavoli et al. [6], which builds on the important lexicographic separation theorem by Martínez-Legaz [49].

The second part of our Theorem 101_{135} can also be obtained as a consequence of an earlier result by Martínez-Legaz and Vincente-Pérez [50, Corollary 3.5], considering that lexicographic sets of desirable gambles are hemispaces and relying on the representation of lexicographic probability systems as stochastic matrices. This result was also used by Benavoli et al. [6] in their study of the connection between sets of desirable gambles and sets of lexicographic probability systems. It makes use of the aforementioned *lexicographic* separation theorem [49], which is arguably more directly suited for our purpose than the more general separation results (see Theorem 98_{133}) we borrowed from Walley [82]. However, we feel that there is value in our more direct proof, since it is more directly tailored to, and 'translated' in, the language of sets of desirable gambles.

We can also characterise the maximal sets of desirable gambles elegantly using lexicographic probability systems.[7] Introduce $\ker E_p$

$$\ker E_p \coloneqq \bigcap_{k=1}^{\ell} \ker E_{p_k} \tag{4.8}$$

as the intersection of the kernels associated with the expectation operators associated with p: it is the set of gambles that have expectation zero for every expectation operator in p.

Proposition 103. *Consider any lexicographic probability system p with ℓ layers, and let $D_p \coloneqq \{f \in \mathcal{L} : 0 \prec_p f\}$ be a (not necessarily coherent) set of desirable gambles.*[8] *Then $\ker E_p = (D_p \cup -D_p)^c$. As a consequence, the following two statements are equivalent:*
(i) $D_p \in \hat{\mathbf{D}}$;
(ii) $\ker E_p = \{0\}$.
In any of these equivalent cases, $\ell \geq n$, and p has no non-trivial Savage-null events.

Proof. For the first statement, that $\ker E_p = (D_p \cup -D_p)^c$, consider any f in $\mathcal{L}$ and the following equivalences:

$$\begin{aligned} f \in \ker E_p \Leftrightarrow E_p(f) = 0 &\Leftrightarrow (0 \not\prec_p f \text{ and } f \not\prec_p 0) \\ &\Leftrightarrow (0 \not\prec_p f \text{ and } 0 \not\prec_p -f) \\ &\Leftrightarrow (f \notin D_p \text{ and } -f \notin D_p) \\ &\Leftrightarrow (f \in D_p^c \text{ and } f \in -D_p^c) \Leftrightarrow f \in D_p^c \cap -D_p^c = (D_p \cup -D_p)^c, \end{aligned}$$

where third equivalence follows from Proposition 97_{132}.

We now prove that $\ker E_p = \{0\}$ implies that p has no non-trivial Savage-null events. Assume *ex absurdo* that p has a non-trivial Savage-null event. By Proposition 96_{132}, there is some x in $\mathcal{X}$ such that $p_k(x) = 0$, implying that $E_{p_k}(\mathbb{I}_{\{x\}}) = 0$, for all k in $\{1, \ldots, \ell\}$. But then $\mathbb{I}_{\{x\}} \in \ker E_p$, contradicting $\ker E_p = \{0\}$.

To prove that (i) implies (ii), assume that $\ker E_p \neq \{0\}$. Because every E_{p_k} is a linear operator, $0 \in \ker E_p$, and therefore $f \in \ker E_p$ for some $f \in \mathcal{L} \setminus \{0\}$. We have just shown that $\ker E_p = D_p^c \cap -D_p^c$, so $f \notin D_p$ and $-f \notin D_p$ for some f in $\mathcal{L} \setminus \{0\}$. Proposition 51_{59} guarantees that then indeed $D_p \notin \hat{\mathbf{D}}$.

To prove that (ii) implies (i), assume that $\ker E_p = \{0\}$. Then we know already that p has no non-trivial Savage-null events, so by Theorem 101_{135}, D_p is a coherent set of desirable gambles. Since $\ker E_p = \{0\}$, we see that $E_p(f) \neq 0$, so $0 <_{\mathrm{L}} E_p(f)$—and

[7] The characterisation in Proposition 103 can also be obtained as a consequence of work by Benavoli et al. [6, Proposition 8], by noting that the lexicographic probability systems p for which $\ker E_p = \{0\}$ are exactly those whose corresponding stochastic matrices are orthonormal.

[8] Since in Theorem 101_{135} we only have defined such sets of desirable gambles for lexicographic probability systems that have no non-trivial Savage-null events, we need to define it here in this more general context as well.

hence $0 \prec_p f$, so $f \in D_p$—or $0 <_{\mathrm{L}} E_p(-f)$—and hence $0 \prec_p -f$, so $-f \in D_p$—for every f in $\mathcal{L} \setminus \{0\}$. Use Proposition 51_{59} to infer that then D_p is indeed a maximal set of desirable gambles.

To show, finally, that then $\ell \geq n$, assume that $\ker E_p = \bigcap_{k=1}^{\ell} \ker E_{p_k} = \{0\}$, so $\dim\ker E_p = 0$. Note that $\dim\ker E_{p_k} = n-1$ for every k in $\{1,\dots,\ell\}$, and therefore, by the dimension theorem [5, Theorem 2.18]

$$\begin{aligned}\dim(\ker E_{p_1} \cap \ker E_{p_2}) &= \dim\ker E_{p_1} + \dim\ker E_{p_2} - \dim(\ker E_{p_1} + \ker E_{p_2})\\ &\geq (n-1)+(n-1)-n = n-2,\end{aligned}$$

and similarly

$$\begin{aligned}&\dim(\ker E_{p_1} \cap \ker E_{p_2} \cap \ker E_{p_2})\\ &\qquad= \dim(\ker E_{p_1} \cap \ker E_{p_2}) + \dim\ker E_{p_3} - \dim((\ker E_{p_1} \cap \ker E_{p_2}) + \ker E_{p_3})\\ &\qquad\geq (n-2)+(n-1)-n = n-3.\end{aligned}$$

Going on in this way, we eventually find that $\dim\bigcap_{k=1}^{\ell} \ker E_{p_k} \geq n-\ell$. This shows that $\dim E_p = 0$ indeed implies that $\ell \geq n$. □

4.2.3 *Savage-null events revisited*

That there are *no non-trivial Savage-null events* in this representation should not surprise us. To gain some insight as to why, we consider a lexicographic probability system $p = (p_1,\dots,p_\ell)$ that *has* some non-trivial Savage-null event E, so conditioning on E is ill-defined. However, there are coherent sets of desirable gambles D that correspond with it, in the sense that $f \prec_p g \Rightarrow f \lessdot_D g$ (or equivalently, $0 \prec_p f \Rightarrow f \in D$) for all f and g in $\mathcal{L}$: indeed, consider[9]

$$D'_p := \mathcal{L}_{>0} \cup \{f \in \mathcal{L} : 0 \prec_p f\} = \mathcal{L}_{>0} \cup \bigcup_{k=1}^{\ell}\Big\{f \in \bigcap_{j=1}^{k-1} \ker E_{p_j} : E_{p_k}(f) > 0\Big\}, \quad (4.9)$$

which is coherent due to Lemma 104.

Lemma 104. *Consider D'_p as defined in Equation* (4.9). *Then D'_p is a coherent set of desirable gambles, and $f \prec_p g \Rightarrow f \lessdot_{D'_p} g$ for all f and g in $\mathcal{L}$.*

Proof. We first prove that D'_p is coherent. For Axiom $\mathrm{D1}_{57}$, recall that $E_{p_k}(0) = 0$ for all k in $\{1,\dots,\ell\}$, and that $0 \notin \mathcal{L}_{>0}$, so indeed $0 \notin D'_p$. Axiom $\mathrm{D2}_{57}$ is trivially satisfied, because of the definition (4.9) of D'_p. For Axiom $\mathrm{D3}_{57}$, consider any f in $\mathcal{L}$ and λ in $\mathbb{R}_{>0}$ and recall that $f \in \bigcap_{j=1}^{k-1} \ker E_{p_j} \Leftrightarrow \lambda f \in \bigcap_{j=1}^{k-1} \ker E_{p_j}$, $E_{p_k}(f) > 0 \Leftrightarrow E_{p_k}(\lambda f) > 0$, and $f > 0 \Leftrightarrow \lambda f > 0$, so $f \in D$ implies that indeed $\lambda f \in D$. Finally, for Axiom $\mathrm{D4}_{57}$, consider any f and g in D'_p. If $f > 0$ and $g > 0$ then $f+g > 0$ so $f+g \in D'_p$. If $f > 0$ and there is some k in $\{1,\dots,\ell\}$ such that $g \in \bigcap_{j=1}^{k-1} \ker E_{p_j}$ and $E_{p_k}(g) > 0$, then, since

[9] We let $\bigcap\varnothing := \mathcal{L}$ in this expression.

$E_{p_j}(f+g) \geq E_{p_j}(g)$ for all j in $\{1,\ldots,\ell\}$, we see that there is some $k' \leq k$ in $\{1,\ldots,\ell\}$ such that $f+g \in \bigcap_{j=1}^{k'-1} \ker E_{p_j}$ and $E_{p_{k'}}(f+g) > 0$, so $f+g \in D'_p$. If the role of f and g are switched, then a similar reasoning shows that, again, $f+g \in D'_p$. Finally, if both f and g belong to $\bigcup_{k=1}^{\ell}\{h \in \bigcap_{j=1}^{k-1} \ker E_{p_j} : E_{p_k}(h) > 0\}$, then let k' be the smallest number in $\{1,\ldots,\ell\}$ for which $E_{p_{k'}}(f) > 0$ and k'' the smallest number in $\{1,\ldots,\ell\}$ for which $E_{p_{k''}}(g) > 0$. Define $k \coloneqq \min\{k',k''\}$, then $E_{p_k}(f+g) > 0$ and $E_{p_j}(f+g) = 0$ for all j in $\{1,\ldots,k-1\}$, so indeed $f+g \in D'_p$.

The second statement is immediate from the definition in Equation (4.9). □

D'_p extends $<_p$ in the *least informative way*: it is the smallest coherent set of desirable gambles such that $\prec_{D'_p} \supseteq <_p$. Since conditioning is never problematic for coherent sets of desirable gambles [31], using D'_p we *can* condition on every non-trivial event—even the Savage-null event E: conditioning on E yields the vacuous set of desirable gambles.

To develop our understanding even further, we look at $\ker E_p$ as defined in Equation (4.8)$_{139}$. Since every $\ker E_{p_k}$ is a $n-1$-dimensional linear subspace, $\ker E_p$ is a linear subspace whose dimension is not higher than $n-1$. Since for every x in E, $\mathbb{I}_{\{x\}} \in \ker E_{p_k}$ for every k in $\{1,\ldots,\ell\}$, we have that $\mathbb{I}_{\{x\}} \in \ker E_p$, so $\ker E_p$ includes $\operatorname{span}\{\mathbb{I}_{\{x\}} : x \in E\}$, and therefore $\dim \ker E_p \geq |E|$. We distinguish between two cases: $\dim \ker E_p \geq 2$ and $\dim \ker E_p = 1$.

If $\dim \ker E_p \geq 2$ then the following lemma guarantees that D'_p cannot be a lexicographic set of desirable gambles.

Lemma 105. *If* $\dim \ker E_p \geq 2$ *then* D'_p *is no lexicographic set of desirable gambles.*

Proof. If $|E| = 1$—say $E = \{x\}$—then consider any $f \neq 0$ in $\ker E_p$ such that $f(x) = 0$. This is always possible since $\dim \ker E_p \geq 2$. Let $g_1 \coloneqq \mathbb{I}_{\{x\}} + f$ and $g_2 \coloneqq \mathbb{I}_{\{x\}} - f$. We claim that $g_1 \notin \mathcal{L}_{>0}$ and $g_2 \notin \mathcal{L}_{>0}$. To see that $g_1 \notin \mathcal{L}_{>0}$, assume *ex absurdo* that $g_1 = \mathbb{I}_{\{x\}} + f \in \mathcal{L}_{>0}$, and therefore $f(x') \geq 0$ for all x' in $\mathcal{X}$. This implies that $E_{p_k}(f) \geq 0$ for all k in $\{1,\ldots,\ell\}$. Since $f \neq 0$ and $f(x) = 0$, there is some x^* in $\mathcal{X} \setminus \{x\}$ for which $f(x^*) > 0$. But since $x^* \notin E = \{x\}$, we find that $E_{p_k}(f) > 0$ for some k in $\{1,\ldots,\ell\}$, contradicting that $f \in \ker E_p$. The proof that $g_2 \notin \mathcal{L}_{>0}$ is completely similar. Since both $\mathbb{I}_{\{x\}}$ and f are elements of $\ker E_p$, so are g_1 and g_2, and therefore $E_{p_k}(g_1) = 0 = E_{p_k}(g_2)$ for all k in $\{1,\ldots,\ell\}$. This tells us that $g_1 \in D'^c_p$ and $g_2 \in D'^c_p$, but their sum $g_1 + g_2 = 2\mathbb{I}_{\{x\}} \in \mathcal{L}_{>0}$, so $f+g \notin D'^c_p$. Hence D'_p is not lexicographic.

If $|E| \geq 2$, consider any two different x and y in E and let $g_1 \coloneqq 2\mathbb{I}_{\{x\}} - \mathbb{I}_{\{y\}}$ and $g_2 \coloneqq 2\mathbb{I}_{\{y\}} - \mathbb{I}_{\{x\}}$. Then $g_1 \notin \mathcal{L}_{>0}$, $g_2 \notin \mathcal{L}_{>0}$, and $E_{p_k}(g_1) = 0 = E_{p_k}(g_2)$ for all k in $\{1,\ldots,\ell\}$, so $g_1 \in D'^c_p$ and $g_2 \in D'^c_p$, but their sum $g_1 + g_2 = \mathbb{I}_{\{x\}} + \mathbb{I}_{\{y\}} \in \mathcal{L}_{>0}$, so $f+g \notin D'^c_p$. Hence, here too, D'_p is not lexicographic. □

In its turn, this non-lexicographic D'_p can be extended to some lexicographic set of desirable gambles in a number of ways. For instance, the extension should contain at least one of g_1 and g_2 (defined in the proof of Lemma 105), indicating already one choice to be made.

On the other hand, if $\dim\ker E_p = 1$ then necessarily $|E| = 1$—say $E = \{x\}$—and then $\ker E_p = \operatorname{span}\{\mathbb{I}_{\{x\}}\}$. Then D'_p *is* a lexicographic set of desirable gambles, despite the fact that p has non-trivial Savage-null events, by the following lemma.

Lemma 106. *If* $\dim\ker E_p = 1$ *then* D'_p *is a lexicographic set of desirable gambles.*

Proof. Consider any f and g in D'^c_p. Then $f \notin \mathcal{L}_{>0}$, $g \notin \mathcal{L}_{>0}$, $0 \not\prec_p f$ and $0 \not\prec_p g$. By Proposition 97_{132}(i) we infer that $f \not\prec_p f+g$ and hence, using the transitivity of $\not\prec_p$ [see Proposition 95_{131}], $0 \not\prec_p f+g$. Assume *ex absurdo* that $f+g \notin D'^c_p$, then it can now only be that $f+g \in \mathcal{L}_{>0}$ and therefore $E_{p_k}(f+g) \geq 0$ for every k in $\{1,\dots,\ell\}$, and $f(x^*)+g(x^*) > 0$ for some x^* in $\mathcal{X}$. If $x^* \notin E$ then $E_{p_k}(f+g) > 0$ for some k in $\{1,\dots,\ell\}$ and therefore $0 \prec_p f+g$, a contradiction. This tells us that necessarily $x^* = x$, and also that $f(z)+g(z) = 0$ for all z in $\mathcal{X} \setminus \{x^*\}$ We infer from this that $E_{p_k}(f+g) = 0$ for all k in $\{1,\dots,\ell\}$, whence $f+g \in \ker E_p = \operatorname{span}\{\mathbb{I}_{\{x\}}\}$. There are now two possibilities. The first is that both f and g belong to $\operatorname{span}\{\mathbb{I}_{\{x\}}\}$. Since $f+g \in \mathcal{L}_{>0}$, this implies that either $f \in \operatorname{posi}\{\mathbb{I}_{\{x\}}\}$ or $g \in \operatorname{posi}\{\mathbb{I}_{\{x\}}\}$. But then $f \in \mathcal{L}_{>0}$ or $g \in \mathcal{L}_{>0}$, a contradiction. The second possibility is that neither f nor g belong to $\operatorname{span}\{\mathbb{I}_{\{x\}}\} = \ker E_p$. This implies that there is some k_1 in $\{1,\dots,\ell\}$ such that $E_{p_j}(f) = 0$ for all j in $\{1,\dots,k_1-1\}$ and $E_{p_{k_1}}(f) \neq 0$, and therefore $E_{p_{k_1}}(f) < 0$, because $0 \not\prec_p f$. Similarly, there is some k_2 in $\{1,\dots,\ell\}$ such that $E_{p_j}(g) = 0$ for all j in $\{1,\dots,k_2-1\}$ and $E_{p_{k_2}}(g) \neq 0$, and therefore $E_{p_{k_2}}(g) < 0$, because $0 \not\prec_p g$. If we now let $k \coloneqq \min\{k_1,k_2\}$, we see that $E_{p_k}(f+g) < 0$, whence $f+g \notin \ker E_p$, a contradiction. □

Interestingly, when we start out with some p that has non-trivial Savage-null events, this argument leads us to its least informative coherent extension D'_p, which *is* a lexicographic set of desirable gambles: it is representable by a lexicographic probability system that has *no* non-trivial Savage-null events, by Theorem 101_{135}.

When $\dim\ker E_p \geq 2$, we have seen that D'_p is not lexicographic, but it can be extended in a number of ways to a lexicographic set of desirable gambles. When $\dim\ker E_p = 1$, then its smallest coherent extension D'_p is also lexicographic. By Lemma 107, then $\lessdot_{D'_p} = \prec_{p'}$ where p is the lexicographic probability system $(p_1,\dots,p_{\ell-1},p_\ell,p_{\ell+1})$ and $p_{\ell+1}$ is any probability mass function such that $\ker E_p \cap \ker E_{p_{\ell+1}} = \{0\}$, for instance $p_{\ell+1} = \mathbb{I}_{\{x\}}$. So we have found a lexicographic probability system p' without non-trivial Savage-null events that represents D'_p: it suffices to add a well-chosen layer. In other words, when importing the language of sets of desirable gambles, we can regard p as having an extra layer that avoids problems with conditioning.

Lemma 107. *Consider any lexicographic probability system $p = (p_1,\dots,p_\ell)$, and assume that $E = \{x\}$ is the only non-trivial Savage-null event. Import the notation above, then $\lessdot_{D'_p} = \prec_{p'}$ where p' is some new lexicographic probability system $p' = (p_1,\dots,p_{\ell-1},p_\ell,p_{\ell+1})$, and $p_{\ell+1}$ is any probability mass function such that* $\ker E_p \cap \ker E_{p_{\ell+1}} = \{0\}$.

Proof. To prove that $\prec_{D'_p} \subseteq \prec_{p'}$, consider any gamble f in $\mathcal{L}$ for which $0 \prec_{D'_p} f$. Then $f \in D'_p$ whence $0 < f$ or $0 \prec_p f$. If $0 \prec_p f$ then $0 \prec_{p'} f$ and the proof is done, so assume that $0 \not\prec_p f$. Then $0 < f$ implies that $f(x') = 0$ for all x' in $\mathcal{X} \setminus \{x\}$—and hence $E_{p_k}(f) = 0$ for all k in $\{1, \dots, \ell\}$—, so $f(x) > 0$. Therefore, $f \in \ker E_p$, so $f \notin \ker E_{p_{\ell+1}}$, whence $E_{p_{\ell+1}}(f) > 0$ and therefore indeed $0 \prec_{p'} f$.

To prove that $\prec_{p'} \subseteq \prec_{D'_p}$, consider any gamble f in $\mathcal{L}$ such that $0 \prec_{p'} f$. Then $0 \prec_p f$, in which case $f \in D'_p$, whence $0 \prec_{D'_p} f$ and the proof is done, or $E_{p_k}(f) = 0$ for all k in $\{1, \dots, \ell\}$ and $E_{p_{\ell+1}}(f) > 0$. Then $f \in \ker E_p = \operatorname{span}\{\mathbb{I}_{\{x\}}\}$, so $f = \lambda \mathbb{I}_{\{x\}}$ for some λ in $\mathbb{R}$. But $E_{p_{\ell+1}}(\lambda \mathbb{I}_{\{x\}}) > 0$ implies that $\lambda > 0$, so $f \in \mathcal{L}_{>0}$, and indeed $f \in D'_p$. □

When $\dim \ker E_p = 1$, there is only one coherent way of defining the initially ill-defined conditionals on $E = \{x\}$. Indeed, when there is only one outcome x, the unique conditional set of desirable gambles on E is the *vacuous* set of desirable gambles. This should be contrasted with the case where $\dim \ker E_p \geq 2$: now there are multiple ways of defining the conditionals on E, all leading to non-vacuous conditional sets of desirable gambles.

4.3 LEXICOGRAPHIC CHOICE FUNCTIONS

Lexicographic probability systems can now also be related to specific types of choice functions, through Proposition 91_{127}: given a coherent set of desirable options D, the most conservative coherent choice function C whose binary choices are represented by $D_C = D$ satisfies the convexity property $\mathrm{C5}_{25}$ if and only if D is a lexicographic set of desirable options. We will call $\overline{\mathbf{C}}_{\mathrm{L}} := \{C_D : D \in \overline{\mathbf{D}}_{\mathrm{L}}\}$ the set of *lexicographic choice functions*.

Looking first at the most conservative coherent choice function that corresponds to D and then checking whether it is 'convex', leads rather restrictively to lexicographic choice functions, and is only possible for lexicographic D: convexity and choice based on Walley–Sen maximality are only compatible for lexicographic binary choice. But suppose we turn things around somewhat, first restrict our attention to all 'convex' coherent choice functions from the outset, and then look at the most conservative such choice function that makes the same binary choices as present in some given D:

$$\inf\{C \in \overline{\mathbf{C}} : C \text{ satisfies Property } \mathrm{C5}_{25} \text{ and } D_C = D\}.$$

We infer from Propositions 40_{48} and 47_{53} that this infimum is still 'convex' and coherent. It will, of course, no longer be lexicographic, unless D is. But the following theorem tells us it still is an infimum of lexicographic choice functions, and therefore adds further weight to our growing suspicion that lexicographic choice functions have a central part in a theory of 'convex' coherent choice functions without Archimedeanity.

Theorem 108. *Consider an arbitrary coherent set of desirable options D. The most conservative coherent choice function C that satisfies Property* $\mathrm{C5}_{25}$ *and*

$D_C = D$ is the infimum of all lexicographic choice functions $C_{D'}$ with D' in $\overline{\mathbf{D}}_{\mathrm{L}}$ such that $D \subseteq D'$:

$$\inf\{C \in \overline{\mathbf{C}} : C \text{ satisfies } \mathrm{C5}_{25} \text{ and } D_C = D\} = \inf\{C_{D'} : D' \in \overline{\mathbf{D}}_{\mathrm{L}} \text{ and } D \subseteq D'\}.$$

Proof. Denote the choice function on the left-hand side by C_{left}, and the one on the right-hand side by C_{right}. Both are coherent, and so by Axiom $\mathrm{C4b}_{20}$ completely characterised by the option sets from which 0 is chosen. Consider any A in $\mathcal{Q}_{\overline{0}}$, then we have to show that $0 \in C_{\text{left}}(\{0\} \cup A) \Leftrightarrow 0 \in C_{\text{right}}(\{0\} \cup A)$.

For the direct implication, we assume that $0 \in C_{\text{left}}(\{0\} \cup A)$, meaning that there is some C^* in $\overline{\mathbf{C}}$ that satisfies Property $\mathrm{C5}_{25}$, $D_{C^*} = D$ and $0 \in C^*(\{0\} \cup A)$. We have to prove that there is some D^* in $\overline{\mathbf{D}}_{\mathrm{L}}$ such that $D \subseteq D^*$ and $D^* \cap A = \emptyset$ [by Proposition 55_{64}], and we will do so by constructing a suitable lexicographic probability system, by a repeated application of an appropriate version of the separating hyperplane theorem [Theorem 98_{133}], as in the proof of Theorem 101_{135}.

To prepare for this, we prove that $\operatorname{posi}(\{0\} \cup A) \cap D = \emptyset$. Indeed, assume *ex absurdo* that $\operatorname{posi}(\{0\} \cup A) \cap D \neq \emptyset$, so there is some $f \in D$ such that $f \in \operatorname{posi}(\{0\} \cup A)$. Then there is some λ in $\mathbb{R}_{>0}$ such that $g := \lambda f \in \operatorname{conv}(\{0\} \cup A)$. Let $A' := A \cup \{g\}$, so $\{0\} \cup A' \subseteq \operatorname{conv}(\{0\} \cup A)$, whence $0 \in C^*(\{0\} \cup A')$ by Property $\mathrm{C5}_{25}$, if we recall that $0 \in C^*(\{0\} \cup A)$. But $f \in D$ implies that $g \in D$, and since $D_{C^*} = D$, also that $g \in D_{C^*}$, or equivalently, $0 \in R^*(\{0, g\})$, by Proposition 55_{64}. Axiom $\mathrm{R3a}_{20}$ then guarantees that $0 \in R^*(\{0\} \cup A')$, a contradiction.

It follows from this observation that we can apply Theorem 98_{133} to show that there is some non-zero linear functional Λ_1 on $\mathcal{L}$ such that

$$\Lambda_1(f) \leq 0 \text{ for all } f \text{ in } \operatorname{posi}(\{0\} \cup A) \text{ and } \Lambda_1(f) \geq 0 \text{ for all } f \text{ in } D. \tag{4.10}$$

[Apply Theorem 98_{133} with $\mathcal{B} = \mathcal{L}$, $\mathcal{W}_2 = \operatorname{posi}(\{0\} \cup A)$ and $\mathcal{W}_1 = D \cup \{0\}$, then $\operatorname{int}(\mathcal{W}_1) \cap \mathcal{W}_2 = \emptyset$ since $\operatorname{int}(\mathcal{W}_1) \subseteq D$, and $0 \in \mathcal{W}_1 \cap \mathcal{W}_2$.] Its kernel $\ker \Lambda_1$ is an $n-1$-dimensional linear space, where n is the dimension of $\mathcal{L}$—the cardinality of $\mathcal{X}$. Since both D and $\ker \Lambda_1$ are convex cones, their intersection $\ker \Lambda_1 \cap D$ is either empty or a convex cone. When $\ker \Lambda_1 \cap D = \emptyset$, we let $\ell := 1$, and stop.

When $\ker \Lambda_1 \cap D \neq \emptyset$, it follows from the same version of the separating hyperplane theorem that there is some non-zero linear functional Λ_2 on $\ker \Lambda_1$ such that

$$\Lambda_2(f) \leq 0 \text{ for all } f \text{ in } \ker \Lambda_1 \cap \operatorname{posi}(\{0\} \cup A) \text{ and } \Lambda_2(f) \geq 0 \text{ for all } f \text{ in } \ker \Lambda_1 \cap D.$$

[Apply Theorem 98_{133} with $\mathcal{B} = \ker \Lambda_1$, $\mathcal{W}_2 = \operatorname{posi}(\{0\} \cup A) \cap \ker \Lambda_1$ and $\mathcal{W}_1 = (\ker \Lambda_1 \cap D) \cup \{0\}$, then $\operatorname{int}(\mathcal{W}_1) \cap \mathcal{W}_2 = \emptyset$ since $\mathcal{W}_2 \subseteq \operatorname{posi}(\{0\} \cup A)$ and $\operatorname{int}(\mathcal{W}_1) \subseteq D$, and $0 \in \mathcal{W}_1 \cap \mathcal{W}_2$.] $\ker \Lambda_2$ is a $n-2$-dimensional linear space. As before, $D \cap \ker \Lambda_2$ is either empty or a non-empty convex cone. If it is empty, let $\ell := 2$; otherwise, repeat the same procedure over and over again, leading to successive non-zero linear functionals Λ_k on $\ker \Lambda_{k-1}$ such that

$$\Lambda_k(f) \leq 0 \text{ for all } f \text{ in } \ker \Lambda_{k-1} \cap \operatorname{posi}(\{0\} \cup A) \text{ and } \Lambda_k(f) \geq 0 \text{ for all } f \text{ in } \ker \Lambda_{k-1} \cap D, \tag{4.11}$$

until eventually we get to the first k such that $D \cap \ker \Lambda_k = \emptyset$, and then let $\ell := k$ and stop. We are guaranteed to stop after at most n repetitions, since $\ker \Lambda_n$ is the 0-dimensional linear space $\{0\}$, for which $D \cap \ker \Lambda_n = \emptyset$. For the last functional Λ_ℓ, we have that

$$\Lambda_\ell(f) > 0 \text{ for all } f \text{ in } D \cap \ker \Lambda_{\ell-1}. \tag{4.12}$$

To see this, recall that by construction $\Lambda_\ell(f) \geq 0$ for all f in $D \cap \ker \Lambda_{\ell-1}$, and that $D \cap \ker \Lambda_\ell = \emptyset$.

In this fashion we obtain ℓ linear functionals $\Lambda_1, \dots, \Lambda_\ell$, each defined on the kernel of the previous functional—except for the domain $\mathcal{L}$ of Λ_1. We now show that we can turn the $\Lambda_1, \dots, \Lambda_\ell$ into expectation operators: positive and normalised linear functionals on the linear space $\mathcal{L}$. Indeed, consider their respective extensions $\Gamma_2, \dots, \Gamma_\ell$ to $\mathcal{L}$ from Lemma 100_{133}, and let $\Gamma_1 \coloneqq \Lambda_1$. They satisfy $\Gamma_k(1) > 0$ for all k in $\{1, \dots, \ell\}$; see Proposition 93_{128} and Lemma 100_{133}(ii). Now consider the real linear functionals on $\mathcal{L}$ defined by $E_k(f) \coloneqq \frac{\Gamma_k(f)}{\Gamma_k(1)}$ for all k in $\{1, \dots, \ell\}$ and f in $\mathcal{L}$. It is obvious from Lemma 100_{133}(i) that these linear functionals are normalised and positive, and therefore expectation operators on $\mathcal{L}$. Indeed, each E_k is the expectation operator associated with the mass function p_k defined by $p_k(x) \coloneqq E_k(\mathbb{I}_{\{x\}})$ for all x in $\mathcal{X}$. In this way, $p \coloneqq (p_1, \dots, p_\ell)$ defines a lexicographic probability system.

We now prove that p has no non-trivial Savage-null events, using Proposition 96_{132}. Assume *ex absurdo* that there is some x^* in $\mathcal{X}$ such that $p_k(x^*) = E_k(\mathbb{I}_{\{x^*\}}) = 0$ for all k in $\{1, \dots, \ell\}$. Then $\mathbb{I}_{\{x^*\}} \in \ker \Gamma_1 = \ker \Lambda_1$ and $\mathbb{I}_{\{x^*\}} \in \ker \Gamma_k$ for all k in $\{2, \dots, \ell\}$. Invoke Lemma 100_{133}(iii) to find that $\mathbb{I}_{\{x^*\}} \in \ker \Lambda_1 \cap \ker \Gamma_2 = \ker \Lambda_2$. Repeated application of this same lemma eventually leads us to conclude that $\mathbb{I}_{\{x^*\}} \in \ker \Lambda_{\ell-1}$ and $\mathbb{I}_{\{x^*\}} \in \ker \Lambda_\ell$. Since also $\mathbb{I}_{\{x^*\}} \in D$ and hence $\mathbb{I}_{\{x^*\}} \in D \cap \ker \Lambda_{\ell-1}$ [Axiom $\mathrm{D2}_{57}$], Equation (4.12) implies that $\Lambda_\ell(\mathbb{I}_{\{x^*\}}) > 0$, a contradiction.

If we now let $D^* \coloneqq \{f \in \mathcal{L} : 0 <_{\mathrm{L}} (E_1(f), \dots, E_\ell(f))\}$, then $D^* \in \overline{\mathbf{D}}_{\mathrm{L}}$ by Theorem 101_{135}. If we can show that $D \subseteq D^*$ and $A \cap D^* = \emptyset$, we are done. So first, consider any f in D. Then $\Lambda_1(f) \geq 0$ by Equation (4.10). If $\Lambda_1(f) > 0$ then also $E_1(f) > 0$ by Lemma 100_{133}(ii), and therefore $f \in D^*$. If $\Lambda_1(f) = 0$ then $\Lambda_2(f) \geq 0$ by Equation (4.11). If $\Lambda_2(f) > 0$ then also $E_2(f) > 0$ by Lemma 100_{133}(ii)&(iv), and therefore $f \in D^*$. We can go on in this way until we get to the first k for which $\Lambda_k(f) > 0$, and therefore also $E_k(f) > 0$ by Lemma 100_{133}(ii)&(iv), whence therefore $f \in D^*$. We are guaranteed to find such a k because we infer from Equation (4.12) that $\Lambda_\ell(f) > 0$. This shows that indeed $D \subseteq D^*$.

Secondly, consider any f in A. Then $\Lambda_1(f) \leq 0$ by Equation (4.10). If $\Lambda_1(f) < 0$ then also $E_1(f) < 0$ by Lemma 100_{133}(ii), and therefore $f \notin D^*$. If $\Lambda_1(f) = 0$ then $\Lambda_2(f) \leq 0$ by Equation (4.11). If $\Lambda_2(f) < 0$ then also $E_2(f) < 0$ by Lemma 100_{133}(ii)&(iv), and therefore $f \notin D^*$. If we go on in this way, only two things can happen: either there is a first k for which $\Lambda_k(f) < 0$, and therefore also $E_k(f) < 0$ by Lemma 100_{133}(ii)&(iv), whence therefore $f \notin D^*$. Or we find that $\Lambda_k(f) \leq 0$, and therefore also $E_k(f) \leq 0$ by Lemma 100_{133}(ii)&(iv), for all $k \in \{1, \dots, \ell\}$, whence again $f \notin D^*$. This shows that indeed $A \cap D^* = \emptyset$.

For the converse implication, assume that $0 \in C_{\mathrm{right}}(\{0\} \cup A)$. We must prove that there is some $\tilde{C}$ in $\overline{\mathbf{C}}$ that satisfies Property $\mathrm{C5}_{25}$, $D_{\tilde{C}} = D$ and $0 \in \tilde{C}(\{0\} \cup A)$. We claim that $\tilde{C} \coloneqq C_{\mathrm{right}}$ does the job. Because we know by assumption that $0 \in C_{\mathrm{right}}(\{0\} \cup A)$, and from Propositions 91_{127} and 47_{53} that C_{right} is coherent and satisfies Property $\mathrm{C5}_{25}$, it only remains to prove that $D_{C_{\mathrm{right}}} = D$. To this end, consider any f in $\mathcal{L}$ and recall the following equivalences:

$$\begin{aligned}
f \in D_{C_{\mathrm{right}}} &\Leftrightarrow 0 \in R_{\mathrm{right}}(\{0,f\}) && \text{[Proposition } 53_{61}\text{]}\\
&\Leftrightarrow (\forall D' \in \overline{\mathbf{D}}_{\mathrm{L}})(D \subseteq D' \Rightarrow 0 \in R_{D'}(\{0,f\})) && \text{[definition of inf]}\\
&\Leftrightarrow (\forall D' \in \overline{\mathbf{D}}_{\mathrm{L}})(D \subseteq D' \Rightarrow f \in D') && \text{[Proposition } 55_{64}\text{]}\\
&\Leftrightarrow f \in D, && \text{[Propositions } 52_{59} \text{ and } 92_{128}\text{]}
\end{aligned}$$

which completes the proof. □

As a consequence of this result, we also have that, for any coherent set of desirable options D,

$$\inf\{C \in \overline{\mathbf{C}} : C \text{ satisfies C}5_{25} \text{ and } D \subseteq D_C\} = \inf\{C_{D'} : D' \in \overline{\mathbf{D}}_{\mathrm{L}} \text{ and } D \subseteq D'\}. \tag{4.13}$$

This also allows us to find the unique least informative choice function that is coherent and satisfies Property C5_{25}, to which we referred in Section 2.6.4_{53}:

Corollary 109. *The unique least informative coherent choice function that satisfies Property* C5_{25}*—and therefore also Property* C6_{25}*—is given by*

$$\begin{aligned}
\inf\{C \in \overline{\mathbf{C}} : C \textit{ satisfies } \mathrm{C}5_{25}\} &= \inf\{C \in \overline{\mathbf{C}} : C \textit{ satisfies } \mathrm{C}5_{25} \textit{ and } \mathrm{C}6_{25}\}\\
&= \inf\{C_D : D \in \overline{\mathbf{D}}_{\mathrm{L}}\}.
\end{aligned}$$

Proof. That

$$\inf\{C \in \overline{\mathbf{C}} : C \text{ satisfies C}5_{25}\} = \inf\{C \in \overline{\mathbf{C}} : C \text{ satisfies C}5_{25} \text{ and C}6_{25}\}$$

follows at once from Proposition 16_{27}. To show that $\inf\{C \in \overline{\mathbf{C}} : C \text{ satisfies C}5_{25}\} = \inf\{C_D : D \in \overline{\mathbf{D}}_{\mathrm{L}}\}$, infer from Propositions 50_{58} and 53_{61} that $D_{\mathrm{v}} \subseteq D_C$ for all C in $\overline{\mathbf{C}}$. Therefore $\inf\{C \in \overline{\mathbf{C}} : C \text{ satisfies C}5_{25}\} = \inf\{C \in \overline{\mathbf{C}} : C \text{ satisfies C}5_{25} \text{ and } D_{\mathrm{v}} \subseteq D_C\}$, which is, using Equation (4.13), equal to $\inf\{C_D : D \in \overline{\mathbf{D}}_{\mathrm{L}} \text{ and } D_{\mathrm{v}} \subseteq D\}$. Proposition 50_{58} guarantees that $D_{\mathrm{v}} \subseteq D$ for every D in $\overline{\mathbf{D}}$, and therefore also in particular for every D in $\overline{\mathbf{D}}_{\mathrm{L}}$. Hence indeed $\inf\{C \in \overline{\mathbf{C}} : C \text{ satisfies C}5_{25}\} = \inf\{C_D : D \in \overline{\mathbf{D}}_{\mathrm{L}}\}$. □

4.4 NO LEXICOGRAPHIC REPRESENTATION FOR BINARY POSSIBILITY SPACES

In the previous sections, we have studied some of the consequences of Property C5_{25} on coherent choice functions. We have characterised the purely binary choice functions that satisfy Property C5_{25}: they are those choice functions that are represented by a lexicographic probability system. But is Property C5_{25} enough to guarantee representation in terms of an appropriately chosen subset of $\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\}$?

Example 20. Consider any mass function p that assigns positive probability to every outcome in $\mathcal{X}$, and its corresponding set of strictly desirable gambles $D \coloneqq \mathcal{L}_{>0} \cup \{f \in \mathcal{L} : E_p(f) > 0\} = \{f \in \mathcal{L} : E_p(f) > 0\}$, where the (second) equality holds because $p(x) > 0$ for every x in $\mathcal{X}$. Clearly, $D^c = \{f \in \mathcal{L} : E_p(f) \leq 0\}$ is a convex cone, so D is a lexicographic set of desirable gambles, and hence, by Proposition 91_{127}, C_D is coherent and satisfies Property $C5_{25}$.

Is C_D representable by a subset of $\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\}$? To answer this, consider an option set A in $\mathcal{Q}_{\overline{0}}$ that consists of $m \geq 1$ gambles $f_1, \dots, f_m$ such that $0 \in \operatorname{posi}(A)$ and $E_p(f_k) = 0$ for every k in $\{1, \dots, m\}$. Then $f_k \notin D$ for every k in $\{1, \dots, m\}$, whence by Proposition 55_{64}, $0 \in C_D(A)$. We claim that $A \cap \hat{D} \neq \emptyset$, for every $\hat{D}$ in $\hat{\mathbf{D}}$. To see this, assume *ex absurdo* that $A \cap \hat{D} = \emptyset$ for some $\hat{D}$ in $\hat{\mathbf{D}}$, then by Proposition 51_{59} $-f_k \in \hat{D}$ for every k in $\{1, \dots, m\}$. Since $0 \in \operatorname{posi}\{f_1, \dots, f_m\}$, we have that $\sum_{k=1}^{m} \lambda_k f_k = 0$ for some $\lambda_1, \dots, \lambda_m$ in $\mathbb{R}_{>0}$. Since, by Axiom $D3_{57}$, $-\lambda_k f_k \in \hat{D}$ for every k in $\{1, \dots, m\}$, we would find by Axiom $D4_{57}$ that their sum $\sum_{k=1}^{m} -\lambda_k f_k = 0$ would belong to $\hat{D}$, in contradiction with Axiom $D1_{57}$. So we see that $A \cap \hat{D} \neq \emptyset$, and therefore $0 \in R_{\hat{D}}(A)$, for every $\hat{D}$ in $\hat{\mathbf{D}}$. This implies that, for every subset $\mathcal{D} \subseteq \hat{\mathbf{D}}$, also $0 \in R_{\mathcal{D}}(A)$. Since we already know that $0 \in C_D(A)$, this means that C_D is not representable by subsets of $\hat{\mathbf{D}}$, even though C_D satisfies Property $C5_{25}$. ◊

This example shows that representation in terms of appropriately chosen $\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\}$ is impossible. But since we have seen in Theorem 108_{143} that lexicographic choice functions seem to fulfil at least some representing role in our theory without Archimedeanity, it seems at least possible that there might be a representation result in terms of $\overline{\mathbf{C}}_{\mathrm{L}}$—in terms of lexicographic choice functions. This brings us to the central question of this section: is, in parallel with the result by Seidenfeld et al. [67], every coherent choice function C on option spaces that satisfies the corresponding Property $C5_{25}$ an infimum of *lexicographic choice functions*, or in other words, is $C = \inf\{C' \in \overline{\mathbf{C}}_{\mathrm{L}} : C \sqsubseteq C'\}$, or equivalently,

$$C(A) = \bigcup \{C'(A) : C' \in \overline{\mathbf{C}}_{\mathrm{L}} \text{ and } C \sqsubseteq C'\} \text{ for all } A \text{ in } \mathcal{Q}?$$

We will show in this section that, unfortunately and perhaps somewhat surprisingly, *this is generally not the case*, by studying in more detail the special case of coherent choice functions on two-dimensional option spaces.

In the remainder of this section, we concentrate on the two-dimensional option space $\mathcal{V} = \mathcal{L}(\mathcal{X})$ of gambles on an uncertain variable that can assume only two possible values $\mathcal{X} \coloneqq \{\mathrm{H}, \mathrm{T}\}$. Note that, by Proposition 16_{27}, the choice functions that we consider—the coherent and 'convex' choice functions—satisfy Property $C6_{25}$. As we have seen in Section 2.9_{76}, coherent choice functions are uniquely determined by their rejection set, and since the choice functions we consider also satisfy Property $C6_{25}$, Proposition 69_{79} guarantees that for binary possibility spaces, its rejection set consists of sets of cardinality two or three.

4.4.1 *An equivalent characterisation: coordinate rejection sets*

It will be useful to give a slightly different characterisation of rejection sets, tailored towards two-dimensional option spaces. Instead of describing the gambles that reject 0 directly, this new characterisation will rather use Axiom C4a$_{20}$ to rescale gambles in the second and fourth quadrants

$$\mathcal{L}_{\mathrm{II}} \coloneqq \{f \in \mathcal{L}(\mathcal{X}) : f(\mathrm{H}) < 0 < f(\mathrm{T})\} \text{ and } \mathcal{L}_{\mathrm{IV}} \coloneqq \{f \in \mathcal{L}(\mathcal{X}) : f(\mathrm{T}) < 0 < f(\mathrm{H})\},$$

obtaining variants that can be described more easily. Indeed, every gamble f_1 in $\mathcal{L}_{\mathrm{II}}$ can be uniquely described as $f_1 = \lambda_1(k_1 - 1, k_1)$ with λ_1 in $\mathbb{R}_{>0}$ and k_1 in $(0,1)$, and similarly, every gamble f_2 in $\mathcal{L}_{\mathrm{IV}}$ as $f_2 = \lambda_2(k_2, k_2 - 1)$ with λ_2 in $\mathbb{R}_{>0}$ and k_2 in $(0,1)$, as indicated by Figure 4.2. With this different characterisation, we will be able to prove useful additional properties, specific to two-dimensional vector spaces.

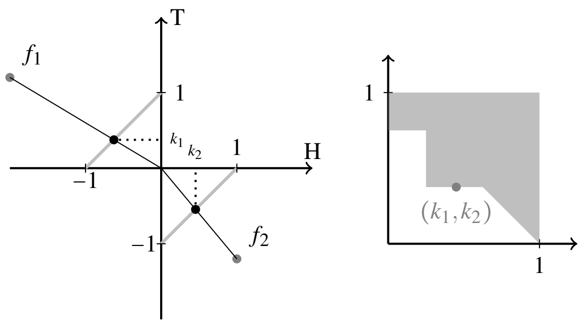


Figure 4.2: Scaling of the gambles (left) and coordinate rejection set (right)

Definition 36 (Coordinate rejection set)**.** *Given any coherent rejection function R, we define its* coordinate rejection set $K_R \subseteq [0,1)^2$ *as*[10,11]

$$K_R \coloneqq \{(k_1, k_2) \in [0,1)^2 : 0 \in R(\{(k_1 - 1, k_1), 0, (k_2, k_2 - 1)\})\}.$$

We will call any subset $K \subseteq [0,1)^2$ a coordinate rejection set. It will be useful to consider a number of potential properties of coordinate rejection sets K:

K1. *monotonicity*: if $(k_1, k_2) \in K$, $k_1' \geq k_1$ and $k_2' \geq k_2$, then also $(k_1', k_2') \in K$, for all (k_1, k_2) and (k_1', k_2') in $[0,1)^2$;

[10] In order to distinguish them from rejection sets, which we denoted generically by $\mathbb{K}$, we will use the generic notation K for coordinate rejection sets. As we have seen in Proposition 69$_{79}$, the binary coordinate rejection set K is related to $\mathbb{K}_2 \cup \mathbb{K}_3$.

[11] I want to remind the reader at this point that in the present context $0 = (0,0)$ is the null vector in $\mathcal{V}$. Both notations will be used interchangeably.

K2. *non-triviality*: $(0,0) \notin K$;

K3. a. for all a, b and c in $[0,1)$ such that $c < a$, $a+b < 1$, $(b,a) \in K$ and $(1-a,c) \in K$:

$$(x,c) \in K \text{ for all } x \text{ in } (b,1) \text{ and } (b,y) \in K \text{ for all } y \text{ in } (c,1);$$

b. for all a and c in $[0,1)$ such that $c < a$, $(0,a) \in K$ and $(1-a,c) \in K$:

$$(0,c) \in K;$$

c. for all a and b in $[0,1)$ such that $0 < a$, $a+b < 1$, $(b,a) \in K$ and $(1-a,0) \in K$:

$$(b,0) \in K;$$

K4. if $k_1 + k_2 > 1$ then $(k_1,k_2) \in K$, for all (k_1,k_2) in $[0,1)^2$.

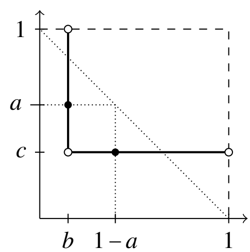


Figure 4.3: Illustration of Property K3a

Properties K2 and K3 imply the following useful property:

Lemma 110. *Consider any coordinate rejection set $K \subseteq [0,1)^2$. If K satisfies Properties* K2 *and* K3*, then*

$$(\forall a \in [0,1])((0,a) \notin K \textit{ or } (1-a,0) \notin K).$$

Proof. If $a = 0$ then $(0,a) = (0,0) \notin K$ by Property K2. Analogously, if $a = 1$ then $(1-a,0) = (0,0) \notin K$ by Property K2. Assume therefore that $a \in (0,1)$, and assume *ex absurdo* that both $(0,a)$ and $(1-a,0)$ are elements of K. Use Property K3b [with $c := 0$; then indeed $c = 0 < a$, $(0,a) \in K$ and $(1-a,c) = (1-a,0) \in K$] to infer that $(0,0) \in K$, which contradicts Property K2. □

The coherence of R—and the two extra Properties R5$_{25}$ and R6$_{25}$—implies a number of corresponding properties of its coordinate rejection set K_R:

Proposition 111. *Consider any coherent rejection function R. Then its coordinate rejection set K_R satisfies Properties* K1 *and* K2*. Furthermore, if R satisfies Property* R6$_{25}$*, then K_R satisfies Property* K3*. Finally, if R satisfies Property* R5$_{25}$*, then K_R satisfies Properties* K3 *and* K4.

Proof. We first prove that K_R satisfies Property $\mathrm{K1}_{148}$. Consider any (k_1,k_2) in K_R, and any (k_1',k_2') in $[0,1)^2$ such that $k_1' \geq k_1$ and $k_2' \geq k_2$. Then $(k_1,k_2) \in K_R$ simply means that $0 \in R(\{(k_1-1,k_1),0,(k_2,k_2-1)\})$, and $k_1' \geq k_1$ and $k_2' \geq k_2$ implies that $(k_1'-1,k_1') \geq (k_1-1,k_1)$ and $(k_2'-1,k_2') \geq (k_2-1,k_2)$. Proposition 30_{41}(ii) tells us that then $0 \in R(\{(k_1'-1,k_1'),0,(k_2',k_2'-1)\})$, whence indeed $(k_1',k_2') \in K_R$.

To prove that K_R satisfies Property $\mathrm{K2}_{\curvearrowleft}$, assume *ex absurdo* that $0 \in K_R$, or equivalently, that $0 \in R(\{(-1,0),0,(0,-1)\})$. Since $(-1,0) < 0$, we infer from Axiom $\mathrm{R2}_{20}$ that $(-1,0) \in R(\{(-1,0),0\})$, and therefore also that $(-1,0) \in R(\{(-1,0),0,(0,-1)\})$, by Axiom $\mathrm{R3a}_{20}$. A similar argument leads from $(0,-1) < 0$ to $(0,-1) \in R(\{(-1,0),0,(0,-1)\})$. Combining these three statements leads to the conclusion that $\{(-1,0),0,(0,-1)\} = R(\{(-1,0),0,(0,-1)\})$, which contradicts Axiom $\mathrm{R1}_{20}$.

Next, assume that R satisfies Property $\mathrm{R6}_{25}$. To prove that K_R then satisfies Property $\mathrm{K3}_{\curvearrowleft}$, we first prove that it satisfies Property $\mathrm{K3a}_{\curvearrowleft}$. Consider any a, b and c in $[0,1)$ and assume that $c < a$, $a+b < 1$, and that (b,a) and $(1-a,c)$ belong to K_R. We are going to prove that $(b,y) \in K_R$ for every y in $(c,1)$; the proof that also $(x,c) \in K_R$ for every x in $(b,1)$ is similar. Consider any λ in $\mathbb{R}_{>0}$, then Property $\mathrm{R6}_{25}$ guarantees that

$$0 \in R(\{(b-1,b),0,\lambda(a,a-1)\}) \text{ and } 0 \in R(\{\lambda(-a,1-a),0,(c,c-1)\}) \tag{4.14}$$

By Axiom R4b, we then find that

$$\begin{cases} -\lambda(a,a-1) \in R(\{(b-\lambda a-1,b-\lambda a+\lambda),-\lambda(a,a-1),0\}) \\ \lambda(a,a-1) \in R(\{0,\lambda(a,a-1),(c+\lambda a,c+\lambda a-\lambda-1)\}), \end{cases}$$

and applying Axiom $\mathrm{R3a}_{20}$ then leads to

$$\begin{aligned} &\{-\lambda(a,a-1),\lambda(a,a-1)\} \\ &\subseteq R(\{(b-\lambda a-1,b-\lambda a+\lambda),-\lambda(a,a-1),0,\lambda(a,a-1),(c+\lambda a,c+\lambda a-\lambda-1)\}). \end{aligned} \tag{4.15}$$

Now use Equations (4.14) and (4.15) together with Axiom $\mathrm{R3a}_{20}$ to infer that

$$\begin{aligned} \{-\lambda(a,a-1),0,\lambda(a,a-1)\} \subseteq R(\{&(b-\lambda a-1,b-\lambda a+\lambda), \\ &-\lambda(a,a-1),0,(c,c-1),\lambda(a,a-1),(c+\lambda a,c+\lambda a-\lambda-1)\}). \end{aligned}$$

Applying Axiom $\mathrm{R3b}_{20}$ leads to

$$\begin{aligned} &-\lambda(a,a-1) \\ &\quad \in R(\{(b-\lambda a-1,b-\lambda a+\lambda),-\lambda(a,a-1),(c,c-1),(c+\lambda a,c+\lambda a-\lambda-1)\}) \end{aligned}$$

and by Axiom R4b this implies that

$$0 \in R(\{(b-1,b),0,(c+\lambda a,c+\lambda a-\lambda-1),(c+2\lambda a,c+2\lambda a-2\lambda-1)\}). \tag{4.16}$$

Let us call $u := (c+\lambda a,c+\lambda a-\lambda-1)$ and $v := (c+2\lambda a,c+2\lambda a-2\lambda-1)$, and $\mu_1 := \frac{1}{c+\lambda a}$ and $\mu_2 := \frac{1}{c+2\lambda a}$; these real numbers are both positive since $0 \leq c < a$ and $\lambda > 0$. Then $0 \in R(\{(b-1,b),0,u,v\})$ by Equation (4.16), and $0 \in R(\{(b-1,b),0,\mu_1 u,\mu_2 v\})$ by

Property R6$_{25}$. But $\mu_1 u < \mu_2 v$ since $\mu_1 u = (1, \frac{c+\lambda a-\lambda-1}{c+\lambda a})$ and $\mu_2 v = (1, \frac{c+2\lambda a-2\lambda-1}{c+2\lambda a})$, and $\frac{c+\lambda a-\lambda-1}{c+\lambda a} < \frac{c+2\lambda a-2\lambda-1}{c+2\lambda a}$ because this statement is equivalent to

$$\begin{aligned} c^2 + 2\lambda ac + \lambda ac + 2\lambda^2 a^2 - \lambda c - 2\lambda^2 a - c - 2\lambda a \\ < c^2 + \lambda ac + 2\lambda ac + 2\lambda^2 a^2 - 2\lambda c - 2\lambda^2 a - c - \lambda a, \end{aligned}$$

which is in turn equivalent to $c < a$, which is one of the assumptions. Then $\mu_1 u \in R(\{\mu_1 u, \mu_2 v\})$ by Axiom R2$_{20}$, whence $\{0, \mu_1 u\} \subseteq R(\{(b-1,b), 0, \mu_1 u, \mu_2 v\})$ by Axiom R3a$_{20}$. Then $0 \in R(\{(b-1,b), 0, \mu_2 v\})$ by Axiom R3b$_{20}$, and $0 \in R(\{(b-1,b), 0, \mu_3 v\})$ by Property R6$_{25}$ with $\mu_3 = \frac{1}{2\lambda+1} > 0$, whence

$$\left(b, \frac{c+2\lambda a}{1+2\lambda}\right) \in K_R.$$

Now, by varying λ in $\mathbb{R}_{>0}$ the number $\frac{c+2\lambda a}{1+2\lambda}$ can take any value in the interval (c,a). We conclude that $(b,y) \in K_R$ for every $y \in (c,1)$, after also recalling that we have already proved that K_R satisfies Property K1$_{148}$.

To prove that K_R satisfies Property K3b$_{149}$, assume that $0 \le c < a < 1$, $(0,a) \in K_R$ and $(1-a,c) \in K_R$. Because K_R already satisfies Property K3a$_{149}$ [with in particular $b := 0$], we know that $(x,c) \in K_R$ for every x in $(0,1)$ and $(0,y) \in K_R$ for every y in $(c,1)$. We have to show that $(0,c) \in K_R$. To this end, consider the gambles $u := (\frac{1-c}{2} - 1, \frac{1-c}{2})$ and $v := (c, c-1)$. Because in particular $(x,c) \in K_R$ for $x = \frac{1-c}{2} \in (0,1)$, we have that $0 \in R(\{u,0,v\})$. Similarly, because in particular $(0,y) \in K_R$ for $y = \frac{1+c}{2} \in (c,1)$, we have that $0 \in R(\{(-1,0),0,-u\})$. Since also $(-1,0) \in R(\{(-1,0),0\})$—and therefore $(-1,0) \in R(\{(-1,0),0,-u\})$ by Axiom R3a$_{20}$—because $(-1,0) < 0$ and by Axiom R2$_{20}$, this leads us to conclude that $\{(-1,0),0\} \subseteq R(\{(-1,0),0,-u\})$, and therefore also $0 \in R(\{0,-u\})$ by Axiom R3b$_{20}$. Hence, $u \in R(\{u,0\})$, by Axiom R4b, and therefore $u \in R(\{u,0,v\})$, by Axiom R3a$_{20}$. Hence $\{0,u\} \subseteq R(\{u,0,v\})$, so Axiom R3b$_{20}$ leads to $0 \in R(\{0,v\})$. Now Axiom R3a$_{20}$ implies that indeed $(0,c) \in K_R$, so Property K3b$_{149}$ is satisfied. Property K3c$_{149}$ can be shown to hold in a similar way.

To conclude, assume that R satisfies Property R5$_{25}$. Since this implies that Property R6$_{25}$ holds, by Proposition 16$_{27}$, we already know that Property K3$_{149}$ is satisfied, so it only remains to prove that K_R satisfies Property K4$_{149}$. Consider any (k_1,k_2) in $[0,1)^2$ such that $k_1+k_2 > 1$. Then $\left(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}\right) > 0$, whence $0 \in R(\{0, \left(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}\right)\})$ by Axiom R2$_{20}$. By Axiom R3a$_{20}$, we get

$$0 \in R\left(\left\{(k_1-1,k_1), 0, (k_2,k_2-1), \left(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}\right)\right\}\right).$$

Since $\left(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}\right) \in \mathrm{conv}(\{(k_1-1,k_1),(k_2,k_2-1)\})$, Property R5$_{25}$ leads us to conclude that $0 \in R(\{(k_1-1,k_1),0,(k_2,k_2-1)\})$, so indeed $(k_1,k_2) \in K_R$. □

4.4.2 *From coordinate rejection sets to rejection functions*

Conversely, we now show how to associate a rejection function with any coordinate rejection set $K \subseteq [0,1)^2$. Taking into account Property K2$_{149}$, we only consider sets K that do not contain 0.

Definition 37. *Given any subset $K \subseteq [0,1)^2 \setminus \{0\}$, we define its corresponding rejection function R_K as follows. We let*

$$R_K(\{0\}) = \varnothing. \tag{4.17}$$

Next, for any A in $\mathcal{Q}_{\overline{0}}$, we let $0 \in R_K(A \cup \{0\})$ if at least one of the following conditions holds:

$$A \cap \mathcal{L}_{>0} \neq \varnothing \tag{4.18}$$

$$(\exists \lambda_1 \in \mathbb{R}_{>0}, (k_1, 0) \in K)\lambda_1(k_1 - 1, k_1) \in A \tag{4.19}$$

$$(\exists \lambda_2 \in \mathbb{R}_{>0}, (0, k_2) \in K)\lambda_2(k_2, k_2 - 1) \in A \tag{4.20}$$

$$(\exists \lambda_1, \lambda_2 \in \mathbb{R}_{>0}, (k_1, k_2) \in K \cap (0,1)^2)\{\lambda_1(k_1 - 1, k_1), \lambda_2(k_2, k_2 - 1)\} \subseteq A, \tag{4.21}$$

and finally, we allow for $R(A)$ to contain non-zero gambles by imposing the following condition:

$$(\forall A \in \mathcal{Q})(\forall f \in A) f \in R_K(A) \Leftrightarrow 0 \in R_K(A - \{f\}). \tag{4.22}$$

The intuition behind this is that the elements of K of the type $(k_1, 0)$ or $(0, k_2)$ determine gambles—$(k_1 - 1, k_1)$ and $(k_2, k_2 - 1)$, respectively—that allow us to reject 0; the other possibility of rejecting 0 is by means of the combined action of a gamble in the second quadrant—$(k_1 - 1, k_1)$ for k_1 in $(0,1)$—and one in the fourth quadrant—$(k_2, k_2 - 1)$ for k_2 in $(0,1)$.

Alternatively, as shown in Lemma 112, we can summarise Conditions (4.19)–(4.21) as

$$\begin{aligned}&(\exists \lambda_1, \lambda_2 \in \mathbb{R}_{>0}, (k_1, k_2) \in K)\\ &\qquad \{\lambda_1(k_1 - 1, k_1), \lambda_2(k_2, k_2 - 1)\} \subseteq A \cup \{(-1,0),(0,-1)\}.\end{aligned} \tag{4.23}$$

Lemma 112. *For any $K \subseteq [0,1)^2 \setminus \{0\}$ and any A in $\mathcal{Q}$, at least one of the Conditions* (4.19)–(4.21) *holds if and only if Condition* (4.23) *holds.*

Proof. If Condition (4.19) holds, then

$$\{\lambda_1(k_1 - 1, k_1), (0,-1)\} \subseteq A \cup \{(-1,0),(0,-1)\},$$

so Condition (4.23) holds with $\lambda_2 \coloneqq 1$ and $k_2 \coloneqq 0$. Similarly, if Condition (4.20) holds, then $\{(-1,0), \lambda_2(k_2, k_2 - 1)\} \subseteq A \cup \{(-1,0),(0,-1)\}$, so Condition (4.23) holds with $\lambda_1 \coloneqq 1$ and $k_1 \coloneqq 0$. If Condition (4.21) holds, then Condition (4.23) holds trivially.

Conversely, assume that Condition (4.23) holds. If both $k_1 \neq 0$ and $k_2 \neq 0$, then Condition (4.21) holds trivially, so assume that either $k_1 = 0$ or $k_2 = 0$—they cannot both be zero, because $0 \notin K$. So assume that $k_1 = 0$ and $k_2 > 0$, then we infer from the assumption that $\{\lambda_1(-1,0), \lambda_2(k_2, k_2 - 1)\} \subseteq A \cup \{(-1,0),(0,-1)\}$. Since $k_2 > 0$ implies that $\lambda_2(k_2, k_2 - 1) \neq (-1,0)$ and $\lambda_2(k_2, k_2 - 1) \neq (0,-1)$ for any choice of $\lambda_2 > 0$, it must be that $\lambda_2(k_2, k_2 - 1) \in A$, so Condition (4.20) holds. The case $k_2 = 0$ and $k_1 > 0$ can be treated similarly, and leads to the conclusion that Condition (4.19) holds. □

Now that we know how to associate with a coordinate rejection set K a rejection function R_K, let us determine which conditions on K ensure the coherence of R_K. We begin by showing that a number of coherence axioms follow directly from the definition, irrespective of the choice of the coordinate rejection set $K \subseteq [0,1)^2 \setminus \{0\}$:

Proposition 113. *Consider any subset $K \subseteq [0,1)^2 \setminus \{0\}$. Then the rejection function R_K given by Definition 37 satisfies Axioms* R2$_{20}$*,* R3a$_{20}$*,* R4a*,* R4b *and Property* R6$_{25}$.

Proof. For Axiom R2$_{20}$, consider any f and g in $\mathcal{L}$ such that $f < g$. Then $0 < g - f$, so we infer from Condition (4.18) that $0 \in R_K(\{0, g-f\})$, and then from Condition (4.22) that indeed $f \in R_K(\{f,g\})$.

For Axiom R3a$_{20}$, assume that $A_1 \subseteq R_K(A_2)$ and $A_2 \subseteq A$. Then we need to prove that $A_1 \subseteq R_K(A)$. Consider any $f \in A_1$, then also $f \in A_2$ and $f \in A$, so we can let $A_2' := A_2 - \{f\}$ and $A' := A - \{f\}$, where $A_2' \subseteq A'$. We then infer from Condition (4.22) that $0 \in R_K(A_2')$, which means that at least one of the Conditions (4.18)–(4.21) holds. But any of these conditions implies that also $0 \in R_K(A')$. Condition (4.22) then guarantees that $f \in R_K(A)$ and therefore that, indeed, $A_1 \subseteq R_K(A)$.

That Axioms R4a and R4b are satisfied follows immediately from Conditions (4.18)–(4.22).

For Property R6$_{25}$, consider any option set $A = \{f_1, \dots, f_n\} \in \mathcal{Q}$, where n is a natural number, and any positive real numbers $\mu_1, \dots, \mu_n$. Assume that $0 \in R_K(\{0\} \cup A)$. First of all, if $f_i \in \mathcal{L}_{>0}$ for some i in $\{1, \dots, n\}$, then also $\mu_i f_i \in \mathcal{L}_{>0}$ since $\mu_i > 0$, whence indeed $0 \in R_K(\{0, \mu_1 f_1, \dots, \mu_n f_n\})$, by Condition (4.18). So assume that $f_i \notin \mathcal{L}_{>0}$ for all i in $\{1, \dots, n\}$. There are now only three possibilities. The first is that there are λ_1 in $\mathbb{R}_{>0}$ and $(k_1, 0)$ in K such that $\lambda_1(k_1 - 1, k_1) = f_i$ for some i in $\{1, \dots, n\}$. Then $(\lambda_1 \mu_i)(k_1 - 1, k_1) = \mu_i f_i \in \{\mu_1 f_1, \dots, \mu_n f_n\}$, and Condition (4.19) guarantees that indeed $0 \in R_K(\{0, \mu_1 f_1, \dots, \mu_n f_n\})$. The second possibility is that there are λ_2 in $\mathbb{R}_{>0}$ and $(0, k_2)$ in K such that $\lambda_2(k_2, k_2 - 1) = f_j$ for some j in $\{1, \dots, n\}$. Then $(\lambda_2 \mu_j)(k_2, k_2 - 1) = \mu_j f_j \in \{\mu_1 f_1, \dots, \mu_n f_n\}$, and Condition (4.20) guarantees that indeed $0 \in R_K(\{0, \mu_1 f_1, \dots, \mu_n f_n\})$. And the final possibility is that there are λ_1 and λ_2 in $\mathbb{R}_{>0}$ and (k_1, k_2) in $K \cap (0,1)^2$ such that $\lambda_1(k_1 - 1, k_1) = f_i$ and $\lambda_2(k_2, k_2 - 1) = f_j$ for some i and j in $\{1, \dots, n\}$. Then $(\lambda_1 \mu_i)(k_1 - 1, k_1) = \mu_i f_i$ and $(\lambda_2 \mu_j)(k_2, k_2 - 1) = \mu_j f_j$, and Condition (4.21) guarantees that indeed $0 \in R_K(\{0, \mu_1 f_1, \dots, \mu_n f_n\})$. □

It turns out that Properties K1$_{148}$–K3$_{149}$ are sufficient for coherence:

Proposition 114. *Consider any subset K of $[0,1)^2$ that satisfies Properties* K1$_{148}$–K3$_{149}$*. Then the rejection function R_K given by Definition 37 satisfies Axioms* R3b$_{20}$ *and* R1$_{20}$.

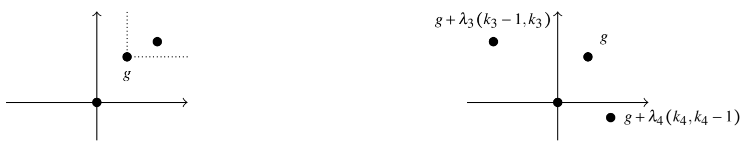
Proof. We begin by proving that R_K satisfies Axiom R3b$_{20}$. Assume *ex absurdo* that it does not, then Proposition 25$_{39}$ guarantees that there are A in $\mathcal{Q}$ and g in $A \setminus \{0\}$ such that $\{0, g\} \subseteq R_K(A)$ and $0 \notin R_K(A \setminus \{g\})$ [because Proposition 113 guarantees that R_K satisfies, amongst other things, Axiom R4].

Because $0 \in R_K(A)$, we infer from Definition 37_{152} and Lemma 112_{152} that there are two possibilities: (i) $A \cap \mathcal{L}_{>0} \neq \varnothing$, or (ii) $\{\lambda_1(k_1-1,k_1), \lambda_2(k_2,k_2-1)\} \subseteq A \cup \{(-1,0),(0,-1)\}$ for some λ_1 and λ_2 in $\mathbb{R}_{>0}$ and some (k_1,k_2) in K.

Case (i) Case (ii)

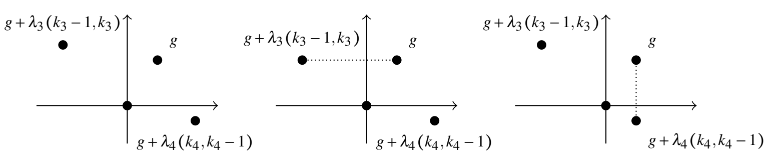
We first deal with case (i). Here we can assume without loss of generality that $A \cap \mathcal{L}_{>0} = \{g\}$ because, otherwise $A \smallsetminus \{g\} \cap \mathcal{L}_{>0} \neq \varnothing$ and we could apply Condition $(4.18)_{152}$ to conclude that $0 \in R_K(A \smallsetminus \{g\})$, a contradiction. We will use the notation $g = (x,y) > 0$.

Because also $g \in R_K(A)$, Condition $(4.22)_{152}$ guarantees that $0 \in R_K(A - \{g\})$, and a similar argument as before shows that there are now two possibilities: (i.a) $(A - \{g\}) \cap \mathcal{L}_{>0} \neq \varnothing$; and (i.b) $\{\lambda_3(k_3-1,k_3), \lambda_4(k_4,k_4-1)\} \subseteq (A-\{g\}) \cup \{(-1,0),(0,-1)\}$ for some λ_3 and λ_4 in $\mathbb{R}_{>0}$ and some (k_3,k_4) in K.



Case (i.a) Case (i.b)

But in fact (i.a) is impossible, because it would contradict our earlier conclusion that $A \cap \mathcal{L}_{>0} = \{g\}$. So we can restrict our attention to case (i.b) with $(A-\{g\}) \cap \mathcal{L}_{>0} = \varnothing$. There are now 3 possibilities: (i.b.1) $k_3 \neq 0 \neq k_4$ corresponding to Condition $(4.21)_{152}$, (i.b.2) $k_3 = 0 \neq k_4$ corresponding to Condition $(4.20)_{152}$, and (i.b.3) $k_3 \neq 0 = k_4$ corresponding to Condition $(4.19)_{152}$—$k_3 = 0 = k_4$ is impossible because $0 \notin K$.



Case (i.b.1) Case (i.b.2) Case (i.b.3)

For case (i.b.1), it must be that $\{\lambda_3(k_3-1,k_3), \lambda_4(k_4,k_4-1)\} \subseteq A - \{g\}$, or equivalently, $\{g+\lambda_3(k_3-1,k_3), g+\lambda_4(k_4,k_4-1)\} \subseteq A$, so

$$\{(\lambda_3k_3+x-\lambda_3, \lambda_3k_3+y), (\lambda_4k_4+x, \lambda_4k_4+y-\lambda_4)\} \subseteq A.$$

Since $\lambda_3 > 0$, we have that $(\lambda_3k_3+x-\lambda_3, \lambda_3k_3+y) \neq g$ and, similarly, since $\lambda_4 > 0$, we have that $(\lambda_4k_4+x, \lambda_4k_4+y-\lambda_4) \neq g$. Therefore,

$$\{(\lambda_3k_3+x-\lambda_3, \lambda_3k_3+y), (\lambda_4k_4+x, \lambda_4k_4+y-\lambda_4)\} \subseteq A \smallsetminus \{g\}.$$

But $(A \setminus \{g\}) \cap \mathcal{L}_{>0} = \varnothing$, so $(\lambda_3 k_3 + x - \lambda_3, \lambda_3 k_3 + y) \notin \mathcal{L}_{>0}$ and $(\lambda_4 k_4 + x, \lambda_4 k_4 + y - \lambda_4) \notin \mathcal{L}_{>0}$. First, we focus on $(\lambda_3 k_3 + x - \lambda_3, \lambda_3 k_3 + y)$ and wonder in what quadrant it lies. Its second component $\lambda_3 k_3 + y$ is positive, since $\lambda_3 > 0$, $k_3 > 0$ and $y \geq 0$. Therefore, since it is no element of $\mathcal{L}_{>0}$, it must lie in the second quadrant, so its first component $\lambda_3 k_3 + x - \lambda_3$ must be negative: $\lambda_3 k_3 < \lambda_3 - x$. But $\lambda_3 k_3 > 0$, so $0 < \lambda_3 - x$, and combining this with $-y \leq 0$, we find that $-y < \lambda_3 - x$, so $y - x + \lambda_3 > 0$. Next, we turn to $(\lambda_4 k_4 + x, \lambda_4 k_4 + y - \lambda_4)$ and wonder it what quadrant it lies. Its first component $\lambda_4 k_4 + x$ is positive, since $\lambda_4 > 0$, $k_4 > 0$ and $x \geq 0$. Therefore, since it is no element of $\mathcal{L}_{>0}$, it must lie in the fourth quadrant, so its second component $\lambda_4 k_4 + y - \lambda_4$ must be negative: $\lambda_4 k_4 < \lambda_4 - y$. But $\lambda_4 k_4 > 0$, so $0 < \lambda_4 - y$. Combining this with $-x \leq 0$, we find that $-x < \lambda_4 - y$, so $x - y + \lambda_4 > 0$. In summary, we have shown that $\lambda_3^* \coloneqq y - x + \lambda_3 > 0$ and $\lambda_4^* \coloneqq x - y + \lambda_4 > 0$, and therefore

$$\left\{(y-x+\lambda_3)\Big(\frac{\lambda_3 k_3 + x - \lambda_3}{y-x+\lambda_3}, \frac{\lambda_3 k_3 + y}{y-x+\lambda_3}\Big), (x-y+\lambda_4)\Big(\frac{\lambda_4 k_4 + x}{x-y+\lambda_4}, \frac{\lambda_4 k_4 + y - \lambda_4}{x-y+\lambda_4}\Big)\right\} \subseteq A \setminus \{g\}.$$

If we let $k_3^* \coloneqq \frac{\lambda_3 k_3 + y}{y-x+\lambda_3}$ and $k_4^* \coloneqq \frac{\lambda_4 k_4 + x}{x-y+\lambda_4}$, then $\frac{\lambda_3 k_3 + x - \lambda_3}{y-x+\lambda_3} = k_3^* - 1$ and $\frac{\lambda_4 k_4 + y - \lambda_4}{x-y+\lambda_4} = k_4^* - 1$, so

$$\{\lambda_3^*(k_3^* - 1, k_3^*), \lambda_4^*(k_4^*, k_4^* - 1)\} \subseteq A \setminus \{g\}. \tag{4.24}$$

We claim that $k_3^* > k_3$. Indeed, infer from $k_3 \in (0,1)$ and $(x,y) > 0$ that $k_3 x + (1-k_3)y > 0$, and therefore

$$y > k_3 y - k_3 x \Rightarrow \lambda_3 k_3 + y > \lambda_3 k_3 + k_3 y - k_3 x = k_3(y - x + \lambda_3) \Rightarrow k_3^* = \frac{\lambda_3 k_3 + y}{y - x + \lambda_3} > k_3.$$

Similarly, we claim that $k_4^* > k_4$. Indeed, $k_4 \in (0,1)$ and $(x,y) > 0$ imply that $(1-k_4)x + k_4 y > 0$, and therefore

$$x > k_4 x - k_4 y \Rightarrow \lambda_4 k_4 + x > \lambda_4 k_4 + k_4 x - k_4 y = k_4(x - y + \lambda_4) \Rightarrow k_4^* = \frac{\lambda_4 k_4 + x}{x - y + \lambda_4} > k_4.$$

Therefore $(k_3^*, k_4^*) > (k_3, k_4)$. Since we already know that K is increasing [Property K1$_{148}$] and $(k_3, k_4) \in K$, we conclude from this that $(k_3^*, k_4^*) \in K$, and therefore also, by Condition (4.21)$_{152}$ and Equation (4.24), that $0 \in R_K(A \setminus \{g\})$, a contradiction.

For case (i.b.2), it must be that $\lambda_4(k_4, k_4 - 1) \in A - \{g\}$, or equivalently, that $g + \lambda_4(k_4, k_4 - 1) \in A$, so $(\lambda_4 k_4 + x, \lambda_4 k_4 + y - \lambda_4) \in A$. Since $\lambda_4 > 0$ and $k_4 > 0$, we may conclude that $(\lambda_4 k_4 + x, \lambda_4 k_4 + y - \lambda_4) \neq g$, and therefore

$$(\lambda_4 k_4 + x, \lambda_4 k_4 + y - \lambda_4) \in A \setminus \{g\}.$$

But $(A \setminus \{g\}) \cap \mathcal{L}_{>0} = \varnothing$, so $(\lambda_4 k_4 + x, \lambda_4 k_4 + y - \lambda_4)$ is no element of $\mathcal{L}_{>0}$. Its first component $\lambda_4 k_4 + x$ is positive, since $\lambda_4 > 0$, $k_4 > 0$ and $y \geq 0$. It must therefore lie in the fourth quadrant, so its second component $\lambda_4 k_4 + y - \lambda_4$ must be negative: $\lambda_4 k_4 < \lambda_4 - y$. But $0 < \lambda_4 k_4$, and therefore $0 < \lambda_4 - y$. Combining this with $-x \leq 0$, we conclude that $-x < \lambda_4 - y$, so $\lambda_4^* \coloneqq x - y + \lambda_4 > 0$, and therefore

$$(x - y + \lambda_4)\Big(\frac{\lambda_4 k_4 + x}{x - y + \lambda_4}, \frac{\lambda_4 k_4 + y - \lambda_4}{x - y + \lambda_4}\Big) \in A \setminus \{g\}.$$

If we let $k_4^* := \frac{\lambda_4 k_4 + x}{x - y + \lambda_4}$, then $\frac{\lambda_4 k_4 + y - \lambda_4}{x - y + \lambda_4} = k_4^* - 1$, so

$$\lambda_4^*(k_4^*, k_4^* - 1) \in A \setminus \{g\}. \tag{4.25}$$

We claim that $k_4^* > k_4$. Indeed, infer from $k_4 \in (0,1)$ and $(x,y) > 0$ that $(1-k_4)x + k_4 y > 0$, and therefore

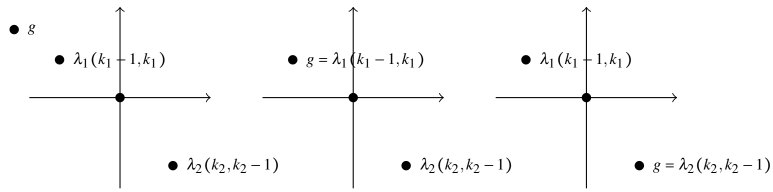
$$x > k_4 x - k_4 y \Rightarrow \lambda_4 k_4 + x > \lambda_4 k_4 + k_4 x - k_4 y = k_4(x - y + \lambda_4) \Rightarrow k_4^* = \frac{\lambda_4 k_4 + x}{x - y + \lambda_4} > k_4.$$

Therefore $(0, k_4^*) > (0, k_4)$. Since we already know that K is increasing [Property K1$_{148}$] and $(0, k_4) \in K$, we conclude from this that $(0, k_4^*) \in K$, and therefore also, by Condition (4.20)$_{152}$ and Equation (4.25), that $0 \in R_K(A \setminus \{g\})$, a contradiction.

For case (i.b.3), we obtain a contradiction in a similar fashion as in case (i.b.2).

We see that we get a contradiction is all cases (i.b.1)–(i.b.3), so case (i.b) always leads to a contradiction, as did case (i.a). This allows us to conclude that case (i) always leads to a contradiction.

We now turn to case (ii), where we assume that $A \cap \mathcal{L}_{>0} = \varnothing$ and that there are λ_1 and λ_2 in $\mathbb{R}_{>0}$ and (k_1, k_2) in K such that $\{\lambda_1(k_1 - 1, k_1), \lambda_2(k_2, k_2 - 1)\} \subseteq A \cup \{(-1,0),(0,-1)\}$. Here we distinguish between three possibilities: (ii.a) $g \notin \{\lambda_1(k_1 - 1, k_1), \lambda_2(k_2, k_2 - 1)\}$, (ii.b) $g = \lambda_1(k_1 - 1, k_1)$, and (ii.c) $g = \lambda_2(k_2, k_2 - 1)$. They are mutually exclusive because it is impossible that $\lambda_1(k_1 - 1, k_1) = \lambda_2(k_2, k_2 - 1)$ [because that would imply $\lambda_1 + \lambda_2 = 0$].



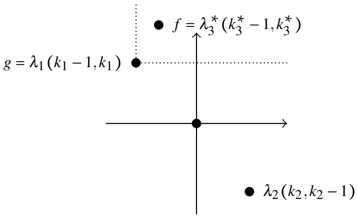
Case (ii.a) Case (ii.b) Case (ii.c)

But we see at once that case (ii.a) is impossible, because it implies by Condition (4.23)$_{152}$ that $0 \in R_K(A \setminus \{g\})$, a contradiction. So we now concentrate on the cases (ii.b) and (ii.c), where it is by the way obvious that indeed

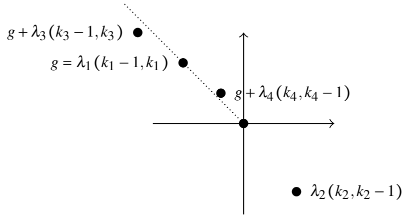
$$A \cap \mathcal{L}_{>0} = \varnothing. \tag{4.26}$$

We begin with the discussion of case (ii.b). We first of all claim that now $k_1 > 0$. Indeed, if $k_1 = 0$ then $(k_1, k_2) = (0, k_2) \in K$, and Property K2$_{149}$ implies that $k_2 > 0$. Since we know that in this case $\lambda_2(k_2, k_2 - 1) \in A \setminus \{g\}$ [since $g = \lambda_1(k_1 - 1, k_1) \neq \lambda_2(k_2, k_2 - 1)$], Condition (4.20)$_{152}$ guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

So we may assume that $k_1 > 0$, and the assumption that $g \in R_K(A)$, or in other words, that $0 \in R_K(A - \{g\})$, leaves us with two possibilities: that (ii.b.1) $(A - \{g\}) \cap \mathcal{L}_{>0} \neq \varnothing$, or that (ii.b.2) $\{\lambda_3(k_3 - 1, k_3), \lambda_4(k_4, k_4 - 1)\} \subseteq (A - \{g\}) \cup \{(-1,0),(0,-1)\}$ for some λ_3 and λ_4 in $\mathbb{R}_{>0}$ and (k_3, k_4) in K [use Condition (4.22)$_{152}$, Definition 37$_{152}$ and Lemma 112$_{152}$].



Case (ii.b.1)



Case (ii.b.2)

For case (ii.b.1), there is some $h \coloneqq (x', y') > 0$ such that $f \coloneqq g + h \in A$. Since the second component $\lambda_1 k_1 + y'$ of f is positive and $f \notin \mathcal{L}_{>0}$ by Equation (4.26), we find that f must lie in the second quadrant, and therefore its first component $\lambda_1 k_1 - \lambda_1 + x'$ is negative: $\lambda_1 k_1 < \lambda_1 - x'$ and therefore $\lambda_3^* \coloneqq \lambda_1 - x' + y' > 0$. If we now let $k_3^* \coloneqq \frac{\lambda_1 k_1 + y'}{\lambda_1 - x' + y'}$, then $f = \lambda_3^*(k_3^* - 1, k_3^*)$. Moreover, $k_3^* < 1$ because this is equivalent to $\lambda_1 k_1 - \lambda_1 + x' < 0$, which we have already found to be true. Similarly, $k_3^* \geq k_1$ because this is equivalent to $x' k_1 + y'(1 - k_1) \geq 0$. Then $(k_3^*, k_2) \in K$ because $(k_1, k_2) \in K$ and K is increasing [Property K1$_{148}$]. Since we now know that $\{\lambda_3^*(k_3^* - 1, k_3^*), \lambda_2(k_2, k_2 - 1)\} \subseteq A \setminus \{g\}$, Condition (4.21)$_{152}$ guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

For case (ii.b.2), $\{g + \lambda_3(k_3 - 1, k_3), g + \lambda_4(k_4, k_4 - 1)\} \subseteq A \cup \{g + (-1, 0), g + (0, -1)\}$, or in other words,

$$\{(\lambda_1 k_1 + \lambda_3 k_3 - \lambda_1 - \lambda_3, \lambda_1 k_1 + \lambda_3 k_3), (\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4)\} \\ \subseteq A \cup \{g + (-1, 0), g + (0, -1)\} \quad (4.27)$$

We claim that here

$$k_3 < k_1. \quad (4.28)$$

To prove this, assume *ex absurdo* that $k_3 \geq k_1$, then also $k_3^* \coloneqq \frac{\lambda_1 k_1 + \lambda_3 k_3}{\lambda_1 + \lambda_3} \geq k_1 > 0$. Moreover, $k_3^* < 1$ because it is a convex combination of $k_1 < 1$ and $k_3 < 1$, and therefore $(k_3^*, k_2) \in [0, 1)^2 \setminus \{0\}$ and $(k_3^*, k_2) \geq (k_1, k_2)$. Then $(k_3^*, k_2) \in K$ because $(k_1, k_2) \in K$ and K is increasing [Property K1$_{148}$]. Moreover, if we also let $\lambda_3^* \coloneqq \lambda_1 + \lambda_3 > 0$, then Equation (4.27) tells us that

$$\lambda_3^*(k_3^* - 1, k_3^*) = g + \lambda_3(k_3 - 1, k_3) \in A \cup \{g + (-1, 0), g + (0, -1)\},$$

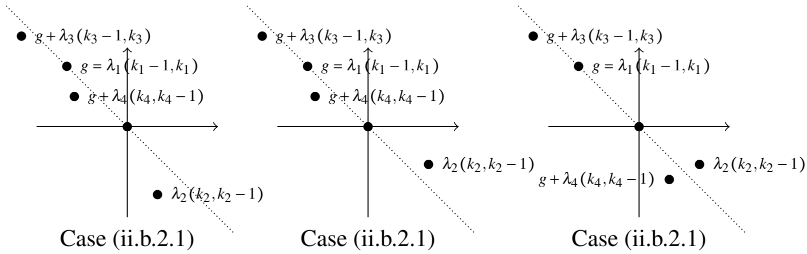
and since we know that $\lambda_3(k_3 - 1, k_3) \notin \{(-1, 0), 0, (0, -1)\}$ [because $\lambda_3 > 0$ and $k_3 \geq k_1 > 0$], this leads us to conclude that $\{\lambda_3^*(k_3^* - 1, k_3^*), \lambda_2(k_2, k_2 - 1)\} \subseteq A \setminus \{g\}$, so Condition (4.21)$_{152}$ together with $(k_3^*, k_2) \in K$ guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

Since Equation (4.28) rules out the possibility that $k_1 = 0$, we find that $k_1 > 0$ as an intermediate result. In the remainder of this case (ii.b), note that nothing depends on whether $k_2 = 0$ or $k_2 > 0$. We can now distinguish between three *distinct* possibilities: (ii.b.2.1) $k_3 > 0$ and $k_4 > 0$, (ii.b.2.2) $k_3 = 0$ and $k_4 > 0$, and (ii.b.2.3) $k_3 > 0$ and $k_4 = 0$, which correspond to Conditions (4.21)$_{152}$, (4.20)$_{152}$ and (4.19)$_{152}$, respectively—$k_3 = 0 = k_4$ is impossible because $0 \notin K$.

In case (ii.b.2.1) we see that $\{\lambda_3(k_3 - 1, k_3), \lambda_4(k_4, k_4 - 1)\} \cap \{(-1, 0), 0, (0, -1)\} = \emptyset$, and therefore Equation (4.27) leads to

$$\{(\lambda_1 k_1 + \lambda_3 k_3 - \lambda_1 - \lambda_3, \lambda_1 k_1 + \lambda_3 k_3), (\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4)\} \subseteq A \setminus \{g\}.$$

We distinguish between two possibilities, which will determine in what quadrants these points lie: $\lambda_4 \le \lambda_1$ and $\lambda_4 > \lambda_1$.



Case (ii.b.2.1) $\lambda_4 \le \lambda_1$ $k_1+k_2<1$ | Case (ii.b.2.1) $\lambda_4 \le \lambda_1$ $k_1+k_2\ge 1$ | Case (ii.b.2.1) $\lambda_4 > \lambda_1$

If $\lambda_4 \le \lambda_1$, then we claim that

$$k_4 \le 1-k_1. \tag{4.29}$$

To prove this, assume *ex absurdo* that $k_4 > 1-k_1$, so $k_1+k_4-1>0$. If $\lambda_1=\lambda_4$, then $(\lambda_1k_1+\lambda_4k_4-\lambda_1,\lambda_1k_1+\lambda_4k_4-\lambda_4)=\lambda_1(k_1+k_4-1,k_1+k_4-1)>0$, contradicting Equation $(4.26)_{156}$, so indeed $\lambda_4<\lambda_1$. We wonder in what quadrant $(\lambda_1k_1+\lambda_4k_4-\lambda_1,\lambda_1k_1+\lambda_4k_4-\lambda_4)\neq 0$ lies. Infer from $k_1+k_4>1$ and $0<\lambda_4<\lambda_1$ that $\lambda_1k_1+\lambda_4k_4-\lambda_4>\lambda_4(k_1+k_4)-\lambda_4>0$. Since $A\cap\mathcal{L}_{>0}=\emptyset$ by Equation $(4.26)_{156}$, we find that $(\lambda_1k_1+\lambda_4k_4-\lambda_1,\lambda_1k_1+\lambda_4k_4-\lambda_4)$ must lie in the second quadrant, and therefore its first component $\lambda_1k_1+\lambda_4k_4-\lambda_1$ must be negative: $\lambda_1k_1+\lambda_4k_4<\lambda_1$. This tells us that $k_4^*\coloneqq\frac{\lambda_1k_1+\lambda_4k_4-\lambda_4}{\lambda_1-\lambda_4}<1$. Moreover, $k_4^*>k_1$ because this is equivalent to $k_4>1-k_1$. Hence $(k_4^*,k_2)\in[0,1)^2\setminus\{0\}$ and $(k_4^*,k_2)>(k_1,k_2)$. This tells us that $(k_4^*,k_2)\in K$ because $(k_1,k_2)\in K$ and K is increasing [Property $\mathrm{K1}_{148}$]. If we now let $\lambda_4^*\coloneqq\lambda_1-\lambda_4>0$, then we see that $\lambda_4^*(k_4^*-1,k_4^*)=(\lambda_1k_1+\lambda_4k_4-\lambda_1,\lambda_1k_1+\lambda_4k_4-\lambda_4)\in A\setminus\{g\}$. Hence also $\{\lambda_4^*(k_4^*-1,k_4^*),\lambda_2(k_2,k_2-1)\}\subseteq A\setminus\{g\}$, and Condition $(4.21)_{152}$ now guarantees that $0\in R_K(A\setminus\{g\})$, a contradiction.

So we see that $0<k_4\le 1-k_1<1$, and hence, because K is increasing [Property $\mathrm{K1}_{148}$], we infer from $(k_3,k_4)\in K$ that $(k_3,1-k_1)\in K$. We distinguish between two further possibilities: $k_1+k_2<1$ and $k_1+k_2\ge 1$.

If $k_1+k_2<1$ then we can use Property $\mathrm{K3a}_{149}$ with $a=1-k_1$, $b=k_3$ and $c=k_2$. Observe that $a+b=1-k_1+k_3<1$ by Equation $(4.28)_{\curvearrowleft}$, that $c=k_2<1-k_1=a$ by assumption, that $(b,a)=(k_3,1-k_1)\in K$ has been proved above, and that $(1-a,c)=(k_1,k_2)\in K$ also by assumption, whence

$$(\forall k_3'\in(k_3,1))(k_3',k_2)\in K.$$

In particular, let $k_3'\coloneqq\frac{\lambda_1k_1+\lambda_3k_3}{\lambda_1+\lambda_3}$. Then $k_3'>\min\{k_1,k_3\}=k_3>0$, where the first inequality follows from $\lambda_1>0$ and $\lambda_3>0$, and the equality from Equation $(4.28)_{\curvearrowleft}$. Moreover, $k_3'<1$ because it is a convex combination of $k_1<1$ and $k_3<1$. Hence $k_3'\in(k_3,1)$ and therefore $(k_3',k_2)\in K$. If we now let $\lambda_3'\coloneqq\lambda_1+\lambda_3>0$, then we see that $\lambda_3'(k_3'-1,k_3')=(\lambda_1k_1+\lambda_3k_3-\lambda_1-\lambda_3,\lambda_1k_1+\lambda_3k_3)\in A\setminus\{g\}$, whence also $\{\lambda_3'(k_3'-1,k_3'),\lambda_2(k_2,k_2-1)\}\subseteq A\setminus\{g\}$, and Condition $(4.21)_{152}$ now guarantees that $0\in R_K(A\setminus\{g\})$, a contradiction.

If $k_1+k_2 \geq 1$ then we have that $k_2 \geq 1-k_1 \geq k_4$, where the second inequality is due to Equation (4.29). Also $k_3^* := \frac{\lambda_1 k_1+\lambda_3 k_3}{\lambda_1+\lambda_3} > \min\{k_1,k_3\} = k_3 > 0$, where the first inequality follows from $\lambda_1 > 0$ and $\lambda_3 > 0$, and the equality from $k_1 > k_3$ [Equation $(4.28)_{157}$]. Moreover, $k_3^* < 1$ because it is a convex combination of $k_1 < 1$ and $k_3 < 1$. This tells us that $(k_3^*,k_2) \in [0,1)^2 \setminus \{0\}$ and $(k_3^*,k_2) > (k_3,1-k_1)$. We then find that $(k_3^*,k_2) \in K$ because $(k_3,1-k_1) \in K$ and K is increasing [Property $\mathrm{K1}_{148}$]. If we now let $\lambda_3^* := \lambda_1 + \lambda_3 > 0$ then we find that $\lambda_3^*(k_3^*-1,k_3^*) = (\lambda_1 k_1+\lambda_3 k_3-\lambda_1-\lambda_3,\lambda_1 k_1+\lambda_3 k_3) \in A \setminus \{0\}$, and therefore also $\{\lambda_3^*(k_3^*-1,k_3^*),\lambda_2(k_2,k_2-1)\} \subseteq A \setminus \{g\}$, and Condition $(4.21)_{152}$ now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

If $\lambda_4 > \lambda_1$, then we claim that

$$k_4 \leq 1-k_1.$$

To prove this, assume *ex absurdo* that $k_4 > 1-k_1$. We wonder in what quadrant the vector $(\lambda_1 k_1+\lambda_4 k_4-\lambda_1,\lambda_1 k_1+\lambda_4 k_4-\lambda_4)$ lies. Infer from $k_1+k_4 > 1$, $k_4 > 0$ and $\lambda_4 > \lambda_1 > 0$ that $\lambda_1 k_1+\lambda_4 k_4-\lambda_1 > \lambda_1(k_1+k_4)-\lambda_1 > 0$. Since $A \cap \mathcal{L}_{>0} = \emptyset$ by Equation $(4.26)_{156}$, we find that $(\lambda_1 k_1+\lambda_4 k_4-\lambda_1,\lambda_1 k_1+\lambda_4 k_4-\lambda_4)$ must lie in the fourth quadrant, and therefore its second component $\lambda_1 k_1+\lambda_4 k_4-\lambda_4$ must be negative: $\lambda_1 k_1+\lambda_4 k_4 < \lambda_4$. This tells us that $k_4^* := \frac{\lambda_1 k_1+\lambda_4 k_4-\lambda_1}{\lambda_4-\lambda_1} < 1$. Moreover, $k_4^* > k_4$ because this is equivalent to $k_4 > 1-k_1$. Let $k_3^* := \frac{\lambda_1 k_1+\lambda_3 k_3}{\lambda_1+\lambda_3}$. Then $k_3^* > \min\{k_1,k_3\} = k_3 > 0$, where the first inequality follows $\lambda_1 > 0$ and $\lambda_3 > 0$ and the equality from $k_1 > k_3$ [Equation $(4.28)_{157}$]. Moreover, $k_3^* < 1$ because it is a convex combination of $k_1 < 1$ and $k_3 < 1$. This tells us that $(k_3^*,k_4^*) \in [0,1)^2 \setminus \{0\}$ and that $(k_3^*,k_4^*) > (k_3,k_4)$. Hence $(k_3^*,k_4^*) \in K$ because $(k_3,k_4) \in K$ and K is increasing [Property $\mathrm{K1}_{148}$]. If we now let $\lambda_3^* := \lambda_1+\lambda_3 > 0$ and $\lambda_4^* := \lambda_4-\lambda_1 > 0$, then $\lambda_3^*(k_3^*-1,k_3^*) = (\lambda_1 k_1+\lambda_3 k_3-\lambda_1-\lambda_3,\lambda_1 k_1+\lambda_3 k_3)$ and $\lambda_4^*(k_4^*,k_4^*-1) = (\lambda_1 k_1+\lambda_4 k_4-\lambda_1,\lambda_1 k_1+\lambda_4 k_4-\lambda_4)$, and therefore $\{\lambda_3^*(k_3^*-1,k_3^*),\lambda_4^*(k_4^*,k_4^*-1)\} \subseteq A \setminus \{g\}$, so Condition $(4.21)_{152}$ now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

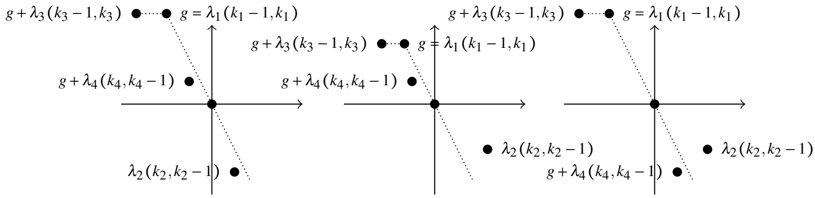
So we see that $0 < k_4 \leq 1-k_1 < 0$, and hence, because K is increasing, we infer from $(k_3,k_4) \in K$ that $(k_3,1-k_1) \in K$. We now have the same two possibilities $k_1+k_2 < 1$ and $k_1+k_2 \geq 1$ as before, and for each of them, we can construct a contradiction in exactly the same way as for the case when $\lambda_4 \leq \lambda_1$.

This shows that we always arrive at a contradiction in case (ii.b.2.1).

In case (ii.b.2.2) we see that $\lambda_4(k_4,k_4-1) \notin \{(-1,0),0,(0,-1)\}$, and therefore Equation $(4.27)_{157}$ leads to

$$(\lambda_1 k_1+\lambda_4 k_4-\lambda_1,\lambda_1 k_1+\lambda_4 k_4-\lambda_4) \in A \setminus \{g\}.$$

We distinguish between two possibilities, which will determine in what quadrant this point lies: $\lambda_4 \leq \lambda_1$ or $\lambda_4 > \lambda_1$.



Case (ii.b.2.2) $\lambda_4 \le \lambda_1$ $k_1+k_2<1$ | Case (ii.b.2.2) $\lambda_4 \le \lambda_1$ $k_1+k_2 \ge 1$ | Case (ii.b.2.2) $\lambda_4 > \lambda_1$

If $\lambda_4 \le \lambda_1$, then we claim that

$$k_4 \le 1-k_1. \tag{4.30}$$

To prove this, assume *ex absurdo* that $k_4 > 1-k_1$, so $k_1+k_4-1>0$. If $\lambda_1=\lambda_4$, then $(\lambda_1k_1+\lambda_4k_4-\lambda_1, \lambda_1k_1+\lambda_4k_4-\lambda_4) = \lambda_1(k_1+k_4-1, k_1+k_4-1) > 0$, a contradiction with Equation (4.26)$_{156}$, so we may assume that $\lambda_4 < \lambda_1$. We now wonder in what quadrant the vector $(\lambda_1k_1+\lambda_4k_4-\lambda_1, \lambda_1k_1+\lambda_4k_4-\lambda_4) \ne 0$ lies. We infer from $k_1>0$, $\lambda_1>\lambda_4>0$ and $k_1+k_4>1$ that $\lambda_1k_1+\lambda_4k_4-\lambda_4 > \lambda_4(k_1+k_4)-\lambda_4>0$. Since $A \cap \mathcal{L}_{>0} = \varnothing$ by Equation (4.26)$_{156}$, we find that $(\lambda_1k_1+\lambda_4k_4-\lambda_1, \lambda_1k_1+\lambda_4k_4-\lambda_4)$ must lie in the second quadrant, and therefore its first component $\lambda_1k_1+\lambda_4k_4-\lambda_1$ must be negative: $\lambda_1k_1+\lambda_4k_4<\lambda_1$. This tells us that $k_4^* := \frac{\lambda_1k_1+\lambda_4k_4-\lambda_4}{\lambda_1-\lambda_4} < 1$. Moreover, $k_4^* > k_1$ because this is equivalent to $k_4 > 1-k_1$. Hence $(k_4^*, k_2) \in [0,1)^2 \setminus 0$ and $(k_4^*, k_2) > (k_1, k_2)$. This tells us that $(k_4^*, k_2) \in K$ because $(k_1,k_2) \in K$ and K is increasing [Property K1$_{148}$]. If we now let $\lambda_4^* := \lambda_1-\lambda_4>0$, then we see that $\lambda_4^*(k_4^*-1, k_4^*) = (\lambda_1k_1+\lambda_4k_4-\lambda_1, \lambda_1k_1+\lambda_4k_4-\lambda_4) \in A \setminus \{g\}$. Hence also $\{\lambda_4^*(k_4^*-1,k_4^*), \lambda_2(k_2,k_2-1)\} \subseteq A \setminus \{g\}$, and Condition (4.21)$_{152}$ now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

So we see that $0<k_4 \le 1-k_1<1$, so $(0,1-k_1) \in [0,1)^2 \setminus \{0\}$ and $(0,1-k_1) > (0,k_4)$ and hence, because K is increasing [Property K1$_{148}$], we infer from $(0,k_4)=(k_3,k_4) \in K$ that also $(0,1-k_1) \in K$. We distinguish between two further possibilities: $k_1+k_2<1$ and $k_1+k_2 \ge 1$.

If $k_1+k_2<1$ then we can use Property K3b$_{149}$ with $a=1-k_1$ and $c=k_2$. Observe that $c=k_2<1-k_1=a$ by assumption, that $(0,a)=(0,1-k_1) \in K$ was derived above, and that $(1-a,c)=(k_1,k_2) \in K$ also by assumption, and therefore we find that $(0,k_2) \in K$. Since $\lambda_2(k_2,k_2-1) \in A \setminus \{g\}$, Condition (4.20)$_{152}$ now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

If $k_1+k_2 \ge 1$ then we have that $k_2 \ge 1-k_1 \ge k_4$, where the second inequality is due to Equation (4.30). Then $(0,k_2) \in K$ because $(0,k_4) \in K$ and K is increasing [Property K1$_{148}$]. Since $\lambda_2(k_2,k_2-1) \in A \setminus \{g\}$, Condition (4.20)$_{152}$ now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

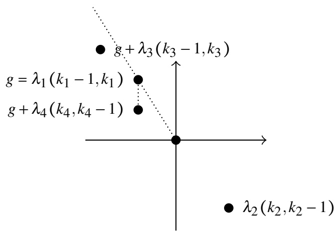
If $\lambda_4 > \lambda_1$, then we claim that, here too,

$$k_4 \le 1-k_1.$$

To prove this, assume *ex absurdo* that $k_4 > 1-k_1$. We wonder in what quadrant the vector $(\lambda_1k_1+\lambda_4k_4-\lambda_1, \lambda_1k_1+\lambda_4k_4-\lambda_4)$ lies. Infer from $0<1-k_1<k_4$ and $0<\lambda_1<\lambda_4$ that $\lambda_1k_1+\lambda_4k_4-\lambda_1 > \lambda_1(k_1+k_4)-\lambda_1>0$. Since $A \cap \mathcal{L}_{>0} = \varnothing$ by Equation (4.26)$_{156}$, we find that the vector $(\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4)$ must lie in the fourth quadrant, and therefore its second component $\lambda_1 k_1 + \lambda_4 k_4 - \lambda_4$ must be negative: $\lambda_1 k_1 + \lambda_4 k_4 < \lambda_4$. This tells us that $k_4^* := \frac{\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1}{\lambda_4 - \lambda_1} < 1$. Moreover, $k_4^* > k_4$ because this is equivalent to $k_4 > 1 - k_1$. Hence $(0, k_4^*) \in [0,1)^2 \setminus \{0\}$ and $(0, k_4^*) > (0, k_4)$. This tells us that $(0, k_4^*) \in K$ because $(0, k_4) \in K$ and K is increasing [Property K1$_{148}$]. If we now let $\lambda_4^* := \lambda_4 - \lambda_1 > 0$, then we see that $\lambda_4^*(k_4^*, k_4^* - 1) = (\lambda_1 k_1 + \lambda_4 k_4 - \lambda_1, \lambda_1 k_1 + \lambda_4 k_4 - \lambda_4) \in A \setminus \{g\}$, and Condition (4.20)$_{152}$ now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

So we see that $0 < k_4 \leq 1 - k_1 < 0$, and hence, because K is increasing, we infer from $(k_3, k_4) \in K$ that $(k_3, 1 - k_1) \in K$. We now have the same two possibilities $k_1 + k_2 < 1$ and $k_1 + k_2 \geq 1$ as before, and for each of them, we can construct a contradiction in exactly the same way as for the case when $\lambda_4 \leq \lambda_1$.

We conclude that case (ii.b.2.2) always leads to a contradiction.



Case (ii.b.2.3)

In case (ii.b.2.3) we see that $\lambda_3(k_3 - 1, k_3) \notin \{(-1,0), 0, (0,-1)\}$, and therefore Equation (4.27)$_{157}$ leads to $(\lambda_1 k_1 + \lambda_3 k_3 - \lambda_1 - \lambda_3, \lambda_1 k_1 + \lambda_3 k_3) \in A \setminus \{g\}$, or if we let $\lambda_3^* := \lambda_1 + \lambda_3 > 0$ and $k_3^* := \frac{\lambda_1 k_1 + \lambda_3 k_3}{\lambda_1 + \lambda_3} > 0$,

$$\lambda_3^*(k_3^* - 1, k_3^*) \in A \setminus \{g\}.$$

Observe that also $k_3^* < 1$ because it is a convex combination of $k_1 < 1$ and $k_3 < 1$. This tells us that $(k_3^*, 0) \in [0,1)^2 \setminus \{0\}$. Moreover, we have that $k_3^* > \min\{k_1, k_3\} = k_3 > 0$ [the strict inequality holds because $\lambda_1 > 0$ and $\lambda_3 > 0$, and the equality holds because $k_1 > k_3$ by Equation (4.28)$_{157}$]. Hence $(k_3^*, 0) > (k_3, 0)$ and therefore $(k_3^*, 0) \in K$, because also $(k_3, 0) \in K$ and K is increasing [Property K1$_{148}$]. Since $\lambda_3^*(k_3^* - 1, k_3^*) \in A \setminus \{g\}$, Condition (4.19)$_{152}$ now guarantees that $0 \in R_K(A \setminus \{g\})$, a contradiction.

We have now found a contradiction in cases (ii.b.2.1)–(ii.b.2.3), which tells us that case (ii.b.2) always leads to a contradiction. Since case (ii.b.1) also led to a contradiction, we may conclude that case (ii.b) always leads to a contradiction.

The discussion of the last remaining case (ii.c) is completely similar to that of case (ii.b): we can distinguish between similar cases, and in each of them we can construct a contradiction in the same manner, by exchanging the roles of k_1 and k_2, and of k_3 and k_4.

Since we have now arrived at a contradiction in all possible cases, we conclude that R_K indeed satisfies Axiom R3b$_{20}$.

We finish the proof by establishing that R_K also satisfies Axiom R1$_{20}$. Since we have already shown that R_K satisfies Axiom R4b [see Proposition 113$_{153}$] and Axiom R3b$_{20}$ [see the argumentation above], by Corollary 26$_{39}$ it suffices to show that $0 \notin R_K(\{0\})$. By Condition (4.17)$_{152}$, this is indeed the case. □

Corollary 115. *Consider any subset K of $[0,1)^2$ that satisfies Properties* K1$_{148}$–K3$_{149}$*. Then the rejection function R_K given by Definition 37$_{152}$ is coherent and satisfies Property* R6$_{25}$.

Proof. This is an immediate consequence of Propositions Proposition 113$_{153}$ and Proposition 114$_{153}$. □

4.4.3 *The correspondence between rejection functions and coordinate rejection sets*

We conclude from the preceding discussion that any coherent rejection function determines a coordinate rejection set via Definition 36$_{148}$, which, in turn, can be used to determine a rejection function via Definition 37$_{152}$. Our next proposition shows that these two procedures commute, or, in other words, that a coherent rejection function is uniquely determined by its associated coordinate rejection set, and the other way around. In order to get there, we first observe that the proof of Proposition 69$_{79}$ immediately implies the following lemma:

Lemma 116. *Consider any coherent rejection function R on $\mathcal{Q}(\mathcal{L}(\{\mathrm{H},\mathrm{T}\}))$ that satisfies Property* R6$_{25}$*. Consider the option sets $\{f_1,\dots,f_m\} \subseteq \mathcal{L}_{\mathrm{II}}$ and $\{g_1,\dots,g_n\} \subseteq \mathcal{L}_{\mathrm{IV}}$, for some m and n in $\mathbb{N}$. Then the following equivalences hold for any i in $\arg\max\{\frac{f_k(\mathrm{T})}{f_k(\mathrm{T})-f_k(\mathrm{H})} : k \in \{1,\dots,m\}\}$ and any j in $\arg\max\{\frac{g_k(\mathrm{H})}{g_k(\mathrm{H})-g_k(\mathrm{T})} : k \in \{1,\dots,n\}\}$:*

(i) $0 \in R(\{0,f_1,\dots,f_m,g_1,\dots,g_n\}) \Leftrightarrow 0 \in R(\{0,f_i,g_j\})$;
(ii) $0 \in R(\{0,g_1,\dots,g_n\}) \Leftrightarrow 0 \in R(\{0,g_j\})$;
(iii) $0 \in R(\{0,f_1,\dots,f_m\}) \Leftrightarrow 0 \in R(\{0,f_i\})$.

Incidentally, Proposition 16$_{27}$ ensures that this lemma applies in particular to coherent rejection functions that satisfy Property R5$_{25}$. We are now ready to prove the characterisation of rejection functions in terms of coordinate rejection sets, as the counterpart of Proposition 65$_{77}$ for the characterisation in terms of rejection sets.

Proposition 117. *For any coherent rejection function R on $\mathcal{Q}(\mathcal{L}(\{\mathrm{H},\mathrm{T}\}))$ that satisfies Property* R6$_{25}$*, $R = R_{K_R}$. Conversely, for any coordinate rejection set K satisfying Properties* K1$_{148}$–K3$_{149}$*, $K = K_{R_K}$.*

Proof. For the first statement, assume that R is coherent and satisfies Property R6$_{25}$. Then we infer from Proposition 111$_{149}$ that K_R satisfies Properties K1$_{148}$–K3$_{149}$, and therefore Corollary 115 guarantees that R_{K_R} is coherent and satisfies Property R6$_{25}$ as well. To prove that $R = R_{K_R}$, we consider any A in $\mathcal{Q}$ and f in A, and show that $f \in R(A) \Leftrightarrow f \in R_{K_R}(A)$. Since both R and R_{K_R} satisfy Axiom R4b$_{20}$ [Proposition 113$_{153}$], we can assume without loss of generality that $f = 0$.

For the direct implication, assume that $0 \in R(A)$. If $A \cap \mathcal{L}_{>0} \neq \emptyset$ then $0 \in R_{K_R}(A)$ by Condition (4.18)$_{152}$. If $A \cap \mathcal{L}_{>0} = \emptyset$ then $0 \in R(A)$ implies that $g(\mathrm{H}) > 0$ or $g(\mathrm{T}) > 0$

for some g in A. To see this: if *ex absurdo* $A \subseteq \mathcal{L}_{\leq 0}$ then $A \setminus \{0\} \subseteq R(A)$ by Proposition 31_{42}, and therefore also $R(A) = A$, contradicting Axiom $\mathrm{R1}_{20}$. If we use the notation $\mathcal{L}_{\mathrm{II}} \cap A = \{g_1, \dots, g_m\}$ and $\mathcal{L}_{\mathrm{IV}} \cap A = \{g'_1, \dots, g'_n\}$ with m and n in $\mathbb{Z}_{\geq 0}$, this tells us that $\max\{n,m\} > 0$. Also, because of Proposition 31_{42} we may assume without loss of generality that $A \cap \mathcal{L}_{<0} = \emptyset$. By Lemma 116 we infer that there are three possibilities:

(i) $0 \in R(\{0,\tilde{g},\tilde{g}'\})$, and hence $0 \in R(\{0,h,h'\})$ [use Property $\mathrm{R6}_{25}$], when $n \geq 1$ and $m \geq 1$, with $\tilde{g} \in A \cap \mathcal{L}_{\mathrm{II}}$ and $\tilde{g}' \in A \cap \mathcal{L}_{\mathrm{IV}}$;

(ii) $0 \in R(\{0,\tilde{g}\})$, and hence $0 \in R(\{0,h\})$ [use Property $\mathrm{R6}_{25}$], when $n > 0 = m$, with $\tilde{g} \in A \cap \mathcal{L}_{\mathrm{II}}$;

(iii) $0 \in R(\{0,\tilde{g}'\})$, and hence $0 \in R(\{0,h'\})$ [use Property $\mathrm{R6}_{25}$], when $m > 0 = n$, with $\tilde{g}' \in A \cap \mathcal{L}_{\mathrm{IV}}$,

where we let, to ease the notation, $h \coloneqq \frac{1}{\tilde{g}(\mathrm{T})-\tilde{g}(\mathrm{H})}\tilde{g}$ and $h' \coloneqq \frac{1}{\tilde{g}'(\mathrm{H})-\tilde{g}'(\mathrm{T})}\tilde{g}'$. For each of these possible cases, we find respectively:

(i) $(h(\mathrm{T}),h'(\mathrm{H})) \in K_R$, which tells us that $0 \in R_{K_R}(\{0,h,h'\})$ by Condition $(4.21)_{152}$, and hence indeed $0 \in R_{K_R}(\{0,\tilde{g},\tilde{g}'\})$, because R_{K_R} satisfies Property $\mathrm{R6}_{25}$ by Proposition 113_{153};

(ii) $(h(\mathrm{T}),0) \in K_R$ [use a suitable and by now familiar combination of Axioms $\mathrm{R2}_{20}$, $\mathrm{R3a}_{20}$ and $\mathrm{R3b}_{20}$], from which we infer that $0 \in R_{K_R}(\{0,\tilde{g}\})$ by Condition $(4.20)_{152}$;

(iii) $(0,h'(\mathrm{H})) \in K_R$ [use a similar, suitable and by now familiar combination of Axioms $\mathrm{R2}_{20}$, $\mathrm{R3a}_{20}$ and $\mathrm{R3b}_{20}$], from which we infer that $0 \in R_{K_R}(\{0,\tilde{g}'\})$ by Condition $(4.19)_{152}$.

In all three cases we can now conclude that, indeed, $0 \in R_{K_R}(A)$, by Axiom $\mathrm{R3a}_{20}$.

For the converse implication, assume that $0 \in R_{K_R}(A)$. If $A \cap \mathcal{L}_{>0} \neq \emptyset$, then $0 \in R(A)$ by Axioms $\mathrm{R2}_{20}$ and $\mathrm{R3a}_{20}$, so assume that $A \cap \mathcal{L}_{>0} = \emptyset$. There are three possibilities:

If Condition $(4.19)_{152}$ holds, then there is some k_1 in $(0,1)$ and some λ_1 in $\mathbb{R}_{>0}$ such that $(k_1,0) \in K_R$ and $\lambda_1(k_1-1,k_1) \in A$. The first statement means that $0 \in R(\{(k_1-1,k_1),0,(0,-1)\})$, whence, after applying a familiar combination of Axioms $\mathrm{R2}_{20}$, $\mathrm{R3a}_{20}$ and $\mathrm{R3b}_{20}$, also $0 \in R(\{(k_1-1,k_1),0\})$. Applying Property $\mathrm{R6}_{25}$, the second statement, and Axiom $\mathrm{R3a}_{20}$ now leads us to deduce that indeed $0 \in R(A)$.

If Condition $(4.20)_{152}$ holds, a similar argument leads us to the conclusion that, here too, $0 \in R(A)$.

Finally, if Condition $(4.21)_{152}$ holds, then $\{\lambda_1(k_1-1,k_1),\lambda_2(k_2,k_2-1)\} \subseteq A$ for some (k_1,k_2) in $K_R \cap (0,1)^2$ and λ_1 and λ_2 in $\mathbb{R}_{>0}$. (k_1,k_2) in $K_R \cap (0,1)^2$ implies that $0 \in R(\{(k_1-1,k_1),0,(k_2,k_2-1)\})$, and therefore also $0 \in R(\{\lambda_1(k_1-1,k_1),0,\lambda_2(k_2,k_2-1)\})$ by Property $\mathrm{R6}_{25}$. Hence $0 \in R(A)$ by Axiom $\mathrm{R3a}_{20}$. This concludes the proof of the first statement.

For the second statement, assume that K satisfies Properties $\mathrm{K1}_{148}$–$\mathrm{K3}_{149}$, then we infer from Corollary 115 that R_K is coherent and satisfies Property $\mathrm{R6}_{25}$. Proposition 111_{149} then guarantees that K_{R_K} satisfies Properties $\mathrm{K1}_{148}$–$\mathrm{K3}_{149}$ as well. To show that $K = K_{R_K}$, consider any (ℓ_1,ℓ_2) in $[0,1)^2 \setminus \{0\}$. First assume that $(\ell_1,\ell_2) \in K_{R_K}$, meaning that $0 \in R_K(\{(\ell_1-1,\ell_1),0,(\ell_2,\ell_2-1)\})$, by the definition of a coordinate rejection set of a rejection function. We have to prove that this implies that $(\ell_1,\ell_2) \in K$. The definition of R_K [Definition 37_{152}] now tells us that Condition $(4.18)_{152}$, Condition $(4.19)_{152}$, Condition $(4.20)_{152}$, or Condition $(4.21)_{152}$ must obtain, with $A \coloneqq \{(\ell_1-1,\ell_1),(\ell_2,\ell_2-1)\}$. Since $(\ell_1,\ell_2) \in [0,1)^2 \setminus \{0\}$, we infer that Condition $(4.18)_{152}$ cannot be fulfilled, and we therefore have three remaining: (a) Condition $(4.19)_{152}$, (b) Condition $(4.20)_{152}$, or (c) Condition $(4.21)_{152}$ is satisfied.

In case (a) there are λ_1 in $\mathbb{R}_{>0}$ and $(k_1,0)$ in K such that $\lambda_1(k_1-1,k_1)\in A$. But, because $A=\{(\ell_1-1,\ell_1),(\ell_2,\ell_2-1)\}$ with $(\ell_1,\ell_2)\in[0,1)^2\smallsetminus\{0\}$, this implies that $\lambda_1=1$ and $k_1=\ell_1$. To see this, recall that $\lambda_1(k_1-1,k_1)$ lies in the second quadrant, while (ℓ_2,ℓ_2-1) lies in the fourth quadrant, so the only remaining possibility is that $\lambda_1(k_1-1,k_1)=(\ell_1-1,\ell_1)$, implying that indeed $\lambda_1=1$ and $k_1=\ell_1$. This guarantees that $(\ell_1,0)\in K$ and, since K is increasing [Property $\mathrm{K1}_{148}$], indeed also that $(\ell_1,\ell_2)\in K$.

In case (b) there are λ_2 in $\mathbb{R}_{>0}$ and $(0,k_2)$ in K such that $\lambda_2(k_2,k_2-1)\in A$. But, because $A=\{(\ell_1-1,\ell_1),(\ell_2,\ell_2-1)\}$ with $(\ell_1,\ell_2)\in[0,1)^2\smallsetminus\{0\}$, this implies that $\lambda_2=1$ and $k_2=\ell_2$. To see this, recall that $\lambda_2(k_2,k_2-1)$ lies in the fourth quadrant, while (ℓ_1,ℓ_1-1) lies in the second quadrant, so the only remaining possibility is that $\lambda_2(k_2,k_2-1)=(\ell_2,\ell_2-1)$, implying that indeed $\lambda_2=1$ and $k_2=\ell_2$. This guarantees that $(0,\ell_2)\in K$ and, since K is increasing [Property $\mathrm{K1}_{148}$], indeed also that $(\ell_1,\ell_2)\in K$.

Finally, in case (c) there are λ_1 and λ_2 in $\mathbb{R}_{>0}$, and (k_1,k_2) in $K\cap(0,1)^2$ such that $\{\lambda_1(k_1-1,k_1),\lambda_2(k_2,k_2-1)\}\subseteq A$. But, because $A=\{(\ell_1-1,\ell_1),(\ell_2,\ell_2-1)\}$ with $(\ell_1,\ell_2)\in[0,1)^2\smallsetminus\{0\}$, this implies that $\lambda_1=\lambda_2=1$, $k_1=\ell_1$ and $k_2=\ell_2$. To see this, observe that $\lambda_1(k_1-1,k_1)$ and (ℓ_1-1,ℓ_1) are the only elements in the second quadrant—and therefore must be equal—and that $\lambda_2(k_2,k_2-1)$ and (ℓ_2,ℓ_2-1) are the only elements in the fourth quadrant—and therefore must be equal. In a similar way as above, this then implies that indeed $\lambda_1=\lambda_2=1$, $k_1=\ell_1$ and $k_2=\ell_2$. Hence, indeed, $(\ell_1,\ell_2)\in K$.

Conversely, assume that $(\ell_1,\ell_2)\in K$, then Condition $(4.23)_{152}$ guarantees that in particular $0\in R_K(\{(\ell_1-1,\ell_1),0,(\ell_2,\ell_2-1)\})$, which implies that $(\ell_1,\ell_2)\in K_{R_K}$. □

4.4.4 *Coordinate rejection sets and convexity*

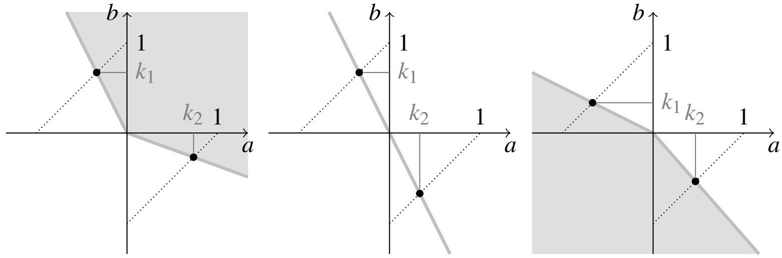
To conclude our preliminary discussion of the relation between coordinate rejection sets and rejection functions, we characterise the conditions under which the rejection function R_K determined by a coordinate rejection set K satisfies the 'convexity' Property $\mathrm{R5}_{25}$. We begin with a lemma that will simplify the argument.

Lemma 118. *Consider (k_1,k_2) in $[0,1)^2$. Let $A:=\{(k_1-1,k_1),0,(k_2,k_2-1)\}$, then*

$$\operatorname{posi}(A)=\begin{cases}B+\mathcal{L}_{\geq 0} & \text{if } k_1+k_2>1\\ B & \text{if } k_1+k_2=1\\ B+\mathcal{L}_{\leq 0} & \text{if } k_1+k_2<1,\end{cases}$$

where $B:=\{\lambda(k_1-1,k_1):\lambda\in\mathbb{R}_{\geq 0}\}\cup\{\lambda(k_2,k_2-1):\lambda\in\mathbb{R}_{\geq 0}\}$.

Proof. Visual proof: see the three possible situations depicted below.



$k_1 + k_2 > 1$ $k_1 + k_2 = 1$ $k_1 + k_2 < 1$

$\square$

In particular, it follows from this result for $A = \{(k_1 - 1, k_1), 0, (k_2, k_2 - 1)\}$ that

$$\mathrm{posi}(A) \cap \mathcal{L}_{>0} = \emptyset \Leftrightarrow k_1 + k_2 \leq 1, \text{ for all } (k_1, k_2) \text{ in } [0,1)^2. \tag{4.31}$$

Proposition 119 (Characterisation of Property R5$_{25}$)**.** *Consider any coordinate rejection set $K \subseteq [0,1)^2 \setminus \{0\}$, and the corresponding rejection function R_K on $\mathcal{Q}(\mathcal{L}(\{\mathrm{H},\mathrm{T}\}))$ that satisfies Properties* K1$_{148}$–K3$_{149}$*. Then the following two statements are equivalent:*

(i) *R_K satisfies Property* R5$_{25}$*,*

(ii) *K satisfies Property* K4$_{149}$.

Proof. We first prove that (i)⇒(ii). Assume that R_K satisfies Property R5$_{25}$, and consider any (k_1, k_2) in $[0,1)^2 \setminus \{0\}$ such that $k_1 + k_2 > 1$. It then follows that $(k_1, k_2) \in (0,1)^2$, and also that $\big(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}\big) > 0$, whence $0 \in R_K(\{0, \big(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}\big)\})$ by Condition (4.18)$_{152}$. By Proposition 113$_{153}$, R_K satisfies Axiom R3a$_{20}$, whence

$$0 \in R_K\left(\left\{(k_1 - 1, k_1), 0, (k_2, k_2 - 1), \left(\frac{k_1 + k_2 - 1}{2}, \frac{k_1 + k_2 - 1}{2}\right)\right\}\right).$$

Also, $\big(\frac{k_1+k_2-1}{2}, \frac{k_1+k_2-1}{2}\big) \in \mathrm{conv}(\{(k_1 - 1, k_1), (k_2, k_2 - 1)\})$. But then Property R5$_{25}$ implies that $0 \in R_K(\{(k_1 - 1, k_1), 0, (k_2, k_2 - 1)\})$, whence indeed $(k_1, k_2) \in K$.

Next, we prove that (ii)⇒(i). Consider arbitrary A and A_1 in $\mathcal{Q}$ such that $A \subseteq A_1 \subseteq \mathrm{conv}(A)$, and let us show that $R_K(A_1) \cap A \subseteq R_K(A)$. Let $A := \{f_1, \dots, f_n\}$ and $A_1 := A \cup \{f_{n+1}, \dots, f_{n+k}\}$ for some n and k in $\mathbb{N}$. Assume that $f_i \in R_K(A_1)$ for some i in $\{1, \dots, n\}$. We then have to prove that $f_i \in R_K(A)$. We can assume without loss of generality that $f_i = 0$, because also $A - \{f_i\} \subseteq A_1 - \{f_i\} \subseteq \mathrm{conv}(A) - \{f_i\} = \mathrm{conv}(A - \{f_i\})$. To ease the notation along, let $\ell_k := \frac{f_k(\mathrm{T})}{f_k(\mathrm{T}) - f_k(\mathrm{H})}$ and $\lambda_k := f_k(\mathrm{T}) - f_k(\mathrm{H})$ for every k such that $f_k \in \mathcal{L}_{\mathrm{II}}$ [there might be no such k] and verify that $\lambda_k > 0$ and $f_k = \lambda_k(\ell_k - 1, \ell_k)$ for every gamble f_k in $A \cap \mathcal{L}_{\mathrm{II}}$. Similarly, for every k in $\{1, \dots, n\}$ such that $f_k \in \mathcal{L}_{\mathrm{IV}}$ [there might be no such k], let $\ell_k := \frac{f_k(\mathrm{H})}{f_k(\mathrm{H}) - f_k(\mathrm{T})}$ and $\lambda_k := f_k(\mathrm{H}) - f_k(\mathrm{T})$; then $\lambda_k > 0$ and $f_k = \lambda_k(\ell_k, \ell_k - 1)$ for every gamble f_k in $A \cap \mathcal{L}_{\mathrm{IV}}$.

First of all, we see that $A \cap \mathcal{L}_{>0} \neq \emptyset$ implies that indeed $0 \in R_K(A)$, by Condition (4.18)$_{152}$. We may therefore in the remainder of this proof assume that $A \cap \mathcal{L}_{>0} = \emptyset$.

Next, we observe that $\operatorname{conv}(A)\cap\mathcal{L}_{>0}\neq\emptyset$ also implies that $0\in R_K(A)$. To see this, if $\operatorname{conv}(A)\cap\mathcal{L}_{>0}\neq\emptyset$—say, $g\in\operatorname{conv}(A)\cap\mathcal{L}_{>0}$, so $g=\sum_{k=1}^n\alpha_k f_k>0$ for some α_1, ..., α_n in $\mathbb{R}_{\geq 0}$ such that $\sum_{k=1}^n\alpha_k=1$—, then, as an intermediate result, $A\cap\mathcal{L}_{\mathrm{II}}\neq\emptyset$ and $A\cap\mathcal{L}_{\mathrm{IV}}\neq\emptyset$. Indeed, assume *ex absurdo* that $A\cap\mathcal{L}_{\mathrm{IV}}=\emptyset$, then, since we have assumed that $A\cap\mathcal{L}_{>0}=\emptyset$, therefore $A\subseteq\mathcal{L}_{\leq 0}\cup\mathcal{L}_{\mathrm{II}}$. But then $g=\sum_{k=1}^n\alpha_k f_k\leq 0$ or $g(\mathrm{H})=\sum_{k=1}^n\alpha_k f_k(\mathrm{H})<0$, contradicting the assumption that $g>0$. Assuming that $A\cap\mathcal{L}_{\mathrm{II}}=\emptyset$ leads to a similar contradiction. It therefore follows that

$$0<g=\sum_{k=1}^n\alpha_k f_k\leq\sum_{\substack{k=1\\ f_k\in\mathcal{L}_{\mathrm{II}}\cup\mathcal{L}_{\mathrm{IV}}}}^n\alpha_k f_k. \tag{4.32}$$

Without loss of generality, we may assume by Lemma 116_{162} that $f_1\in\mathcal{L}_{\mathrm{II}}$ is a gamble in

$$\operatorname{arg\,max}\left\{\frac{h(\mathrm{T})}{h(\mathrm{T})-h(\mathrm{H})}:h\in A\cap\mathcal{L}_{\mathrm{II}}\right\}$$

and that $f_2\in\mathcal{L}_{\mathrm{IV}}$ is a gamble in

$$\operatorname{arg\,max}\left\{\frac{h(\mathrm{H})}{h(\mathrm{H})-h(\mathrm{T})}:h\in A\cap\mathcal{L}_{\mathrm{IV}}\right\}.$$

We now claim that, as a consequence,

$$\lambda_1' f_1\geq\sum_{\substack{k=1\\ f_k\in\mathcal{L}_{\mathrm{II}}}}^n\alpha_k f_k \text{ and } \lambda_2' f_2\geq\sum_{\substack{k=1\\ f_k\in\mathcal{L}_{\mathrm{IV}}}}^n\alpha_k f_k \text{ for some } \lambda_1' \text{ and } \lambda_2' \text{ in } \mathbb{R}_{\geq 0}. \tag{4.33}$$

To prove the statement involving f_1, observe that $\ell_k\leq\ell_1$ for every k in $\{1,\dots,m\}$ for which $f_k\in A\cap\mathcal{L}_{\mathrm{II}}$, and therefore

$$\begin{aligned}\sum_{\substack{k=1\\ f_k\in\mathcal{L}_{\mathrm{II}}}}^n\alpha_k f_k=\sum_{\substack{k=1\\ f_k\in\mathcal{L}_{\mathrm{II}}}}^n\alpha_k\lambda_k(\ell_k-1,\ell_k)&\leq\sum_{\substack{k=1\\ f_k\in\mathcal{L}_{\mathrm{II}}}}^n\alpha_k\lambda_k(\ell_1-1,\ell_1)\\ &=\sum_{\substack{k=1\\ f_k\in\mathcal{L}_{\mathrm{II}}}}^n\alpha_k\frac{\lambda_k}{\lambda_1}\lambda_1(\ell_1-1,\ell_1)=f_1\sum_{\substack{k=1\\ f_k\in\mathcal{L}_{\mathrm{II}}}}^n\alpha_k\frac{\lambda_k}{\lambda_1},\end{aligned}$$

so we see that $\lambda_1':=\sum_{k\in\{1,\dots,n\},f_k\in\mathcal{L}_{\mathrm{II}}}\alpha_k\frac{\lambda_k}{\lambda_1}\geq 0$ does the job nicely. The statement involving f_2 holds because of a similar argument.

We can now combine Equations (4.32) and (4.33) to infer that $0<\lambda_1' f_1+\lambda_2' f_2$, which can be rewritten as

$$0<(\lambda_1\lambda_1'(\ell_1-1)+\lambda_2\lambda_2'\ell_2,\lambda_1\lambda_1'\ell_1+\lambda_2\lambda_2'(\ell_2-1)).$$

But, this implies that $\ell_1+\ell_2>1$. Indeed, assume *ex absurdo* that $\ell_1+\ell_2\leq 1$, then we infer from the inequality above that both $\ell_2(\lambda_2\lambda_2'-\lambda_1\lambda_1')\geq 0$ and $\ell_1(\lambda_1\lambda_1'-\lambda_2\lambda_2')\geq 0$, and therefore also $\lambda_2\lambda_2\geq\lambda_1\lambda_1'$ and $\lambda_2\lambda_2'\leq\lambda_1\lambda_1'$, where at least one of these inequalities must be strict, a contradiction.

Hence indeed $\ell_1+\ell_2>1$, and applying $(\mathrm{ii})_{\curvearrowleft}$ allows us to infer that $(\ell_1,\ell_2)\in K$ and therefore $0\in R_K(\{f_1,0,f_2\})$. This proves our statement that $\operatorname{conv}(A)\cap\mathcal{L}_{>0}\neq\emptyset$ implies that $0 \in R_K(A)$, so we may also assume without loss of generality that $\operatorname{conv}(A) \cap \mathcal{L}_{>0} = \emptyset$, and return to the main line of the proof.

Since we have assumed that $f_i = 0 \in R_K(A_1)$, Definition 37_{152} tells us that there are four possibilities: one of the four Conditions $(4.18)_{152}$–$(4.21)_{152}$ must hold for A_1.

Condition $(4.18)_{152}$ for A_1 amounts to $A_1 \cap \mathcal{L}_{>0} \neq \emptyset$, contradicting our assumption that $\operatorname{conv}(A) \cap \mathcal{L}_{>0} = \emptyset$, because $A_1 \subseteq \operatorname{conv}(A)$.

If Condition $(4.21)_{152}$ holds for A_1, then $\{\lambda_1^*(k_1^* - 1, k_1^*), \lambda_2^*(k_2^*, k_2^* - 1)\} \subseteq A_1$ for some λ_1^* and λ_2^* in $\mathbb{R}_{>0}$ and (k_1^*, k_2^*) in $K \cap (0,1)^2$. Let $h_1 \coloneqq \lambda_1^*(k_1^* - 1, k_1^*)$ and $h_2 \coloneqq \lambda_2^*(k_2^*, k_2^* - 1)$. Then $A \cap \mathcal{L}_{\mathrm{II}} \neq \emptyset$ and $A \cap \mathcal{L}_{\mathrm{IV}} \neq \emptyset$ [otherwise, if $A \cap \mathcal{L}_{\mathrm{II}} = \emptyset$, since we already know that $A \cap \mathcal{L}_{>0} = \emptyset$, every convex combination of elements of A would have a non-positive value in T, which contradicts that $h_1 = \lambda_1^*(k_1^* - 1, k_1^*) \in A_1 \subseteq \operatorname{conv}(A)$, since $h_1(\mathrm{T}) = \lambda_1^* k_1^* > 0$; a similar argument leads us to conclude that $A \cap \mathcal{L}_{\mathrm{II}} \neq \emptyset$], so we may assume again without loss of generality that f_1 is a gamble in $\arg\max\{\frac{h(\mathrm{T})}{h(\mathrm{T}) - h(\mathrm{H})} : h \in A \cap \mathcal{L}_{\mathrm{II}}\}$ and that f_2 is a gamble in $\arg\max\{\frac{h(\mathrm{H})}{h(\mathrm{H}) - h(\mathrm{T})} : h \in A \cap \mathcal{L}_{\mathrm{IV}}\}$. Since we have assumed that $\operatorname{conv}(A) \cap \mathcal{L}_{>0} = \emptyset$, we see that $\operatorname{conv}(\{h_1, 0, h_2\}) \cap \mathcal{L}_{>0} = \emptyset$—and therefore also $\operatorname{posi}(\{h_1, 0, h_2\}) \cap \mathcal{L}_{>0} = \emptyset$—whence, by Equation $(4.31)_{165}$, $k_1^* + k_2^* \leq 1$. If $(k_1^*, k_2^*) = (\ell_k, \ell_m)$ for some k and m in $\{1, \ldots, n\}$ such that $f_k \in \mathcal{L}_{\mathrm{II}}$ and $f_m \in \mathcal{L}_{\mathrm{IV}}$, then $0 \in R_K(A)$ by Condition $(4.21)_{152}$. If this is not the case, then we distinguish between three possibilities: (i) $k_1^* \neq \ell_k$ for all k in $\{1, \ldots, n\}$ such that $f_k \in \mathcal{L}_{\mathrm{II}}$ and $k_2^* = \ell_m$ for some m in $\{1, \ldots, n\}$ such that $f_m \in \mathcal{L}_{\mathrm{IV}}$, (ii) $k_1^* = \ell_k$ for some k in $\{1, \ldots, n\}$ such that $f_k \in \mathcal{L}_{\mathrm{II}}$ and $k_2^* \neq \ell_m$ for all m in $\{1, \ldots, n\}$ such that $f_m \in \mathcal{L}_{\mathrm{IV}}$, and (iii) $k_1^* \neq \ell_k$ for all k in $\{1, \ldots, n\}$ such that $f_k \in \mathcal{L}_{\mathrm{II}}$ and $k_2^* \neq \ell_m$ for all m in $\{1, \ldots, n\}$ such that $f_m \in \mathcal{L}_{\mathrm{IV}}$.

In case (i), we already find that $\lambda(k_2^*, k_2^* - 1) \in A$ for some λ in $\mathbb{R}_{>0}$. If $k_1^* \leq \ell_1$, then $(k_1^*, k_2^*) \in K$ implies that $(\ell_1, k_2^*) \in K$ because K is increasing. Since we know that $f_1 = \lambda_1(\ell_1 - 1, \ell_1) \in A$, this guarantees that $0 \in R_K(A)$, by Condition $(4.21)_{152}$. If $k_1^* > \ell_1$, then we claim that necessarily also $\ell_1 + \ell_2 > 1$, and therefore $(\ell_1, \ell_2) \in K$ by Property K4_{149}, so indeed $0 \in R_K(A)$ by Condition $(4.21)_{152}$. To see that $\ell_1 + \ell_2 > 1$, assume *ex absurdo* that (a) $\ell_1 + \ell_2 < 1$ or (b) $\ell_1 + \ell_2 = 1$.

If (a) $\ell_1 + \ell_2 < 1$, then we infer from Lemma 118_{164} and $k_1^* > \ell_1$ that $h_1 \notin \operatorname{posi}(\{f_1, 0, f_2\}) = \operatorname{posi}(A)$ [this equality holds because every element f_k of $A \subseteq \mathcal{L}_{>0}^c$ belongs either to $\mathcal{L}_{\leq 0} \subseteq \operatorname{posi}(\{f_1, 0, f_2\})$, to $\mathcal{L}_{\mathrm{II}}$—and then $\ell_k \leq \ell_1$ so $f_k \in \operatorname{posi}(\{f_1, 0, f_2\})$—, or to $\mathcal{L}_{\mathrm{IV}}$—and then $\ell_k \leq \ell_2$ so $f_k \in \operatorname{posi}(\{f_1, 0, f_2\})$], so *a fortiori* also $h_1 \notin \operatorname{conv}(A)$, a contradiction.

If (b) $\ell_1 + \ell_2 = 1$, then we infer from Lemma 118_{164} that $\operatorname{posi}(\{f_1, f_2\}) = \operatorname{span}(\{f_1\})$. Since it follows from the assumptions that every element of A is dominated by some element of $\operatorname{posi}(\{f_1, f_2\}) = \operatorname{span}(\{f_1\})$, we see that $\operatorname{posi}(A) \subseteq \operatorname{span}(\{f_1\}) + \mathcal{L}_{\leq 0}$. Since $k_1^* > \ell_1$, we conclude that $h_1 \notin \operatorname{span}(\{f_1\}) + \mathcal{L}_{\leq 0}$, so *a fortiori* also $h_1 \notin \operatorname{conv}(A)$, again a contradiction.

In case (ii), a completely similar argument leads us to conclude that $0 \in R_K(A)$ here as well.

In case (iii) there are, again, three possibilities: (α) $k_1^* < \ell_1$ and $k_2^* < \ell_2$, so $(\ell_1, \ell_2) \in K$ because K is increasing, and therefore $0 \in R_K(A)$ by Condition $(4.21)_{152}$; (β) $k_1^* > \ell_1$ and $k_2^* < \ell_2$, and its symmetric counterpart $k_1^* < \ell_1$ and $k_2^* > \ell_2$; and (γ) $k_1^* > \ell_1$ and $k_2^* > \ell_2$, and therefore $\ell_1 + \ell_2 < k_1^* + k_2^* \leq 1$, so $\ell_1 + \ell_2 < 1$ and Lemma 118_{164} guarantee that $h_1 \notin \operatorname{posi}(\{f_1, 0, f_2\}) = \operatorname{posi}(A)$, and therefore *a fortiori* $h_1 \notin \operatorname{conv}(A)$, a contradiction. It therefore suffices to consider case (β), and show that $k_1^* > \ell_1$ and $k_2^* < \ell_2$ implies that $0 \in R_K(A)$, since the case that $k_1^* < \ell_1$ and $k_2^* > \ell_2$ can be covered by a completely symmetrical argument. So assume that $k_1^* > \ell_1$ and $k_2^* < \ell_2$. Since $h_1 \in \operatorname{conv}(A) \subseteq \operatorname{posi}(A)$, Lemma 118_{164} and $k_1^* > \ell_1$ guarantee that necessarily $\ell_1 + \ell_2 > 1$, so $(\ell_1, \ell_2) \in K$ by Property $\mathrm{K4}_{149}$, and therefore once again $0 \in R_K(A)$, by Condition $(4.21)_{152}$.

If Condition $(4.20)_{152}$ holds for A_1, then $\lambda_2^*(k_2^*, k_2^* - 1) \in A_1$ for some λ_2^* in $\mathbb{R}_{>0}$ and $(0, k_2^*) \in K$. Let $h_2 := \lambda_2^*(k_2^*, k_2^* - 1)$. Then $A \cap \mathcal{L}_{\mathrm{IV}} \neq \emptyset$ [otherwise, if $A \cap \mathcal{L}_{\mathrm{IV}} = \emptyset$, since we already know that $A \cap \mathcal{L}_{>0} = \emptyset$, every convex combination of elements of A would have a non-positive value in H, which contradicts that $h_2 = \lambda_2^*(k_2^*, k_2^* - 1) \in A_1 \subseteq \operatorname{conv}(A)$, since $h_2(\mathrm{H}) = \lambda_2^* k_2^* > 0$], and we may therefore assume without loss of generality that f_2 is a gamble in $\arg\max\{\frac{h(\mathrm{H})}{h(\mathrm{H}) - h(\mathrm{T})} : h \in A \cap \mathcal{L}_{\mathrm{IV}}\}$. If $k_2^* = \ell_m$ for some m in $\{1, \dots, n\}$ such that $f_m \in \mathcal{L}_{\mathrm{IV}}$, then $0 \in R_K(A)$ by Condition $(4.20)_{152}$. If this is not the case, then $k_2^* \neq \ell_m$ for all m in $\{1, \dots, n\}$ such that $f_m \in \mathcal{L}_{\mathrm{IV}}$, and in particular also $k_2^* \neq \ell_2$. If $k_2^* < \ell_2$, then $(0, k_2^*) \in K$ implies that $(0, \ell_2) \in K$ because K is increasing, any we may therefore again conclude that $0 \in R_K(A)$, by Condition $(4.20)_{152}$. If $k_2^* > \ell_2$, then $A \cap \mathcal{L}_{\mathrm{II}} \neq \emptyset$ [otherwise, if $A \cap \mathcal{L}_{\mathrm{II}} = \emptyset$, since we already know that $A \cap \mathcal{L}_{>0} = \emptyset$, by Lemma 118_{164} we have that $\operatorname{posi}(A) \subseteq \operatorname{posi}(\{(-1,0), 0, f_2\})$, and since $h_2 \in \mathcal{L}_{\mathrm{IV}} \cap \operatorname{conv}(A)$, this implies that $k_2^* = \frac{h_2(\mathrm{H})}{h_2(\mathrm{H}) - h_2(\mathrm{T})} \leq \frac{f_2(\mathrm{H})}{f_2(\mathrm{H}) - f_2(\mathrm{T})} = \ell_2$, a contradiction]. We may therefore assume without loss of generality that f_1 is a gamble in $\arg\max\{\frac{h(\mathrm{T})}{h(\mathrm{T}) - h(\mathrm{H})} : h \in A \cap \mathcal{L}_{\mathrm{II}}\}$. Because $h_2 \in \operatorname{conv}(A)$, we find that $\ell_1 + \ell_2 > 1$: if not, then $\ell_1 + \ell_2 \leq 1$, and by a completely similar reasoning as under (a) and (b) above, we find that $h_2 \notin \operatorname{conv}(A)$, a contradiction. But $\ell_1 + \ell_2 > 1$ implies that $(\ell_1, \ell_2) \in K$ by Property $\mathrm{K4}_{149}$, and therefore we again conclude that $0 \in R_K(A)$, by Condition $(4.21)_{152}$.

Finally, if Condition $(4.19)_{152}$ holds for A_1, then a completely similar argument to the one above for Condition $(4.19)_{152}$ leads us to conclude again that $0 \in R_K(A)$. □

The results in this section so far can be succinctly summarised as follows:

Theorem 120. *Consider a two-dimensional option space $\mathcal{V}$. There is a one-to-one correspondence between coherent rejection functions on $\mathcal{V}$ satisfying Property* $\mathrm{R6}_{25}$ *and subsets of* $[0,1)^2$ *satisfying Properties* $\mathrm{K1}_{148}$–$\mathrm{K3}_{149}$.

Moreover, there is a one-to-one correspondence between coherent rejection functions on $\mathcal{V}$ satisfying Property $\mathrm{R5}_{25}$ *and subsets of* $[0,1)^2$ *satisfying Properties* $\mathrm{K1}_{148}$–$\mathrm{K4}_{149}$.

4.4.5 *Counterexample*

Let us call *lexicographic* coordinate rejection set a coordinate rejection set corresponding to a lexicographic choice function. In order to find a coordinate rejection set that is no infimum of such lexicographic coordinate rejection sets, we first need to find out what these lexicographic coordinate rejection sets look like. Recall from Proposition 94_{129} that all the lexicographic coherent sets of desirable gambles on a binary possibility space $\{\mathrm{H}, \mathrm{T}\}$ are given by

$$\overline{\mathbf{D}}_{\mathrm{L}} := \{D_\rho, D_\rho^{\mathrm{H}}, D_\rho^{\mathrm{T}} : \rho \in (0,1)\} \cup \{D_0, D_1\} = \{D_\rho : \rho \in (0,1)\} \cup \hat{\mathbf{D}},$$

and the lexicographic rejection functions on $\mathcal{L}(\{\mathrm{H},\mathrm{T}\})$ are $\overline{\mathbf{R}}_{\mathrm{L}} = \{R_D : D \in \overline{\mathbf{D}}_{\mathrm{L}}\}$. We determine the corresponding coordinate rejection sets. For any D in $\overline{\mathbf{D}}_{\mathrm{L}}$, we let K_D be the coordinate rejection set that corresponds to the rejection function R_D. For any ρ in $(0,1)$ and $(k_1,k_2) \in [0,1)^2$, observe that

$$\begin{aligned}(k_1,k_2) \in K_{D_\rho} &\Leftrightarrow 0 \in R_{D_\rho}(\{(k_1-1,k_1),0,(k_2,k_2-1)\})\\ &\Leftrightarrow \{(k_1-1,k_1),(k_2,k_2-1)\} \cap D_\rho \neq \emptyset\\ &\Leftrightarrow (k_1-1,k_1) \in D_\rho \text{ or } (k_2,k_2-1) \in D_\rho\\ &\Leftrightarrow k_1 > \rho \text{ or } k_2 > 1-\rho,\end{aligned} \tag{4.34}$$

and similarly,

$$\begin{aligned}(k_1,k_2) \in K_{D_\rho^{\mathrm{H}}} &\Leftrightarrow (k_1-1,k_1) \in D_\rho^{\mathrm{H}} \text{ or } (k_2,k_2-1) \in D_\rho^{\mathrm{H}}\\ &\Leftrightarrow k_1 > \rho \text{ or } k_2 \geq 1-\rho,\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}(k_1,k_2) \in K_{D_\rho^{\mathrm{T}}} &\Leftrightarrow (k_1-1,k_1) \in D_\rho^{\mathrm{T}} \text{ or } (k_2,k_2-1) \in D_\rho^{\mathrm{T}}\\ &\Leftrightarrow k_1 \geq \rho \text{ or } k_2 > 1-\rho.\end{aligned} \tag{4.36}$$

Finally, also for D_0 and D_1,

$$\begin{aligned}(k_1,k_2) \in K_{D_0} &\Leftrightarrow (k_1-1,k_1) \in D_0 \text{ or } (k_2,k_2-1) \in D_0\\ &\Leftrightarrow k_1 > 0\end{aligned} \tag{4.37}$$

and

$$\begin{aligned}(k_1,k_2) \in K_{D_1} &\Leftrightarrow (k_1-1,k_1) \in D_1 \text{ or } (k_2,k_2-1) \in D_1\\ &\Leftrightarrow k_2 > 0.\end{aligned} \tag{4.38}$$

Look at Figure 4.4$_\curvearrowright$ for a graphical representation of these coordinate rejection sets for a generic ρ.

We also need to keep in mind that taking infima of rejection functions corresponds to taking *intersections* of coordinate rejection sets: consider any collection $\mathcal{R}$ of coherent rejection functions and any (k_1,k_2) in $[0,1)^2$, then

$$\begin{aligned}(k_1,k_2) \in K_{\inf \mathcal{R}} &\Leftrightarrow 0 \in \bigcap_{R \in \mathcal{R}} R(\{(k_1-1,k_1),0,(k_2,k_2-1)\})\\ &\Leftrightarrow (\forall R \in \mathcal{R}) 0 \in R(\{(k_1-1,k_1),0,(k_2,k_2-1)\})\\ &\Leftrightarrow (\forall R \in \mathcal{R})(k_1,k_2) \in K_R\\ &\Leftrightarrow (k_1,k_2) \in \bigcap_{R \in \mathcal{R}} K_R,\end{aligned}$$

so $\bigcap_{R \in \mathcal{R}} K_R$ is the coordinate rejection set that corresponds to the rejection function $\inf \mathcal{R}$.

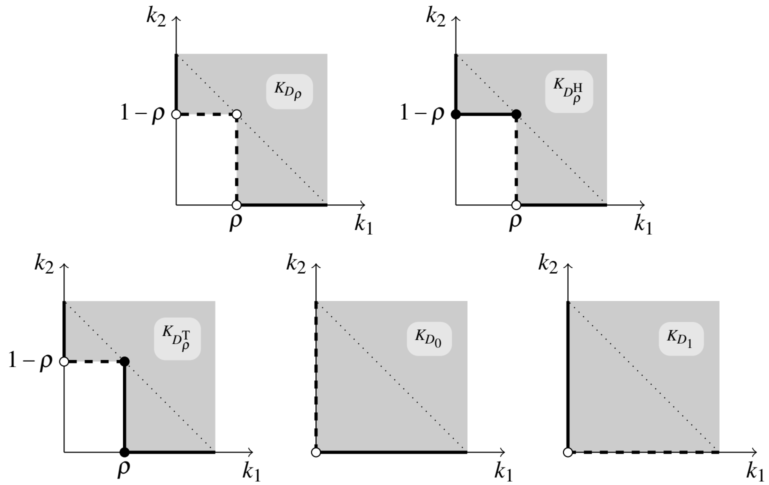


Figure 4.4: Illustration of lexicographic coordinate rejection sets

We are now, finally, ready to provide an example of a coordinate rejection set that satisfies Properties K1$_{148}$–K4$_{149}$ but is no intersection of lexicographic coordinate rejection sets.

Example 21. Consider any ℓ_1 and ℓ_2 in $(0,1)$ such that $\ell_1+\ell_2<1$, and the coordinate rejection set $K_{\ell_1,\ell_2} \subseteq [0,1)^2$ depicted in Figure 4.5, and defined by

$$\begin{aligned}K_{\ell_1,\ell_2} &:= \big\{(k_1,k_2)\in[0,1)^2 : k_1+k_2>1 \text{ or} \\ &\qquad\qquad (k_1\geq\ell_1 \text{ and } k_2\geq\ell_2 \text{ and } (k_1,k_2)\neq(\ell_1,\ell_2))\big\} \\ &= \big\{(k_1,k_2)\in[0,1)^2 : k_1+k_2>1 \text{ or } (k_1,k_2)>(\ell_1,\ell_2)\big\}. \end{aligned} \tag{4.39}$$

We show that it corresponds to a rejection function that is coherent and satisfies Property R5$_{25}$. By Theorem 120$_{168}$ it suffices to show that K_{ℓ_1,ℓ_2} satisfies Properties K1$_{148}$–K4$_{149}$. That it satisfies Properties K1$_{148}$, K2$_{149}$ and K4$_{149}$ is clear from its definition. We show that it also satisfies Property K3$_{149}$. Note that $(0,a)\notin K_{\ell_1,\ell_2}$ and $(1-a,0)\notin K_{\ell_1,\ell_2}$ for all a in $[0,1]$, so the Properties K3b$_{149}$ and K3c$_{149}$ are trivially satisfied for K_{ℓ_1,ℓ_2}. It therefore only remains to prove that Property K3a$_{149}$ is satisfied for K_{ℓ_1,ℓ_2}. Consider any a, b and c in $[0,1)$ such that $c<a$, $a+b<1$, $(b,a)\in K_{\ell_1,\ell_2}$ and $(1-a,c)\in K_{\ell_1,\ell_2}$. We need to show that then

$$(x,c)\in K_{\ell_1,\ell_2} \text{ for all } x \text{ in } (b,1) \text{ and } (b,y)\in K_{\ell_1,\ell_2} \text{ for all } y \text{ in } (c,1),$$

so consider any x in $(b,1)$ and y in $(c,1)$. Since $(b,a)\in K_{\ell_1,\ell_2}$ and $a+b<1$, Equation (4.39) tells us that $(b,a)>(\ell_1,\ell_2)$, so $x>b\geq\ell_1$. Similarly, since

$(1-a,c) \in K_{\ell_1,\ell_2}$ and $c < a$ (or equivalently, $1-a+c<1$), Equation (4.39) tells us that $(1-a,c) > (\ell_1,\ell_2)$, so $y > c \geq \ell_2$. Then $(x,c) > (\ell_1,\ell_2)$ and $(b,y) > (\ell_1,\ell_2)$, whence indeed $(x,c) \in K_{\ell_1,\ell_2}$ and $(b,y) \in K_{\ell_1,\ell_2}$. So we see that K_{ℓ_1,ℓ_2} satisfies Properties K1$_{148}$–K4$_{149}$. It therefore corresponds to a coherent and 'convex' rejection function.

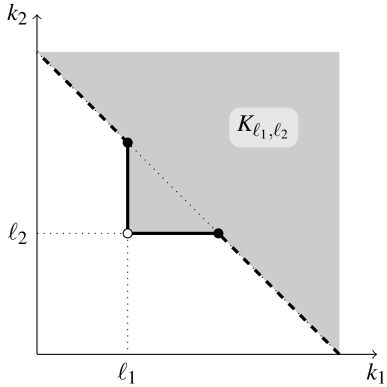


Figure 4.5: The rejection set K_{ℓ_1,ℓ_2}

We show that K_{ℓ_1,ℓ_2} is no intersection of lexicographic coordinate rejection sets. Assume *ex absurdo* that it is an intersection $\bigcap K_{\mathcal{D}}$ of some non-empty collection of lexicographic coordinate rejection sets $K_{\mathcal{D}} \coloneqq \{K_D : D \in \mathcal{D}\}$, with $\mathcal{D} \subseteq \overline{\mathbf{D}}_{\mathrm{L}}$. Then, since $(\ell_1,\ell_2) \notin K_{\ell_1,\ell_2}$, there must be some D in $\mathcal{D}$ such that $(\ell_1,\ell_2) \notin K_D$. There are a number of possibilities: (i) $D = D_\rho$ for some ρ in $(0,1)$, (ii) $D = D_\rho^{\mathrm{H}}$ for some ρ in $(0,1)$, or (iii) $D = D_\rho^{\mathrm{T}}$ for some ρ in $(0,1)$—$D \in \{D_0, D_1\}$ is impossible since (ℓ_1,ℓ_2) belong to both K_{D_0} [by Equation (4.37)$_{169}$] and K_{D_1} [by Equation (4.38)$_{169}$].

In case (i), since $(\ell_1,\ell_2) \notin K_{D_\rho}$, we infer from Equation (4.34)$_{169}$ that $\ell_1 \leq \rho$ and $\ell_2 \leq 1-\rho$, or in other words, that $\rho \in [\ell_1, 1-\ell_2]$. From $\ell_1+\ell_2 < 1$, we infer that $\ell_1 < \rho$ or $\ell_2 < 1-\rho$. We consider the case that $\ell_1 < \rho$; if $\ell_2 < 1-\rho$, a symmetrical argument leads to a similar result. From Equation (4.34)$_{169}$ we infer, using $\ell_2 \leq 1-\rho$, that on the one hand $(\rho,\ell_2) \notin K_{D_\rho}$. On the other hand, we infer from $(\rho,\ell_2) > (\ell_1,\ell_2)$ that $(\rho,\ell_2) \in K_{\ell_1,\ell_2}$, by Equation (4.39). This leads us to conclude that $K_{\ell_1,\ell_2} \neq K_{D_\rho}$.

In case (ii), then, since $(\ell_1,\ell_2) \notin K_{D_\rho^{\mathrm{H}}}$, we infer from Equation (4.35)$_{169}$ that $\ell_1 \leq \rho$ and $\ell_2 < 1-\rho$, or in other words, that $\rho \in [\ell_1, 1-\ell_2)$. This implies that $\ell_2 < \frac{1-\rho+\ell_2}{2} < 1-\rho$: indeed, $\frac{1-\rho+\ell_2}{2}$ is a convex mixture of ℓ_2 and $1-\rho$. From Equation (4.35)$_{169}$, we infer, using $\frac{1-\rho+\ell_2}{2} < 1-\rho$, that on the one hand $(\ell_1, \frac{1-\rho+\ell_2}{2}) \notin K_{D_\rho^{\mathrm{H}}}$. On the other hand, we infer from $(\ell_1, \frac{1-\rho+\ell_2}{2}) > (\ell_1,\ell_2)$

that $(\ell_1, \frac{1-\rho+\ell_2}{2}) \in K_{\ell_1,\ell_2}$, by Equation (4.39)$_{170}$. This leads us to conclude that $K_{\ell_1,\ell_2} \neq K_{D^{\mathrm{H}}_\rho}$.

In case (iii), a completely symmetrical argument leads to the conclusion that $K_{\ell_1,\ell_2} \neq K_{D^{\mathrm{H}}_\rho}$.

This tells us that none of the remaining possibilities can obtain, a contradiction. $\Diamond$

To conclude, the rejection function that corresponds to K_{ℓ_1,ℓ_2} is coherent and satisfies Property R5$_{25}$ by Theorem 120$_{168}$, but it is no infimum of lexicographic rejection functions. This answers the initial question in this section—is $C = \inf\{C' \in \overline{\mathbf{C}}_{\mathrm{L}} : C \sqsubseteq C'\}$ for every coherent choice function C that satisfies Property C5$_{25}$?—in the negative: in the restrictive case of two possible outcomes, we have found a counterexample.

4.5 DISCUSSION

In the first part of this chapter—Sections 4.1$_{127}$–4.3$_{143}$—we have investigated the implications of Seidenfeld et al.'s [67] convexity axiom. We have obtained a nice representation of purely binary 'convex' choice functions in terms of lexicographic probability systems, which we have studied in some detail.

The central question of this chapter, however, was whether there is a representation for choice functions in terms of maximal (or other) 'representing models' that are easy to describe. We have shown that, when considering $\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\}$ as the 'representing' models, coherence alone is not sufficient, and, moreover, that adding Property C5$_{25}$ does not help, even if we resort to the larger set $\overline{\mathbf{C}}_{\mathrm{L}}$ of lexicographic choice functions rather than the maximal ones $\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\}$ as possible representing models.

There are several open problems that the discussion in this chapter now allows us to identify. First and foremost, the characterisation of the maximal coherent choice functions $\hat{\mathbf{C}}$ is a very important open problem. Let us elaborate a bit. In Proposition 62$_{70}$ we have already identified a subset of $\hat{\mathbf{C}}$: every maximal coherent set of desirable options $\hat{D}$, induces a choice function $C_{\hat{D}}$ that is maximal: $\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\} \subseteq \hat{\mathbf{C}}$. Consider the seemingly straightforward proposition that, for any maximal coherent choice function $\hat{C}$, its corresponding set of desirable options $D_{\hat{C}}$ is maximal as well: $\{D_{\hat{C}} : \hat{C} \in \hat{\mathbf{C}}\} \subseteq \hat{\mathbf{D}}$. If we were able to establish this proposition, it would imply that $C_{D_{\hat{C}}} = \hat{C}$ for every $\hat{C}$ in $\hat{\mathbf{C}}$. To see why, consider any $\hat{C}$ in $\hat{\mathbf{C}}$, then assuming that $D_{\hat{C}}$ is a maximal set of desirable options, by Proposition 62$_{70}$ $C_{D_{\hat{C}}}$ is a maximal choice function. But $C_{D_{\hat{C}}} \sqsubseteq \hat{C}$ by Corollary 59$_{67}$, and therefore $C_{D_{\hat{C}}} = \hat{C}$, since $C_{D_{\hat{C}}}$ is a maximal choice function. This would allow us to conclude that every maximal coherent choice function is purely binary, and therefore imply that $\{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\} = \hat{\mathbf{C}}$. However, the proof of the above-mentioned proposition has proved to be very elusive, and it therefore remains to be seen whether all maximal coherent choice functions are purely binary.

There is another open problem related to this: it remains to be established whether the coherent choice functions constitute a dually atomic complete meet-semilattice, and therefore a strong belief structure. As already mentioned in Example 19126, if we could establish that $\hat{\mathbf{C}} = \{C_{\hat{D}} : \hat{D} \in \hat{\mathbf{D}}\}$, then the coherent choice functions would have no representation in terms of maximal choice functions, and would therefore not constitute a strong belief structure.

5

INDIFFERENCE AND SYMMETRY

In this chapter, we add a different type of assessment to the picture: that of *indifference*. We will see what it means to be indifferent between two options, and how an uncertainty model can represent such indifference.

First, we move in Section 5.1 to a definition and closer investigation of the property of indifference between two options. This is, by the way, also a crucial step towards enabling uncertainty models to represent symmetry [27], which we allude to in Section 5.8_{191}. For example, as we will see in Chapter 8_{247}, when modelling exchangeability and proving de Finetti-type representation theorems, an appropriate notion of indifference is essential. After the definition of indifference and some of its properties are in place, we derive representation results for desirability (Section 5.3_{178}) and for choice models (Section 5.4_{179}). For desirability, such a representation already exists (see, for instance, References [20, 31, 58]), but we give an equivalent formulation in terms of *equivalence classes* of options, as we perceive this as more elegant, and as it introduces the notations we will need later on for choice models. Eventually, in Section 5.9_{193} we will find the natural extension of a direct assessment (as in Chapter 3_{89}) combined with an indifference assessment.

For choice functions, indifference was introduced by Seidenfeld [63], and for sets of desirable gambles by De Cooman and Quaeghebeur [31]. This chapter is based on earlier work of mine, together with Gert de Cooman, Enrique Miranda and Erik Quaeghebeur [78].

5.1 INDIFFERENCE AND DESIRABILITY

For sets of desirable gambles, there is a systematic way of modelling indifference [20, 31, 58]. In what follows, we recall how it works, but formulate everything in terms of the more abstract notion of options, rather than gambles.

In addition to a subject's set of desirable options D—the options he strictly prefers to the zero option—we also consider the options that he considers to be *equivalent* to the zero option. We call these options *indifferent*. A set of indifferent options I is simply a subset of $\mathcal{V}$, but as before with desirable options, we pay special attention to *coherent* sets of indifferent options.

Definition 38 (Coherent set of indifferent options)**.** *We call a set of indifferent options I* coherent *if for all u and v in $\mathcal{V}$ and all λ in $\mathbb{R}$:*

I1. $0 \in I$;

I2. *if* $u \in \mathcal{V}_{>0} \cup \mathcal{V}_{<0}$ *then* $u \notin I$;

I3. *if* $u \in I$ *then* $\lambda u \in I$;

I4. *if* $u, v \in I$ *then* $u + v \in I$.

We collect all coherent sets of indifferent options in $\bar{\mathbf{I}}$.

Taken together, Axioms I3 and I4 are equivalent to imposing that $\operatorname{span}(I) = I$, and due to Axiom I1, I is non-empty and therefore a linear subspace of $\mathcal{V}$.

The interaction between indifferent and desirable options is subject to rationality criteria as well: they should be compatible with one another.

Definition 39 (Compatibility for desirability)**.** *Given a set of desirable options D and a coherent set of indifferent options I, we call D* compatible *with I if $D + I \subseteq D$.*

The idea behind this is that adding an indifferent option to a desirable one should result in a desirable option. Since $D \subseteq D + I$, due to Axiom I1, compatibility of D and I is equivalent to $D + I = D$: the set of indifferent gambles acts as a neutral element with respect to the Minkowski addition.

The smallest such compatible coherent set of indifferent options is always the trivial one, given by the null space $I = \{0\}$. The idea behind Definition 39 is that adding an indifferent option to a desirable option does not make it non-desirable, or also, that adding a desirable option to an indifferent one makes it desirable.

An immediate consequence of compatibility between a coherent set of desirable options D and a coherent set of indifferent options I is that $D \cap I = \emptyset$, meaning that no option can be assessed as desirable—strictly preferred to the zero option—and indifferent—equivalent to the zero option—at the same time. To see this, if *ex absurdo* $D \cap I \neq \emptyset$, then there would be some u in D such that $u \in I$. But then, by Axiom I3, also $-u \in I$, so compatibility of D with I would imply that $u + (-u) = 0 \in D$, which contradicts Axiom D1_{57}.

5.2 INDIFFERENCE AND QUOTIENT SPACES

In order to introduce indifference for choice functions, we build on a coherent set of indifferent options I, as defined in Definition 38. Two options u and v are considered to be indifferent to a subject whenever $v-u$ is indifferent, or in other words whenever $v-u \in I$. The underlying idea of our notion of indifference will be that we identify indifferent options, and choose between equivalence classes of indifferent options, rather than between single options. We formalise this intuition below.

We collect all options that are indifferent to an option $u \in \mathcal{V}$ into the *equivalence class*, or quotient class

$$[u] \coloneqq \{v \in \mathcal{V} : v-u \in I\} = \{u\}+I.$$

We also denote the set $[u]$ as u/I. Of course, $[0]=\{0\}+I=I$ is a linear subspace, and the $[u]=\{u\}+I$ are affine subspaces of $\mathcal{V}$. The set of all these equivalence classes is the *quotient space*

$$\mathcal{V}/I \coloneqq \{[u] : u \in \mathcal{V}\} = \{\{u\}+I : u \in \mathcal{V}\} = \{u/I : u \in \mathcal{V}\}.$$

This quotient space is a vector space under the vector addition, defined by

$$[u]+[v] = \{u\}+I+\{v\}+I = \{u+v\}+I = [u+v] \text{ for all } u \text{ and } v \text{ in } \mathcal{V},$$

and the scalar multiplication, defined by

$$\lambda[u] = \lambda(\{u\}+I) = \{\lambda u\}+I = [\lambda u], \text{ for all } u \text{ in } \mathcal{V} \text{ and } \lambda \text{ in } \mathbb{R}.$$

$[0]=I$ is the additive identity of $\mathcal{V}/I$.

That we identify indifferent options, and therefore express preferences between equivalence classes of indifferent options, essentially means that we define choice functions on $\mathcal{Q}(\mathcal{V}/I)$. But in order to characterise coherence for such choice functions, we need to introduce a convenient vector ordering on $\mathcal{V}/I$ that is appropriately related to the vector ordering on $\mathcal{V}$; see Section 2.3_{19}. For two elements $[u]$ and $[v]$ of $\mathcal{V}/I$, we define

$$[u] \preceq [v] \Leftrightarrow (\exists w \in I)u \preceq v+w, \tag{5.1}$$

and as usual, the strict variant of the vector ordering on $\mathcal{V}/I$ is characterised by

$$[u] \prec [v] \Leftrightarrow ([u] \preceq [v] \text{ and } [u] \neq [v]) \text{ for all } [u] \text{ and } [v] \text{ in } \mathcal{V}/I.$$

We begin by showing that this is indeed a vector ordering:

Proposition 121. *The ordering $\preceq$ on $\mathcal{V}/I$ is a vector ordering, and $[u] \prec [v] \Leftrightarrow (\exists w \in I)u \prec v+w$ for any u and v in $\mathcal{V}$.*

Proof. For the first statement, we show that $\preceq$ is a partial order—meaning that it is reflexive, antisymmetric and transitive—that satisfies the two characteristic Properties (2.1)$_{10}$ and (2.2)$_{10}$ of a vector ordering. To see that $\preceq$ is reflexive, use $w = 0$ in Equation (5.1). For antisymmetry, consider any $[u]$ and $[v]$ in $\mathcal{V}/I$ such that $[u] \preceq [v]$ and $[v] \preceq [u]$, and therefore $u \preceq v + w$ and $v \preceq u + w'$ for some w and w' in I. This implies, by repeatedly applying Equation (2.1)$_{10}$ and the transitivity of the vector ordering on $\mathcal{V}$ that $w + w' \succeq 0$. Now, by Axiom I4$_{176}$, $w + w' \in I$, and therefore, by Axiom I2$_{176}$, $w + w' = 0$, implying that $u \preceq v + w$ and $v \preceq u - w$, and therefore also $v + w \preceq u$. Hence indeed $u = v + w$ and therefore $[u] = [v]$, by the antisymmetry of the vector ordering on $\mathcal{V}$. For transitivity, consider any $[u]$, $[v]$ and $[w]$ in $\mathcal{V}/I$ such that $[u] \preceq [v]$ and $[v] \preceq [w]$, and therefore $u \preceq v + v'$ and $v \preceq w + w'$ for some v' and w' in I. Then $v + v' \preceq w + w' + v'$ by Equation (2.1)$_{10}$, whence also $u \preceq w + w' + v'$ by the transitivity of the vector ordering on $\mathcal{V}$. By coherence [Axiom I4$_{176}$] also $v' + w' \in I$, whence indeed $[u] \preceq [w]$ by Equation (5.1)$_{\curvearrowleft}$.

Next, we prove that $\preceq$ satisfies Equations (2.1)$_{10}$ and (2.2)$_{10}$. Consider any $[u_1]$ and $[u_2]$ in $\mathcal{V}/I$ such that $[u_1] \preceq [u_2]$, implying that $u_1 \preceq u_2 + u_2'$ for some $u_2' \in I$. For Equation (2.1)$_{10}$, consider any v in $\mathcal{V}$, then also $u_1 + v \preceq u_2 + v + u_2'$, implying that indeed $[u_1] + [v] \preceq [u_2] + [v]$. The converse implication follows analogously, by adding $-v$ rather than v. For Equation (2.2)$_{10}$, consider any λ in $\mathbb{R}_{>0}$, then also $\lambda u_1 \preceq \lambda u_2 + \lambda u_2'$, implying that indeed $\lambda[u_1] \preceq \lambda[u_2]$ because $\lambda u_2' \in I$ by Axiom I3$_{176}$. The converse implication follows analogously, by multiplying with $\frac{1}{\lambda}$ rather than λ.

We now turn to the second statement. For necessity, consider any u and v in $\mathcal{V}$ such that $[u] \prec [v]$, so $[u] \neq [v]$ and $u \preceq v + w$ for some w in I. $[u] \neq [v]$ implies that $u \neq v + w$ for all w in I. Taken together with $u \preceq v + w$ for some w, this implies that indeed $u \prec v + w$ for some w. For sufficiency, consider any u and v in $\mathcal{V}$ such that $u \prec v + w$ for some w in I. Assume *ex absurdo* that $[u] \nprec [v]$, meaning that $[u] = [v]$ or $u \npreceq v + w$ for all w in I. The latter possibility is incompatible with the assumption that $u \prec v + w$ for some w in I, leaving only the first possibility, which is equivalent to $u = v + w'$ for some w' in I. Then $v + w' \prec v + w$, implying that $0 \prec w - w'$ by Equation (2.1)$_{10}$. This contradicts the coherence of I [Axiom I2$_{176}$] because $w - w'$ is an element of I by Axiom I4$_{176}$. $\square$

We use the notation $A/I \coloneqq \{[u] : u \in A\} = \{u/I : u \in A\}$ for the option set of equivalence classes $[u]$ associated with the options u in an option set A in $\mathcal{Q}(\mathcal{V})$. The map $\bullet/I$ is an onto map from $\mathcal{Q}(\mathcal{V})$ to $\mathcal{Q}(\mathcal{V}/I)$ that preserves set inclusion:

$$(\forall A_1, A_2 \in \mathcal{Q}(\mathcal{V}))A_1 \subseteq A_2 \Rightarrow A_1/I \subseteq A_2/I. \tag{5.2}$$

5.3 QUOTIENT SPACES AND SETS OF DESIRABLE OPTIONS

We use the quotient space $\mathcal{V}/I$ to prove interesting characterisations of indifference for sets of desirable options.

Proposition 122. *A set of desirable options $D \subseteq \mathcal{V}$ is compatible with a coherent set of indifferent options I if and only if there is some* (representing) *set of desirable options $D' \subseteq \mathcal{V}/I$ such that $D = \{u : [u] \in D'\} = \bigcup D'$. Moreover, the representing set of desirable options is unique and given by $D' = D/I \coloneqq \{[u] : u \in D\}$.*

Proof. For necessity, observe that

$$D+I=\{u+v : u\in D \text{ and } v\in I\}=\bigcup\{\{u\}+I : u\in D\}=\bigcup\{[u] : u\in D\}=\bigcup D/I,$$

and that compatibility with I guarantees that $D+I=D$. For sufficiency, assume that there is some set of desirable options $D'\subseteq \mathcal{V}/I$ such that $D=\{u : [u]\in D'\}$. Consider any u in D and any v in $I=[0]$, then $u+v\in[u]+[0]=[u+0]=[u]$, and therefore indeed $u+v\in D$. Then also $D/I=\{[u] : u\in D\}=\{[u] : [u]\in D'\}=D'$. □

This, together with the definition of compatibility, shows that the correspondence between sets of desirable options on $\mathcal{V}$ and (their representing) sets of desirable options on $\mathcal{V}/I$ is one-to-one and onto. It also preserves coherence.

Proposition 123. *Consider any set of desirable options $D\subseteq\mathcal{V}$ that is compatible with a coherent set of indifferent options I, and its representing set of desirable options $D/I\subseteq\mathcal{V}/I$. Then D is coherent if and only if D/I is.*

Proof. For the direct implication, assume that $D\subseteq\mathcal{V}$ is coherent. We show that $D/I=\{[u] : u\in D\}\subseteq\mathcal{V}/I$ satisfies the Axioms D1_{57}–D4_{57}. For Axiom D1_{57}, assume *ex absurdo* that $[0]\in D/I$, implying that $u\in D$ for some u in $[0]=I$, but then $u\in D\cap I$ and hence $D\cap I$ is non-empty, a contradiction. For Axiom D2_{57}, consider any $[u]$ in $\mathcal{V}/I$ such that $[0]\prec[u]$, meaning that $0\prec u+w$ for some w in I. Then $u+w\in D$ for some w in I, by coherence of D [Axiom D2_{57}], implying that indeed $[u]=[u]+I=[u]+[w]=[u+w]\in D/I$. For Axiom D3_{57}, assume that $[u]\in D/I$, and consider any λ in $\mathbb{R}_{>0}$. Then $u\in D$, implying that $\lambda u\in D$ by coherence of D [Axiom D3_{57}], whence indeed $\lambda[u]=[\lambda u]\in D/I$. For Axiom D4_{57}, assume that $[u]$ and $[v]$ belong to D/I, implying that u and v belong to D. Then $u+v\in D$ by coherence of D [Axiom D4_{57}], whence indeed $[u]+[v]=[u+v]\in D/I$.

For the converse implication, assume that $D/I\subseteq\mathcal{V}/I$ is coherent. We show that $D=\{u : [u]\in D/I\}\subseteq\mathcal{V}$ satisfies the Axioms D1_{57}–D4_{57}. For Axiom D1_{57}, infer that $0\notin D$ since $[0]\notin D/I$ by coherence of D/I [Axiom D1_{57}]. For Axiom D2_{57}, consider any u in $\mathcal{V}$ such that $0\prec u$. Then $[0]\prec[u]$ by taking for example $w=0$ in Proposition 121, implying that $[u]\in D/I$ by coherence of D/I [Axiom D2_{57}], whence indeed $u\in D$. For Axiom D3_{57}, assume that $u\in D$, and consider any λ in $\mathbb{R}_{>0}$. Then $[u]\in D/I$, implying that $\lambda[u]=[\lambda u]\in D/I$ by coherence of D/I [Axiom D3_{57}], whence indeed $\lambda u\in D$. For Axiom D4_{57}, assume that u and v belong to D, implying that $[u]$ and $[v]$ belong to D/I. Then $[u]+[v]\in D/I$ by coherence of D/I [Axiom D4_{57}], implying that indeed $u+v\in D$. □

5.4 QUOTIENT SPACES AND CHOICE FUNCTIONS

The discussion above inspires us to combine indifference with choice functions in the following manner: given a coherent set of indifferent options I, we say that a choice function is compatible with it when it is determined by its restriction to the quotient space that I induces.

Definition 40 (Compatibility for choice models)**.** *We call a choice function C on $\mathcal{V}$* compatible *with a coherent set of indifferent options I if there is some* representing *choice function* C' *on* $\mathcal{V}/I$ *such that* $C(A) = \{u \in A : [u] \in C'(A/I)\}$ *for all* A *in* $\mathcal{Q}(\mathcal{V})$.

Compatibility with a set of indifferent options can be defined in terms of rejection functions and choice relations as well. We call a rejection function R on $\mathcal{V}$ compatible with I if there is a representing rejection function R' on $\mathcal{V}/I$ such that $R(A) = \{u \in A : [u] \in R'(A/I)\}$ for all A in $\mathcal{Q}(\mathcal{V})$. Similarly, we call a choice relation $\lhd$ on $\mathcal{V}$ compatible with I if there is some representing choice relation $\lhd'$ on $\mathcal{V}/I$ such that $A_1 \lhd A_2 \Leftrightarrow A_1/I \lhd' A_2/I$ for all A_1 and A_2 in $\mathcal{Q}(\mathcal{V})$.

Proposition 124. *Consider any coherent set of indifferent options* $I \subseteq \mathcal{V}$*, and any corresponding choice function* C*, rejection function* R *and choice relation* $\lhd$ *on* $\mathcal{V}$*. Then the following three statements are equivalent:*

(i) C *is compatible with* I*;*

(ii) R *is compatible with* I*;*

(iii) $\lhd$ *is compatible with* I.

Proof. We will prove the following circular chain of implications: (i)⇒(ii)⇒(iii)⇒(i).

To show that (i) implies (ii), because C and R correspond, note that $R(A) = A \setminus C(A)$ for all A in $\mathcal{Q}(\mathcal{V})$. Consider any A in $\mathcal{Q}(\mathcal{V})$. Assume that C is compatible with I, so $C(A) = \{u \in A : [u] \in C'(A/I)\}$ for some choice function C' on $\mathcal{V}/I$, and therefore $R(A) = \{u \in A : [u] \notin C'(A/I)\}$. If we let R' be rejection function on $\mathcal{V}/I$ corresponding to C', then $R(A) = \{u \in A : [u] \notin R'(A/I)\}$, whence R is indeed compatible with I, with representing rejection function R'.

To show that (ii) implies (iii), because R and $\lhd$ correspond, note that $A_1 \lhd A_2 \Leftrightarrow A_1 \subseteq R(A_1 \cup A_2)$ for all A_1 and A_2 in $\mathcal{Q}(\mathcal{V})$. Consider any A_1 and A_2 in $\mathcal{Q}(\mathcal{V})$. Assume that R is compatible with I, so $R(A_1 \cup A_2) = \{u \in A_1 \cup A_2 : [u] \in R'((A_1 \cup A_2)/I)\} = \{u \in A_1 \cup A_2 : [u] \in R'(A_1/I \cup A_2/I)\}$ for some rejection function R' on $\mathcal{V}/I$. Infer that then

$$
\begin{aligned}
A_1 \lhd A_2 &\Leftrightarrow A_1 \subseteq \{u \in A_1 \cup A_2 : [u] \in R'(A_1/I \cup A_2/I)\} \\
&\Leftrightarrow (\forall u \in A_1)[u] \in R'(A_1/I \cup A_2/I) \\
&\Leftrightarrow A_1/I \subseteq R'(A_1/I \cup A_2/I) \Leftrightarrow A_1/I \lhd' A_2/I
\end{aligned}
$$

where we let $\lhd' \coloneqq \lhd_{R'}$ be the choice relation on $\mathcal{V}/I$ corresponding to R'. Therefore $\lhd$ is indeed compatible with I, with representing choice relation $\lhd'$.

Finally, to show that (iii) implies (i), because $\lhd$ and C correspond, note that $C(A) = A \setminus \bigcup\{A' \subseteq A : A' \lhd A\}$ for all A in $\mathcal{Q}(\mathcal{V})$. Assume that $\lhd$ is compatible with I, so there is some $\lhd'$ on $\mathcal{V}/I$ such that $A_1 \lhd A_2 \Leftrightarrow A_1/I \lhd' A_2/I$ for all A_1 and A_2 in $\mathcal{Q}(\mathcal{V})$. Consider any A in $\mathcal{Q}(\mathcal{V})$ and any u in A. Infer that then

$$
\begin{aligned}
u \in C(A) &\Leftrightarrow u \in A \text{ and } u \notin \bigcup\{A' \subseteq A : A'/I \lhd' A/I\} \\
&\Leftrightarrow u \in A \text{ and } [u] \notin \bigcup\{\tilde{A} \subseteq A/I : \tilde{A} \lhd' A/I\} \\
&\Leftrightarrow u \in A \text{ and } [u] \in A/I \setminus \bigcup\{\tilde{A} \subseteq A/I : \tilde{A} \lhd' A/I\} \\
&\Leftrightarrow u \in A \text{ and } [u] \in C'(A/I),
\end{aligned}
$$

where we let $C' \coloneqq C_{\lhd'}$ be the choice function on $\mathcal{V}/I$ corresponding to C'. Therefore C is indeed compatible with I, with representing choice function C'. □

Definition 40_{179} allows for characterisations that are similar to the ones for desirability in Propositions 122_{178} and 123_{179}. If a choice function on $\mathcal{V}$ is compatible with I then the representing choice function on $\mathcal{V}/I$ is necessarily unique, and we denote it by C/I:

Proposition 125. *For any choice function C on $\mathcal{V}$ that is compatible with some coherent set of indifferent options I, the unique representing choice function C/I on $\mathcal{V}/I$ is given by $C/I(A/I) \coloneqq C(A)/I$ for all A in $\mathcal{Q}(\mathcal{V})$. Hence also*

$$C(A) = A \cap \left(\bigcup C/I(A/I)\right) \textit{ for all } A \textit{ in } \mathcal{Q}(\mathcal{V}).$$

Proof. Let C' be any representing choice function, and consider any A in $\mathcal{Q}(\mathcal{V})$, then

$$C(A)/I = \{[u] : u \in C(A)\} = \{[u] : u \in A, [u] \in C'(A/I)\} = A/I \cap C'(A/I) = C'(A/I). \quad \square$$

Similarly, the unique representing rejection function R/I on $\mathcal{V}/I$ of a rejection function R on $\mathcal{V}$ that is compatible with I, is given by $R/I(A/I) \coloneqq R(A)/I$ for all A in $\mathcal{Q}(\mathcal{V})$, and the unique representing choice relation $\lhd/I$ on $\mathcal{V}/I$ of a choice relation $\lhd$ on $\mathcal{V}$ that is compatible with I, is (trivially) given by

$$A_1/I(\lhd/I)A_2/I \Leftrightarrow A_1 \lhd A_2, \text{ for all } A_1 \text{ and } A_2 \text{ in } \mathcal{Q}(\mathcal{V}).$$

Proposition 126. *Consider any coherent set of indifferent options $I \subseteq \mathcal{V}$, and any choice function C on $\mathcal{V}$ compatible with I, any rejection function R on $\mathcal{V}$ compatible with I, and any choice relation $\lhd$ on $\mathcal{V}$ compatible with I. Then the following two statements are equivalent:*

(i) *C, R and $\lhd$ correspond;*
(ii) *C/I, R/I and $\lhd/I$ correspond.*

Proof. That (i) implies (ii), follows from the proof of Proposition 124, so it suffices to show that (ii) implies (i). Consider any choice function C, rejection function R and choice relation $\lhd$ on $\mathcal{V}$, all compatible with I, and assume that C/I, R/I and $\lhd/I$ correspond. We will show that then C, R and $\lhd$ correspond; by the discussion in Section $2.2.2_{15}$ it suffices to show that $R_\lhd = R_C = R$. To prove that $R_C = R$, consider any A in $\mathcal{Q}(\mathcal{V})$ and any u in $\mathcal{V}$, and infer the following chain of equivalences:

$$\begin{aligned}
u \in C(A) &\Leftrightarrow u \in A \text{ and } [u] \in C/I(A/I) && \text{by Proposition 125}\\
&\Leftrightarrow u \in A \text{ and } [u] \notin R/I(A/I) && \text{because } C/I \text{ and } R/I \text{ correspond}\\
&\Leftrightarrow u \in A \text{ and } u \notin R(A) && \text{by the compatibility of } R \text{ with } I\\
&\Leftrightarrow u \in A \setminus R(A).
\end{aligned}$$

To prove that $R_{\lhd} = R$, consider any A in $\mathcal{Q}(\mathcal{V})$ and any u in $\mathcal{V}$, and infer the following chain of equivalences:

$$
\begin{aligned}
u \in R_{\lhd}(A) &\Leftrightarrow u \in \bigcup\{A' \subseteq A : A' \lhd A\} && \text{by Definition } 5_{17}\\
&\Leftrightarrow (\exists A' \subseteq A) u \in A' \text{ and } A' \lhd A\\
&\Leftrightarrow u \in A \text{ and } \{u\} \lhd A && \text{by Definition } 3_{15}\\
&\Leftrightarrow u \in A \text{ and } \{[u]\} \lhd/I\, A/I && \text{by the compatibility of } \lhd \text{ with } I\\
&\Leftrightarrow u \in A \text{ and } [u] \in R/I(A/I) && \text{because } R/I \text{ and } \lhd/I \text{ correspond}\\
&\Leftrightarrow u \in R(A) && \text{by the compatibility of } R \text{ with } I. \quad \square
\end{aligned}
$$

Proposition $125_{\curvearrowleft}$, together with the definition of compatibility, shows that the correspondence between choice functions on $\mathcal{V}$ and (their representing) choice functions on $\mathcal{V}/I$ is one-to-one and onto. It also preserves coherence:

Proposition 127. *Consider any choice function C on $\mathcal{V}$ that is compatible with a coherent set of indifferent options I, and its representing choice function C/I on $\mathcal{V}/I$. Then C is coherent if and only if C/I is.*

Before we prove Proposition 127, it will be useful to first establish four technical lemmas about equivalence classes.

Lemma 128. *Consider any two option sets A_1 in A_2 in $\mathcal{Q}(\mathcal{V})$, then $A_2/I \setminus A_1/I \subseteq (A_2 \setminus A_1)/I$.*

Proof. Consider any $\tilde{u}$ in $A_2/I \setminus A_1/I$, then $\tilde{u} = [v_2]$ for some v_2 in A_2 and $\tilde{u} \neq [v_1]$ for all v_1 in A_1, implying that indeed $\tilde{u} = [v_2]$ for some v_2 in $A_2 \setminus A_1$. $\square$

Lemma 129. *Consider any choice function C on $\mathcal{V}$ that is compatible with a coherent set of indifferent options I, and any option sets A_1 and A_2 in $\mathcal{Q}(\mathcal{V})$ such that $A_1 \subseteq A_2$. Then $A_1 \cap C(A_2) = \emptyset \Leftrightarrow A_1/I \cap C/I(A_2/I) = \emptyset$.*

Proof. First, assume that $A_1 \cap C(A_2) \neq \emptyset$, and consider any v in $A_1 \cap C(A_2)$. Then $[v] \in A_1/I$ and $[v] \in C(A_2)/I = C/I(A_2/I)$, by Proposition $125_{\curvearrowleft}$, so $A_1/I \cap C/I(A_2/I) \neq \emptyset$. Conversely, assume that $A_1/I \cap C/I(A_2/I) \neq \emptyset$, and consider any $\tilde{u}$ in $A_1/I \cap C/I(A_2/I)$. Then there is some v in A_1 such that $\tilde{u} = [v]$, and we infer from Proposition $125_{\curvearrowleft}$ that $\tilde{u} \in C(A_2)/I$, so also $v \in C(A_2)$. $\square$

Lemma 130. *Consider any rejection function R on $\mathcal{V}$ that is compatible with a coherent set of indifferent options I, and any option sets A_1 and A_2 in $\mathcal{Q}(\mathcal{V})$ such that $A_1 \subseteq A_2$. Then $A_1 \subseteq R(A_2) \Leftrightarrow A_1/I \subseteq R/I(A_2/I)$.*

Proof. This follows at once from Lemma 129, taking into account that C and R correspond if and only if C/I and R/I do. $\square$

Lemma 131. *For all $\tilde{A}$ in $\mathcal{Q}(\mathcal{V}/I)$, there is some A in $\mathcal{Q}(\mathcal{V})$ such that $A/I = \tilde{A}$.*

Proof. For every element $\tilde{u}$ in the finite set $\tilde{A}$, we consider some vector u in $\mathcal{V}$ such that $[u] = \tilde{u}$ [this is always possible since $\mathcal{V}/I = \{[u] : u \in \mathcal{V}\}$], and collect these options in the option set A. Then A belongs to $\mathcal{Q}(\mathcal{V})$ because it is a finite subset of $\mathcal{V}$, and $A/I = \{[u] : u \in A\} = \tilde{A}$. □

Proof of Proposition 127. For the direct implication, assume that C is coherent. We show that C/I satisfies Axioms C1$_{20}$–C4b$_{20}$.

For Axiom C1$_{20}$, assume *ex absurdo* that $C/I(\tilde{A}) = \varnothing$ for some $\tilde{A}$ in $\mathcal{Q}(\mathcal{V}/I)$. Consider any option set A in $\mathcal{Q}(\mathcal{V})$ such that $A/I = \tilde{A}$ [there always is such an A, due to Lemma 131], then $C(A) = \{u \in A : [u] \in C/I(\tilde{A})\} = \{u \in A : [u] \in \varnothing\} = \varnothing$, contradicting the coherence of C [Axiom C1$_{20}$].

For Axiom C2$_{20}$, consider any $[u]$ and $[v]$ in $\mathcal{V}/I$ such that $[u] \prec [v]$, meaning that $u \prec v + w$ for some w in I, by Proposition 121$_{177}$. Then $u \notin C(\{u, v+w\})$ by coherence of C [Axiom C2$_{20}$], and therefore indeed $[u] \notin C/I(\{[u],[v+w]\}) = C/I(\{[u],[v]\})$, where the equality holds because w belongs to I.

Axioms C3a$_{20}$ and C3b$_{20}$ are more easy to establish in their form of Axioms R3a$_{20}$ and R3b$_{20}$, so we consider the rejection function R corresponding to C, which, by Proposition 124$_{180}$ is compatible with I. By Proposition 126$_{181}$ therefore R/I is the rejection function corresponding to C/I, so it suffices to prove that R/I satisfies Axioms R3a$_{20}$ and R3b$_{20}$.

For Axiom R3a$_{20}$, consider any $\tilde{A}_1, \tilde{A}_2$ and $\tilde{A}$ in $\mathcal{Q}(\mathcal{V}/I)$ such that $\tilde{A}_1 \subseteq R/I(\tilde{A}_2)$ and $\tilde{A}_2 \subseteq \tilde{A}$. Consider any option set A in $\mathcal{Q}(\mathcal{V})$ such that $A/I = \tilde{A}$, and define the option sets $A_2 := \{u \in A : [u] \in \tilde{A}_2\} \in \mathcal{Q}(\mathcal{V})$ and $A_1 := \{u \in A_2 : [u] \in \tilde{A}_1\} \in \mathcal{Q}(\mathcal{V})$. Then $A_1 \subseteq A_2 \subseteq A$, $A_2/I = \{[v] : v \in A_2\} = \{[v] : v \in A \text{ and } [v] \in \tilde{A}_2\} = A/I \cap \{[v] : [v] \in \tilde{A}_2\} = \tilde{A} \cap \tilde{A}_2 = \tilde{A}_2$ and, similarly, $A_1/I = \{[v] : v \in A_2 \text{ and } [v] \in \tilde{A}_1\} = A_2/I \cap \tilde{A}_1 = \tilde{A}_2 \cap \tilde{A}_1 = \tilde{A}_1$, and therefore $A_1/I \subseteq R/I(A_2/I)$. Since $A_1 \subseteq A_2$, by Lemma 130 therefore $A_1 \subseteq R(A_2)$. We conclude from the coherence of R [Axiom R3a$_{20}$] that $A_1 \subseteq R(A)$, whence $A_1/I \subseteq R(A)/I = R/I(A/I)$. Because $\tilde{A} = A/I$ and $\tilde{A}_1 = A_1/I$, therefore indeed $\tilde{A}_1 \subseteq R/I(\tilde{A})$.

For Axiom R3b$_{20}$, consider any $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}$ in $\mathcal{Q}(\mathcal{V}/I)$ such that $\tilde{A}_1 \subseteq R/I(\tilde{A}_2)$ and $\tilde{A} \subseteq \tilde{A}_1$. Consider any option set A_2 in $\mathcal{Q}(\mathcal{V})$ such that $A_2/I = \tilde{A}_2$, and define the option sets $A_1 := \{u \in A_2 : [u] \in \tilde{A}_1\} \in \mathcal{Q}(\mathcal{V})$ and $A := \{u \in A_1 : [u] \in \tilde{A}\} \in \mathcal{Q}(\mathcal{V})$. Then $A \subseteq A_1 \subseteq A_2$, and, similarly as above, $A_1/I = \{[v] : v \in A_2 \text{ and } [v] \in \tilde{A}_1\} = A_2/I \cap \{[v] : [v] \in \tilde{A}_1\} = \tilde{A}_2 \cap \tilde{A}_1 = \tilde{A}_1$ and $A/I = \{[v] : v \in A_1 \text{ and } [v] \in \tilde{A}\} = A_1/I \cap \tilde{A} = \tilde{A}_1 \cap \tilde{A} = \tilde{A}$, and therefore $A_1/I \subseteq R/I(A_2/I)$. Since $A_1 \subseteq A_2$, by Lemma 130 therefore $A_1 \subseteq R(A_2)$. We conclude from the coherence of R [Axiom R3b$_{20}$] that $A_1 \setminus A \subseteq R(A_2 \setminus A)$, whence $(A_1 \setminus A)/I \subseteq R(A_2 \setminus A)/I$. By Lemma 128 therefore $A_1/I \setminus A/I \subseteq R(A_2 \setminus A)/I$, and, using the compatibility of R with I, we find that then $A_1/I \setminus A/I \subseteq R/I((A_2 \setminus A)/I)$. We now prove that

$$(A_2 \setminus A)/I = A_2/I \setminus A/I.$$

That $(A_2 \setminus A)/I \supseteq A_2/I \setminus A/I$ follows from Lemma 128. To show that $(A_2 \setminus A)/I \subseteq A_2/I \setminus A/I$, consider any $\tilde{u}$ in $(A_2 \setminus A)/I$, meaning that $\tilde{u} = [v]$ for some v in $A_2 \setminus A$. This implies already that $\tilde{u} \in A_2/I$. Assume *ex absurdo* that $\tilde{u} \in A/I = \tilde{A}$. Since $[v] = \tilde{u}$, therefore $[v] \in \tilde{A}$, whence $v \in \{u \in A_2 : [u] \in \tilde{A}\} \subseteq \{u \in A_2 : [u] \in \tilde{A}_1\} = A_1$, and since we already know that $v \in A_2 \setminus A$, therefore $v \in A_1 \cap (A_2 \setminus A) = A_1 \setminus A$. This implies that $\tilde{u} \in (A_1 \setminus A)/I$, so $\tilde{u} = [w]$ for some w in $A_1 \setminus A$. Since $\tilde{u} \in \tilde{A}$, therefore $[w] \in \tilde{A}$, whence $w \in \{u \in A_1 : [u] \in \tilde{A}\} = A$, a contradiction with $w \in A_1 \setminus A$. So we conclude that $(A_2 \setminus A)/I = A_2/I \setminus A/I$, whence $A_1/I \setminus A/I \subseteq R/I(A_2/I \setminus A/I)$. Because $\tilde{A} = A/I$, $\tilde{A}_1 = A_1/I$ and $\tilde{A}_2 = A_2/I$, therefore indeed $\tilde{A}_1 \setminus \tilde{A} \subseteq R/I(\tilde{A}_2 \setminus \tilde{A})$.

For Axioms C4a$_{20}$ and C4b$_{20}$, consider any $\tilde{A}_1$ and $\tilde{A}_2$ in $\mathcal{Q}(\mathcal{V}/I)$ such that $\tilde{A}_1 \subseteq C/I(\tilde{A}_2)$, and consider any λ in $\mathbb{R}_{>0}$ and $\tilde{u}$ in $\mathcal{V}/I$. Consider any option set A_2 in $\mathcal{Q}(\mathcal{V})$ such that $A_2/I = \tilde{A}_2$, let $A_1 := \{u \in A_2 : [u] \in \tilde{A}_1\} \in \mathcal{Q}(\mathcal{V})$, and consider any u in $\mathcal{V}$ such that $[u] = \tilde{u}$. We first prove that then $A_1 \subseteq C(A_2)$. Indeed, consider any $v \in A_1$, meaning that $v \in A_2$ and $[v] \in \tilde{A}_1$, and therefore also $[v] \in C/I(\tilde{A}_2)$. Proposition 125$_{181}$ then guarantees that indeed $u \in C(A_2)$. We now infer from the coherence of C [Axioms C4a$_{20}$ and C4b$_{20}$] that $\lambda A_1 \subseteq C(\lambda A_2)$ and $A_1 + \{u\} \subseteq C(A_2 + \{u\})$. Hence indeed $\lambda\tilde{A}_1 = \lambda A_1/I = (\lambda A_1)/I \subseteq C/I((\lambda A_2)/I) = C/I(\lambda A_2/I) = C/I(\lambda\tilde{A}_2)$ and $\tilde{A}_1 + \{\tilde{u}\} = A_1/I + \{[u]\} = (A_1 + \{u\})/I \subseteq C/I((A_2 + \{u\})/I) = C/I(A_2/I + \{\tilde{u}\}) = C/I(\tilde{A}_2 + \{\tilde{u}\})$, where the inclusions follow from Equation (5.2)$_{178}$ and Proposition 125$_{181}$.

For the converse implication, assume that C/I is coherent. We show that C satisfies Axioms C1$_{20}$–C4b$_{20}$.

For Axiom C1$_{20}$, consider any A in $\mathcal{Q}(\mathcal{V})$ and assume *ex absurdo* that $C(A) = \emptyset$. Then it follows from Proposition 125$_{181}$ that $C/I(A/I) = C(A)/I = \emptyset/I = \emptyset$, which contradicts the coherence of C/I [Axiom C1$_{20}$].

For Axiom C2$_{20}$, consider any u and v in $\mathcal{V}$ such that $u \prec v$. Then $[u] \prec [v]$ by letting for example $w = 0$ in Proposition 121$_{177}$. The coherence of C/I [Axiom C2$_{20}$] then guarantees that $[u] \notin C/I(\{[u],[v]\})$, implying that $u \notin C(\{u,v\})$, by Proposition 125$_{181}$.

Axioms C3a$_{20}$ and C3b$_{20}$ are more easy to establish in their form of Axioms R3a$_{20}$ and R3b$_{20}$, so we consider the rejection function R/I corresponding to C/I, and let, as usual, R be given by $R(A) := \{u \in A : [u] \in R/I(A/I)\}$ for all A in $\mathcal{Q}(\mathcal{V})$. Then R is compatible with I, and, by Proposition 126$_{181}$ it is the rejection function corresponding to C/I, so it suffices to prove that R satisfies Axioms R3a$_{20}$ and R3b$_{20}$.

For Axiom R3a$_{20}$, consider any option sets A, A_1 and A_2 in $\mathcal{Q}(\mathcal{V})$ such that $A_2 \subseteq A$ and $A_1 \subseteq R(A_2)$. Lemma 130$_{182}$ then guarantees that $A_1/I \subseteq R/I(A_2/I)$. Because $A_2 \subseteq A$, we have by Equation (5.2)$_{178}$ that $A_2/I \subseteq A/I$, whence $A_1/I \subseteq R/I(A/I)$ by the coherence of R/I [Axiom R3a$_{20}$]. Using Lemma 130$_{182}$ again, we infer that then indeed $A_1 \subseteq R(A)$.

For Axiom R3b$_{20}$, consider any option sets A, A_1 and A_2 in $\mathcal{Q}(\mathcal{V})$ such that $A \subseteq A_1$ and $A_1 \subseteq R(A_2)$. Let

$$\overline{A}_1 := \{u \in A_2 : [u] \in A_1/I\} \supseteq A_1 \text{ and } \check{A} := \{u \in \overline{A}_1 : [u] \cap A_1 \subseteq A\},$$

then $\overline{A}_1/I = A_1/I$. Because R is compatible with I, we infer from Lemma 130$_{182}$ that the statements $A_1 \subseteq R(A_2)$, $\overline{A}_1 \subseteq R(A_2)$ and $\overline{A}_1/I \subseteq R/I(A_2/I)$ are equivalent, and therefore all hold. Observe that also $\check{A} \subseteq \overline{A}_1 \subseteq A_2$, implying that $\check{A}/I \subseteq \overline{A}_1/I \subseteq A_2/I$ by Equation (5.2)$_{178}$. That $\check{A}/I \subseteq \overline{A}_1/I$ and $\overline{A}_1/I \subseteq R/I(A_2/I)$ implies, together with the coherence of R/I [Axiom R3b$_{20}$], that $\overline{A}_1/I \setminus \check{A}/I \subseteq R/I(A_2/I \setminus \check{A}/I)$.

We now prove that $A_2/I \setminus \check{A}/I \subseteq (A_2 \setminus A)/I$. Consider any $\tilde{u}$ in $A_2/I \setminus \check{A}/I$, and assume *ex absurdo* that $\tilde{u} \notin (A_2 \setminus A)/I$, or in other words, that $(\forall w \in A_2)(w \notin A \Rightarrow \tilde{u} \neq [w])$, or equivalently, $(\forall w \in A_2)(\tilde{u} = [w] \Rightarrow w \in A)$. This implies that $\tilde{u} \cap A_2 \subseteq A$, from which we infer on the one hand that $\tilde{u} \cap A_1 \subseteq A$ since $A_1 \subseteq A_2$. On the other hand, we infer that $(\tilde{u} \cap A_2)/I \subseteq A/I$ by Equation (5.2)$_{178}$, whence $\tilde{u} \in A/I$ because $\tilde{u} \in A_2/I$ by assumption, and therefore also $\tilde{u} \in A_1/I = \overline{A}_1/I$. Both observations together imply that $\tilde{u} \in \check{A}/I$, a contradiction with $\tilde{u} \in A_2/I \setminus \check{A}/I$. Therefore indeed $A_2/I \setminus \check{A}/I \subseteq (A_2 \setminus A)/I$, so coherence of R/I [Axiom R3a$_{20}$; with $\tilde{A} := (A_2 \setminus A)/I$, $\tilde{A}_1 := \overline{A}_1/I \setminus \check{A}/I$ and $\tilde{A}_2 := A_2/I \setminus \check{A}/I$] now implies that $\overline{A}_1/I \setminus \check{A}/I \subseteq R/I((A_2 \setminus A)/I)$.

We next prove that $(A_1 \setminus A)/I \subseteq \overline{A}_1/I \setminus \check{A}/I$. Consider any $\tilde{u}$ in $(A_1 \setminus A)/I$, so $\tilde{u} = [v]$ for some v in $A_1 \setminus A$. This already implies that $\tilde{u} \in \overline{A}_1/I$. Assume *ex absurdo* that $\tilde{u} \notin \overline{A}_1/I \setminus \check{A}/I$, so $\tilde{u} \in \check{A}/I$, whence $\tilde{u} \cap A_1 \subseteq A$. But we know that $v \in A_1$ and $v \in \tilde{u}$, so $v \in A$, a contradiction. We conclude form all this that $(A_1 \setminus A)/I \subseteq R/I((A_2 \setminus A)/I)$, whence by Lemma 130_{182}, indeed $A_1 \setminus A \subseteq R(A_2 \setminus A)$.

For Axioms C4a_{20} and C4b_{20}, consider any A_1 and A_2 in $\mathcal{Q}(\mathcal{V})$ such that $A_1 \subseteq C(A_2)$, and consider any λ in $\mathbb{R}_{>0}$ and u in $\mathcal{V}$. Then $A_1/I \subseteq C/I(A_2/I)$ by Equation $(5.2)_{178}$ and Proposition 125_{181}, implying that $(\lambda A_1)/I = \lambda A_1/I \subseteq C/I(\lambda A_2/I) = C/I((\lambda A_2)/I)$ and $(A_1 + \{u\})/I = A_1/I + \{[u]\} \subseteq C/I(A_2/I + \{[u]\}) = C/I((A_2 + \{u\})/I)$, where the inclusions follow from the coherence of C/I [Axioms C4a_{20} and C4b_{20} respectively]. This then implies that indeed $\lambda A_1 \subseteq C(\lambda A_2)$ and $A_1 + \{u\} \subseteq C(A_2 + \{u\})$. □

To conclude this general discussion of indifference for choice functions, we mention that it is closed under arbitrary infima, which enables *conservative inference under indifference*: we can consider the least informative choice function that is compatible with some assessments and is still compatible with a given coherent set of indifferent options.

Proposition 132. *Consider any coherent set of indifferent options I, and any non-empty collection of coherent choice functions* $\{C_i : i \in \mathcal{I}\}$ *that are compatible with I, then its coherent infimum* $\inf\{C_i : i \in \mathcal{I}\}$ *is compatible with I as well, and* $C/I = \inf\{C_i/I : i \in \mathcal{I}\}$.

Proof. Let us denote $C \coloneqq \inf\{C_i : i \in \mathcal{I}\}$, then C is a coherent choice function by Proposition 40_{48}. We show that it is compatible with I. By assumption, and using Proposition 125_{181}, we have for all i in $\mathcal{I}$ that

$$C_i(A) = \{u \in A : [u] \in C_i/I(A/I)\} \text{ for all } A \in \mathcal{Q}(\mathcal{V}).$$

Hence, for all $A \in \mathcal{Q}(\mathcal{V})$:

$$\begin{aligned} C(A) = \bigcup_{i \in \mathcal{I}} C_i(A) &= \bigcup_{i \in \mathcal{I}} \{u \in A : [u] \in C_i/I(A/I)\} \\ &= \{u \in A : (\exists i \in \mathcal{I})[u] \in C_i/I(A/I)\} \\ &= \{u \in A : [u] \in \bigcup_{i \in \mathcal{I}} C_i/I(A/I)\} \\ &= \{u \in A : [u] \in (\inf\{C_i/I : i \in \mathcal{I}\})(A/I)\}, \end{aligned}$$

and the stated result now follows from Propositions 40_{48} and 125_{181}. □

In particular, the least informative coherent choice function C that is compatible with a coherent set of indifferent options I corresponds to the case where the associated choice function C/I on the quotient space $\mathcal{V}/I$ is the vacuous one C_{v}, meaning that

$$C(A) = \{u \in A : [u] \in C_{\text{v}}(A/I)\} = \{u \in A : (\forall v \in A)[u] \nprec [v]\} = A \cap \max A/I$$

for all A in $\mathcal{Q}(\mathcal{V})$.

5.5 THE RELATION WITH OTHER DEFINITIONS OF INDIFFERENCE

Seidenfeld [63] has a rather different approach to combining a notion of indifference with choice functions. After making the necessary translation (see Section 2.4$_{28}$) from horse lotteries to (abstract) options, it can be summarised as follows. Rather than, as we have done above, starting out with a notion of indifference and then looking at which choice functions are compatible with it, they start from a given choice function C, and associate a binary relation $\approx_C$ on $\mathcal{V}$ with it as follows:

$$u \approx_C v \Leftrightarrow (\forall A \in \mathcal{Q})\big(\{u,v\} \subseteq A \Rightarrow (u \in C(A) \Leftrightarrow v \in C(A))\big) \text{ for all } u \text{ and } v \text{ in } \mathcal{V}. \tag{5.3}$$

The idea behind this definition is that two options are considered to be related if both options are either chosen or rejected together, whenever both are available. This relation has the following interesting properties, which are instrumental in elucidating the relationship between Seidenfeld's approach and ours:

Proposition 133. *Let C be a coherent choice function. Then $\approx_C$ is an equivalence relation that furthermore satisfies*

(i) $u \approx_C v \Rightarrow u + w \approx_C v + w$;

(ii) $u \approx_C v \Rightarrow \lambda u \approx_C \lambda v$,

for all u, v and w in $\mathcal{V}$, and all λ in $\mathbb{R}$. As a consequence, the set $I_C \coloneqq \{u \in \mathcal{V} : u \approx_C 0\}$ is a coherent set of indifferent options.

Proof. We first prove that $\approx_C$ is an equivalence relation. The reflexivity and symmetry are an immediate consequence of its definition (5.3). To prove transitivity, consider any u, v and w in $\mathcal{V}$ and assume that $u \approx_C v$ and $v \approx_C w$. Consider any A in $\mathcal{Q}$ and assume that $\{u,w\} \subseteq A$ and $u \in C(A)$. It suffices to prove that then also $w \in C(A)$. Assume *ex absurdo* that $w \notin C(A)$, then we infer from Axiom C3a$_{20}$ that also $w \notin C(A \cup \{v\})$. Since $v \approx_C w$, we infer from Equation (5.3) that then $v \notin C(A \cup \{v\})$, and similarly, since $u \approx_C v$, that $u \notin C(A \cup \{v\})$, so $\{u,v\} \subseteq (A \cup \{v\}) \setminus C(A \cup \{v\})$. Axiom C3b$_{20}$ then tells us that $u \notin C(A)$, a contradiction.

To prove (i), assume that $u \approx_C v$, and consider any $A \in \mathcal{Q}$ such that $\{u+w, v+w\} \subseteq A$ and $u + w \in C(A)$. Then it suffices to prove that also $v + w \in C(A)$. It follows from $u + w \in C(A)$ and Axiom C4b$_{20}$ that $u \in C(A - \{w\})$. Since $u \approx_C v$, we infer from Equation (5.3) that then also $v \in C(A - \{w\})$, whence, again by Axiom C4b$_{20}$, indeed $v + w \in C(A)$.

To prove (ii), assume that $u \approx_C v$. We first prove that then also $-u \approx_C -v$. Indeed, by applying (i) with $w \coloneqq -u - v$, we find that $-v \approx_C -u$. Now use the symmetry of $\approx_C$. Next, consider any $A \in \mathcal{Q}$ such that $\{\lambda u, \lambda v\} \subseteq A$ and $\lambda u \in C(A)$. Then it suffices to prove that also $\lambda v \in C(A)$. The proof is trivial if $\lambda = 0$. Because we have just proved that both $u \approx_C v$ and $-u \approx_C -v$, we may now assume without loss of generality that $\lambda > 0$. It follows from $\lambda u \in C(A)$ and Axiom C4a$_{20}$ that $u \in C(\frac{1}{\lambda}A)$. Since $u \approx_C v$, we infer from Equation (5.3) that then also $v \in C(\frac{1}{\lambda}A)$, whence, again by Axiom C4a$_{20}$, indeed $\lambda v \in C(A)$.

We complete the proof by showing that I_C is a coherent set of indifferent options. To prove I1$_{176}$, simply observe that $0 \approx_C 0$ by reflexivity of $\approx_C$. To prove I2$_{176}$, it suffices to consider any $u \in \mathcal{V}_{>0}$, due to (ii). It follows from Axiom C2$_{20}$ that $0 \notin C(\{0,u\})$ and, using Axiom C1$_{20}$, that also $u \in C(\{0,u\})$, so we infer from Equation (5.3) that $0 \not\approx_C u$, whence indeed $u \notin I_C$. To prove I3$_{176}$, simply use (ii). To prove I4$_{176}$, simply use (i) and the transitivity of $\approx_C$. □

The coherent set of indifferent options I_C turns out to be the largest that C is compatible with:

Proposition 134. *Consider any coherent set of indifferent options I and any coherent choice function C on $\mathcal{V}$. Then C is compatible with I if and only if I is a linear subspace of I_C.*

Proof. Recall that I is a linear space, by Definition 38$_{176}$. For necessity, assume that C is compatible with I, and hence

$$(\forall A \in \mathcal{Q})C(A) = \{u \in A : [u] \in C(A)/I\}.$$

We need to prove that $I \subseteq I_C$, so consider any u in I and any $A \supseteq \{0,u\}$ in $\mathcal{Q}$ such that $u \in C(A)$ [There always is such an A, for instance $A := \{0,u\}$, because $u \in I$ and therefore $[u] \in C/I(\{[u]\}) = C/I(\{0,u\}/I) = C(\{0,u\})/I$, by compatibility]. Then $[u] \in C(A)/I$ and because $[u] = [0]$, we find that $[0] \in C(A)/I$, so $0 \in C(A)$. Hence indeed $u \approx_C 0$, by Equation (5.3).

For sufficiency, assume that I is a linear subspace of I_C, so $I \subseteq I_C$, then we need to prove that C is compatible with I. In other words, if we consider any A in $\mathcal{Q}$ and any $u \in A$ such that $[u] \in C(A)/I$, then we must prove that $u \in C(A)$. $[u] \in C(A)/I$ means that there is some $v \in C(A)$ such that $u - v \in I$, and therefore, by assumption, $u \approx_C v$. Since $\{u,v\} \subseteq A$ and $v \in C(A)$, we infer from Equation (5.3) that indeed $u \in C(A)$. □

It follows that if C is compatible with I, then C is automatically also compatible with any subspace of I, and that the largest linear subspace that C is compatible with, is I_C. Also, this proposition shows that the smallest set of indifferent option that is compatible with C, is given by $I = \{0\}$, and that this coherent set of indifferent options is compatible with any coherent choice function. Seidenfeld's approach starts from a choice function, and identifies the coarsest equivalence—or indifference—relation that is compatible with it. Though we have seen that it is related, our approach, because it starts out with an indifference relation, goes the other way around, is more constructive, and is better suited for studying which choice functions are compatible with a given indifference relation, and therefore also for performing conservative inference for choice functions under indifference. We will come back to this idea in Sections 5.7$_{190}$, 5.8$_{191}$ and 5.9$_{193}$.

Proposition 134 also shows that our approach is closely connected to Bradley's [9, Section 2]: for him, for a choice function to '*satisfy*'—in our words: to 'be compatible with'—an equivalence relation, it must be finer than Seidenfeld's $\approx_C$.

Finally, note that a relation analogous to Equation $(5.3)_{186}$ can be established for coherent sets of desirable options D: we simply define $u \approx_D v \Leftrightarrow u \approx_{C_D} v$ for all u and v in $\mathcal{V}$. Then

$$\begin{aligned} u \approx_D v &\Leftrightarrow (\forall A \in \mathcal{Q})\big(\{u,v\} \subseteq A \Rightarrow (0 \in C_D(A - \{u\}) \Leftrightarrow 0 \in C_D(A - \{v\}))\big) \\ &\Leftrightarrow (\forall A \in \mathcal{Q})\big(\{u,v\} \subseteq A \Rightarrow (A - \{u\} \cap D \neq \emptyset \Leftrightarrow A - \{v\} \cap D \neq \emptyset)\big) \end{aligned}$$

for all u and v in $\mathcal{V}$, using Axiom $\mathrm{C4b}_{20}$ for the first equivalence and Proposition 55_{64} for the second one. Because we have defined it as a special $\approx_C$, the binary relation $\approx_D$ is by Proposition 133_{186} an equivalence relation, and $I_D := \{u \in \mathcal{V} : 0 \approx_D u\}$ is a coherent set of indifferent options. As we will see in Lemma 135, $\approx_D$ can be more elegantly represented as:

$$u \approx_D v \Leftrightarrow D + \{u\} = D + \{v\}, \text{ for all } u \text{ and } v \text{ in } \mathcal{V}.$$

Lemma 135. *Consider any coherent set of desirable options D. Then*

$$\begin{aligned} (\forall A \in \mathcal{Q})\big(\{u,v\} \subseteq A \Rightarrow (A - \{u\} \cap D \neq \emptyset \Leftrightarrow A - \{v\} \cap D \neq \emptyset)\big) \\ \Leftrightarrow D + \{u\} = D + \{v\} \end{aligned}$$

for all u and v in $\mathcal{V}$.

Proof. Consider any u and v in $\mathcal{V}$.

For the direct implication, assume that the left-hand side holds. We first prove that this implies that both $u - v \notin D$ and $v - u \notin D$. Indeed, consider the left-hand side for the particular choice $A := \{u,v\}$, leading to $\{0, v-u\} \cap D \neq \emptyset \Leftrightarrow \{0, u-v\} \cap D \neq \emptyset$, or equivalently, $v - u \in D \Leftrightarrow u - v \in D$, because $0 \notin D$ by Axiom $\mathrm{D1}_{57}$. So, if we had that $u - v \in D$ or $v - u \in D$, this would imply that both $u - v$ and $v - u$ would elements of D, and therefore also their sum $0 = u - v + v - u \in D$ by Axiom $\mathrm{D4}_{57}$. This contradicts Axiom $\mathrm{D1}_{57}$.

Next, consider any w in $\mathcal{V}$, and consider the left-hand side for the particular choice $A := \{u,v,w\}$. Then $A \supseteq \{u,v\}$ and therefore $A - \{u\} \cap D \neq \emptyset \Leftrightarrow A - \{v\} \cap D \neq \emptyset$, which can rewritten as $\{0, v-u, w-u\} \cap D \neq \emptyset \Leftrightarrow \{u-v, 0, w-v\} \cap D \neq \emptyset$. But $0 \notin D$ by Axiom $\mathrm{D1}_{57}$, whence $\{v-u, w-u\} \cap D \neq \emptyset \Leftrightarrow \{u-v, w-v\} \cap D \neq \emptyset$. Since we have seen above that neither $u - v \in D$ nor $v - u \in D$, this can in turn be rewritten as $w - u \notin D \Leftrightarrow w - v \notin D$. Because the choice of w in $\mathcal{V}$ was arbitrary, this tells us that $w \in D + \{u\} \Leftrightarrow w \in D + \{v\}$ for all w in $\mathcal{V}$, and therefore indeed $D + \{u\} = D + \{v\}$.

For the converse implication, assume that $D + \{u\} = D + \{v\}$. This immediately allows us to infer that $u - v \notin D$ and $v - u \notin D$. Consider any $A \supseteq \{u,v\}$. If $A = \{u,v\}$ then $A - \{u\} \cap D = \emptyset$ and $A - \{v\} \cap D = \emptyset$, and the proof is done. Let $n := |A| - 2$ and assume therefore that $n \geq 1$. Label the elements of $A \setminus \{u,v\}$ as $w_1, \ldots, w_n$, without loss of generality. Then $A = \{u, v, w_1, \dots, w_n\}$. Infer the following equivalences:

$$\begin{aligned} A - \{u\} \cap D \neq \varnothing &\Leftrightarrow \{0, v-u, w_1-u, \dots, w_n-u\} \cap D \neq \varnothing \\ &\Leftrightarrow \{w_1-u, \dots, w_n-u\} \cap D \neq \varnothing \\ &\Leftrightarrow (\exists i \in \{1, \dots, n\}) w_i - u \in D \\ &\Leftrightarrow (\exists i \in \{1, \dots, n\}) w_i \in D + \{u\} = D + \{v\} \\ &\Leftrightarrow (\exists i \in \{1, \dots, n\}) w_i - v \in D \Leftrightarrow A - \{v\} \cap D \neq \varnothing, \end{aligned}$$

where the second equivalence follows from the fact that $0 \notin D$ [by Axiom D1$_{57}$] and our earlier observation that $v - u \notin D$. □

5.6 THE RELATION WITH DESIRABILITY

There is an interesting relationship between the coherent choice functions and the coherent sets of desirable options that are compatible with a fixed coherent set of indifferent options.

Proposition 136. *Consider any coherent set of indifferent options I.*

(i) *If C is any coherent choice function compatible with I, then the corresponding coherent set of desirable options D_C is also compatible with I, and $D_C/I = D_{C/I}$.*

(ii) *If D is any coherent set of desirable options compatible with I, then the corresponding coherent choice function C_D is also compatible with I, and $C_D/I = C_{D/I}$.*

Proof. We begin with the first statement. Consider any coherent choice function C that is compatible with I. We must prove that $D_C + I \subseteq D_C$. Observe that for any w in $\mathcal{V}$:

$$w \in D_C \Leftrightarrow 0 \notin C(\{0, w\}) \Leftrightarrow [0] \notin C/I(\{[0], [w]\}) \Leftrightarrow [w] \in D_{C/I}, \tag{5.4}$$

where the first equivalence follows from Proposition 53$_{61}$ and the second from the compatibility of C with I. So, consider any v in D_C and any u in I, then $[0] \notin C/I(\{[0], [v]\})$ and $[u] = [0] = I$, implying that $[v+u] = [v] + [u] = [v]$, and therefore $[0] \notin C/I(\{[0], [v+u]\})$, implying that indeed $v + u \in D_C$. The last statement follows directly from Equation (5.4).

We turn now towards the second statement. Consider any coherent set of indifferent options I and any coherent set of desirable options D such that $D + I \subseteq D$. We must prove that $C_D(A) = \{u \in A : [u] \in C_D/I(A/I)\}$ for all A in $\mathcal{Q}(\mathcal{V})$. Due to Proposition 54$_{62}$, we know that $C_D(A) = \{u \in A : (\forall v \in A) v - u \notin D\}$ for all A in $\mathcal{Q}(\mathcal{V})$, whence, by Proposition 122$_{178}$,

$$\begin{aligned} C_D(A) &= \{u \in A : (\forall v \in A)[v] - [u] \notin \{[w] : w \in D\}\} \\ &= \{u \in A : (\forall v \in A)[v] - [u] \notin D/I\} \\ &= \{u \in A : (\forall [v] \in A/I)[v] - [u] \notin D/I\} = \{u \in A : [u] \in C_{D/I}(A/I)\}, \end{aligned}$$

for all A in $\mathcal{Q}(\mathcal{V})$, because $C_{D/I}(A/I) = \{\tilde{u} \in A/I : (\forall \tilde{v} \in A/I) \tilde{v} - \tilde{u} \notin D/I\}$ for all A/I in $\mathcal{Q}(\mathcal{V}/I)$. That $C_D/I = C_{D/I}$ now follows from Proposition 125$_{181}$. □

This correspondence allows us to show an equivalent result of Proposition 134_{187} for desirability:

Corollary 137. *Consider any coherent set of indifferent options I and any coherent set of desirable options $D \subseteq \mathcal{V}$. Then D is compatible with I if and only if I is a linear subspace of I_D.*

Proof. First, note that $I_D = \{u \in \mathcal{V} : 0 \approx_D u\} = \{u \in \mathcal{V} : 0 \approx_{C_D} u\} = I_{C_D}$, and that Proposition 134_{187} implies that C_D is compatible with I if and only if $I \subseteq I_{C_D}$. So it suffices to show that C_D is compatible with I if and only if D is compatible with I. For necessity, assume that C_D is compatible with I. Use Proposition 136(i) to infer that D_{C_D} is compatible with I, and Proposition 58_{66} that $D_{C_D} = D$, whence indeed D is compatible with I. For sufficiency, assume that D is compatible with I. Use Proposition 136(ii) to infer that then indeed C_D is compatible with I. □

5.7 EXAMPLE: FAIR COINS

To exhibit the power and simplicity of our definition of indifference, we reconsider the example of the finite possibility space $\mathcal{X} := \{\mathrm{H}, \mathrm{T}\}$ of Example 10_{85}, where the vector space $\mathcal{V}$ is again the two-dimensional vector space $\mathcal{L}(\mathcal{X})$ of real-valued functions on $\mathcal{X}$, or gambles, and the vector ordering $\leq$ is the usual point-wise ordering of gambles.

We want to express indifference between heads and tails, or in other words between $\mathbb{I}_{\{\mathrm{H}\}}$ and $\mathbb{I}_{\{\mathrm{T}\}}$. This means that $\mathbb{I}_{\{\mathrm{H}\}} - \mathbb{I}_{\{\mathrm{T}\}}$ is considered to be equivalent to the zero gamble, so the linear space of all gambles that are equivalent to zero—or in other words, the set of indifferent gambles (or options)—is then given by

$$I = \{\lambda(\mathbb{I}_{\{\mathrm{H}\}} - \mathbb{I}_{\{\mathrm{T}\}}) : \lambda \in \mathbb{R}\} = \{f \in \mathcal{L}(\mathcal{X}) : E_p(f) = 0\},$$

where E_p is the expectation associated with the uniform mass function $p = (1/2, 1/2)$ on $\{\mathrm{H}, \mathrm{T}\}$, associated with a fair coin: $E_p(f) := \frac{1}{2}(f(\mathrm{H}) + f(\mathrm{T}))$. So, for any gamble f in $\mathcal{L}(\mathcal{X})$—any real-valued function on $\mathcal{X}$:

$$[f] = \{f\} + I = \{g \in \mathcal{L}(\mathcal{X}) : E_p(g) = E_p(f)\},$$

which tells us that the equivalence class $[f]$ can be characterised by the common uniform expectation $E_p(f)$ of its elements. Therefore, $\mathcal{L}(\mathcal{X})/I$ has unit dimension, and we can identify it with the real line $\mathbb{R}$. The vector ordering between equivalence classes is given by, using Equation $(5.1)_{177}$:

$$\begin{aligned}
[f] \leq [g] &\Leftrightarrow (\exists \lambda \in \mathbb{R}) f \leq g + \lambda(\mathbb{I}_{\{\mathrm{H}\}} - \mathbb{I}_{\{\mathrm{T}\}}) \\
&\Leftrightarrow (\exists \lambda \in \mathbb{R})(f(\mathrm{H}) \leq g(\mathrm{H}) + \lambda \text{ and } f(\mathrm{T}) \leq g(\mathrm{T}) - \lambda) \\
&\Leftrightarrow (\exists \lambda \in \mathbb{R}) f(\mathrm{H}) - g(\mathrm{H}) \leq \lambda \leq -f(\mathrm{T}) + g(\mathrm{T}) \\
&\Leftrightarrow f(\mathrm{H}) - g(\mathrm{H}) \leq -f(\mathrm{T}) + g(\mathrm{T}) \Leftrightarrow E_p(f) \leq E_p(g),
\end{aligned}$$

and similarly $[f] \prec [g] \Leftrightarrow E_p(f) < E_p(g)$ for all f and g in $\mathcal{L}(\mathcal{X})$. Hence, the strict vector ordering $\prec$ on $\mathcal{L}(\mathcal{X})/I$ is total, so we infer from the argumentation in Example 6_{69} that there is only one representing choice function, namely the vacuous one. Therefore, there is only one choice function C on $\mathcal{L}(\mathcal{X})$ that is compatible with I, namely, the one that has the vacuous choice function C_{v} on $\mathcal{L}(\mathcal{X})/I$ as its representation C/I. Recall that for any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}))$:

$$\begin{aligned} C_{\mathrm{v}}(A/I) &= \{[u] : (\forall [g] \in A/I)[f] \nprec [g]\} \\ &= \{[u] : (\forall [g] \in A/I)[g] \preceq [f]\} = \{[u] : (\forall [g] \in A/I) E_p(g) \leq E_p(f)\}, \end{aligned}$$

and therefore

$$C(A) \coloneqq \{f \in A : (\forall g \in A) E_p(g) \leq E_p(f)\} = C^{\mathrm{E}}_{\{p\}}(A).$$

The indifference assessment between heads and tails leaves us no choice but to use an E-admissible model for a probability mass function, associated with a fair coin.

The choice function C is therefore based on E-admissibility, but is not compatible with M-admissibility. To see this, consider the set of options $A \coloneqq \{h, 0, -h\}$ with $h \coloneqq (h(\mathrm{H}), h(\mathrm{T})) = (1, -1)$, so $h(\mathrm{H}) + h(\mathrm{T}) = 0$. Hence $C(A) = A$; but no M-admissible choice function will select 0 in A: observe that $0 \notin C_{\hat{D}}(A)$ for all $\hat{D} \in \mathcal{D}$, because $0 \in C_{\hat{D}}(A)$ would imply that $\{h, -h\} \cap \hat{D} = \emptyset$, contradicting that $\hat{D}$ is a maximal set of desirable options by Proposition 51_{59}.

5.8 CHOICE FUNCTIONS AND SYMMETRY

As another example—and a precursor to Chapter 8_{247}—showing how powerful our approach to dealing with choice and indifference is, we will prove a simple and elegant representation result that tells us how to perform conservative inference with choice functions under a permutation symmetry assessment.

We consider a finite possibility space $\mathcal{X}$, where the vector space $\mathcal{V}$ of options is the finite-dimensional vector space $\mathbb{R}^{r\mathcal{X}}$, of $\mathbb{R}^r$-valued functions on $\mathcal{X}$, or *vector-valued* gambles on the outcome of an uncertain variable X in $\mathcal{X}$. The vector ordering $\preceq$ is the usual point-wise ordering of such vector-valued gambles.[1]

We assume there is symmetry lurking behind the uncertain variable X, represented by a group $\mathcal{P}$ of permutations of the set of possible outcomes $\mathcal{X}$—the idea being that a subject assesses that no distinction should be made between an outcome x and its permutations πx, for $\pi \in \mathcal{P}$—or in other words, between the variable X and its permutations πX. If we consider any vector-valued gamble $u(X)$ on the variable X, then the subject will therefore be indifferent between

[1] The reason why we work here with vector-valued gambles here rather than real-valued ones, is explained in some detail in Section 2.4_{28}.

the uncertain vector-valued rewards $u(X)$ and $u(\pi X)$. The smallest coherent[2] set of indifferent options $I_{\mathcal{P}}$ that corresponds to this indifference assessment, is therefore given by

$$I_{\mathcal{P}} \coloneqq \operatorname{span}(\{u - \pi^t u : u \in \mathcal{V} \text{ and } \pi \in \mathcal{P}\}),$$

where we define the linear permutation operator π^t on the linear space of options (vector-valued gambles) $\mathcal{V}$ by $\pi^t u = u \circ \pi$, or in other words

$$(\pi^t u)(x) \coloneqq u(\pi x) \text{ for all } u \text{ in } \mathcal{V}, x \text{ in } \mathcal{X} \text{ and } \pi \text{ in } \mathcal{P}.$$

Let us, for any x in $\mathcal{X}$, define the *permutation invariant atom* $[x]_{\mathcal{P}}$ containing x as

$$[x]_{\mathcal{P}} \coloneqq \{\pi x : \pi \in \mathcal{P}\}.$$

The permutation invariant atoms constitute a partition of $\mathcal{X}$, and we denote the set of all of them by $\mathcal{A}_{\mathcal{P}} \coloneqq \{[x]_{\mathcal{P}} : x \in \mathcal{X}\}$. A vector-valued gamble u is called *$\mathcal{P}$-invariant* if $\pi^t u = u$ for all π in $\mathcal{P}$, and it is not hard to see that this is equivalent to u being constant on the invariant atoms. The set of all $\mathcal{P}$-invariant vector-valued gambles is denoted by $\mathcal{V}_{\mathcal{P}}$, and it is a linear subspace of $\mathcal{V}$ that is clearly isomorphic to the linear space of all vector-valued functions on $\mathcal{A}_{\mathcal{P}}$, whose dimension $r|\mathcal{A}_{\mathcal{P}}|$ is typically much lower than that of $\mathcal{V}$.

A choice function that takes the symmetry assessment into account is—as we have argued—one that is compatible with $I_{\mathcal{P}}$ and all of its linear subspaces. What we will do now, is to investigate how such compatible choice functions can be represented by choice functions on a typically much lower-dimensional option space: symmetry reduces complexity. Most of the work for this has already been done in Definition 40_{179} and Proposition 125_{181}, which indeed states that choice functions compatible with $I_{\mathcal{P}}$ can be represented uniquely by choice functions on the lower-dimensional quotient space $\mathcal{V}/I_{\mathcal{P}}$. The only thing that is left for us to do, therefore, is to take a closer look at this quotient space and its elements.

Let us, to this end, define the transformation $\operatorname{inv}_{\mathcal{P}}$ on $\mathcal{V}$ as follows:

$$\operatorname{inv}_{\mathcal{P}} u \coloneqq \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}} \pi^t u, \text{ for all } u \text{ in } \mathcal{V}. \tag{5.5}$$

It satisfies the following very interesting properties:

Proposition 138. $\operatorname{inv}_{\mathcal{P}}$ *is a linear transformation of* $\mathcal{V}$*, and*

(i) $\operatorname{inv}_{\mathcal{P}} \circ \pi^t = \operatorname{inv}_{\mathcal{P}} = \pi^t \circ \operatorname{inv}_{\mathcal{P}}$ *for all* π *in* $\mathcal{P}$*;*

(ii) $\operatorname{inv}_{\mathcal{P}} \circ \operatorname{inv}_{\mathcal{P}} = \operatorname{inv}_{\mathcal{P}}$*;*

[2]The requirement that $I_{\mathcal{P}} \cap \mathcal{V}_{<0} = \emptyset$—or equivalently $I_{\mathcal{P}} \cap \mathcal{V}_{>0} = \emptyset$—is related to the left amenability of the finite permutation group $\mathcal{P}$ [27, 82], and is easily shown to be satisfied.

(iii) $\ker(\mathrm{inv}_{\mathcal{P}}) = \mathcal{I}_{\mathcal{P}}$;
(iv) $\mathrm{rng}(\mathrm{inv}_{\mathcal{P}}) = \mathcal{V}_{\mathcal{P}}$.
Moreover, for any u and v in $\mathcal{V}$, we have that $v \in u/\mathcal{I}_{\mathcal{P}} \Leftrightarrow \mathrm{inv}_{\mathcal{P}}\, v = \mathrm{inv}_{\mathcal{P}}\, u$.

Proof. That the transformation $\mathrm{inv}_{\mathcal{P}}$ is linear, is immediate from its definition in Equation (5.5).

To prove (i), observe that $\mathrm{inv}_{\mathcal{P}} \circ \pi^t = \frac{1}{|\mathcal{P}|} \sum_{\varpi \in \mathcal{P}} \varpi^t \circ \pi^t = \frac{1}{|\mathcal{P}|} \sum_{\varpi \in \mathcal{P}} (\pi \circ \varpi)^t = \mathrm{inv}_{\mathcal{P}}$, where the last equality holds because $\mathcal{P}$ is a group. For the second identity, observe that $\pi^t \circ \mathrm{inv}_{\mathcal{P}} = \frac{1}{|\mathcal{P}|} \sum_{\varpi \in \mathcal{P}} \pi^t \circ \varpi^t = \frac{1}{|\mathcal{P}|} \sum_{\varpi \in \mathcal{P}} (\varpi \circ \pi)^t = \mathrm{inv}_{\mathcal{P}}$, where the first equality follows form the linear character of π^t and the last equality holds because $\mathcal{P}$ is a group.

To prove (ii), observe that

$$\mathrm{inv}_{\mathcal{P}} \circ \mathrm{inv}_{\mathcal{P}} = \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}} \mathrm{inv}_{\mathcal{P}} \circ \pi^t = \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}} \mathrm{inv}_{\mathcal{P}} = \mathrm{inv}_{\mathcal{P}},$$

where the first equality is due to the linearity of $\mathrm{inv}_{\mathcal{P}}$ and the second is due to (i).

To prove (iii), consider any u in $\ker(\mathrm{inv}_{\mathcal{P}})$. Then $\mathrm{inv}_{\mathcal{P}}\, u = 0$ and therefore $u = u - \mathrm{inv}_{\mathcal{P}}\, u = \frac{1}{|\mathcal{P}|} \sum_{\pi \in \mathcal{P}} (u - \pi^t u)$ is an element of $\mathcal{I}_{\mathcal{P}}$. Conversely, consider any u in $\mathcal{I}_{\mathcal{P}}$, then $u = \sum_{k=1}^n \lambda_k (v_k - \pi_k^t v_k)$ for some n in $\mathbb{N}$, λ_k in $\mathbb{R}$, v_k in $\mathcal{V}$ and π_k in $\mathcal{P}$. But then $\mathrm{inv}_{\mathcal{P}}\, u = \sum_{k=1}^n \lambda_k (\mathrm{inv}_{\mathcal{P}}\, v_k - \mathrm{inv}_{\mathcal{P}}(\pi_k^t v_k)) = 0$, where the first equality is due to the linearity of $\mathrm{inv}_{\mathcal{P}}$ and the last due to (i). Hence indeed $u \in \ker(\mathrm{inv}_{\mathcal{P}})$.

To prove (iv), consider any u in $\mathrm{rng}(\mathrm{inv}_{\mathcal{P}})$. Then $u = \mathrm{inv}_{\mathcal{P}}\, v$ for some v in $\mathcal{V}$, and therefore $\pi^t u = \pi^t (\mathrm{inv}_{\mathcal{P}}\, v) = (\pi^t \circ \mathrm{inv}_{\mathcal{P}}) v = \mathrm{inv}_{\mathcal{P}}\, v = u$ for all π in $\mathcal{P}$, where the third equality follows from (i). Hence indeed $u \in \mathcal{V}_{\mathcal{P}}$. Conversely, consider any u in $\mathcal{V}_{\mathcal{P}}$. Then $\pi^t u = u$ for all π in $\mathcal{P}$, and therefore $u = \mathrm{inv}_{\mathcal{P}}\, u$, whence indeed $u \in \mathrm{rng}(\mathrm{inv}_{\mathcal{P}})$.

For the last statement, simply observe that $v \in u/\mathcal{I}_{\mathcal{P}} \Leftrightarrow v - u \in \mathcal{I}_{\mathcal{P}} \Leftrightarrow \mathrm{inv}_{\mathcal{P}}(v - u) = 0 \Leftrightarrow \mathrm{inv}_{\mathcal{P}}\, v = \mathrm{inv}_{\mathcal{P}}\, u$, where the second equivalence follows from (iii) and the last from the linearity of $\mathrm{inv}_{\mathcal{P}}$. □

The various statements in this proposition tell us that $\mathrm{inv}_{\mathcal{P}}$ is a linear projection operator that maps any vector-valued gamble u to the corresponding uniquely $\mathcal{P}$-invariant member $\mathrm{inv}_{\mathcal{P}}\, u$ of the equivalence class $u/\mathcal{I}_{\mathcal{P}}$, which is essentially a vector-valued gamble on $\mathcal{A}_{\mathcal{P}}$.

By Proposition 125_{181}, every coherent choice function C on $\mathcal{V}$ that is compatible with $\mathcal{I}_{\mathcal{P}}$ therefore has a unique representing coherent choice function $C_{\mathcal{P}}$ on the typically much lower-dimensional linear space of all vector-valued gambles on $\mathcal{A}_{\mathcal{P}}$, with

$$C(A) = \{u \in A : \mathrm{inv}_{\mathcal{P}}\, u \in C_{\mathcal{P}}(\mathrm{inv}_{\mathcal{P}} A)\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{V}).$$

5.9 NATURAL EXTENSION UNDER INDIFFERENCE

In Chapter 3_{89} we have found the natural extension of an assessment $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{V})$: if $\mathcal{B}$ avoids complete rejection, then it has a coherent extension, and the least informative such extension is given by $\mathcal{E}(\mathcal{B}) = R_{\mathcal{B}}$, defined in Equation $(3.1)_{92}$. The type of assessment we considered there—subsets $\mathcal{B}$ of

$\mathcal{Q}_0(\mathcal{V})$—is *direct*, meaning that we directly assess rejections 0 from all the elements of $\mathcal{B}$. Such assessments do not subsume so-called *structural assessments*, such as indifference assessments. We will also deal with other kinds of structural assessments, such as irrelevance, in Section 7.4_{234}. In this section, given a direct assessment $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{V})$ and an indifference assessment I in $\bar{\mathbf{I}}$, we will find the least informative coherent rejection function that extends $\mathcal{B}$ and is compatible with I: we will find the *natural extension of* $\mathcal{B}$ *under* I.

As a first example, consider $\mathcal{B} := \emptyset$, and any I in $\bar{\mathbf{I}}$. The natural extension of this assessment can be found easily: it is the coherent rejection function R, given by

$$R(A) := \{u \in A : [u] \in R_{\mathrm{v}}(A/I)\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{V}),$$

where R_{v} is the vacuous rejection function on $\mathcal{V}/I$. It is the most conservative rejection function on $\mathcal{V}$ that is compatible with I.

5.9.1 *Defining the natural extension under indifference*

The natural extension under indifference, if it is coherent, is the least informative coherent rejection function that extends the assessment $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{V})$ and is compatible with the set of indifferent options I.

Definition 41. *Given any assessment $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{V})$ and any coherent set of indifferent options I, the* natural extension of $\mathcal{B}$ under *I is the rejection function*

$$\begin{aligned}\mathcal{E}_I(\mathcal{B}) :=& \inf\{R \in \overline{\mathbf{R}}(\mathcal{V}) : \\ &(\forall B \in \mathcal{B}) 0 \in R(B) \textit{ and } (\forall A \in \mathcal{Q}(\mathcal{V}))R(A) = \{u \in A : [u] \in R(A)/I\}\} \\ =& \inf\{R \in \overline{\mathbf{R}}(\mathcal{V}) : R \textit{ extends } \mathcal{B} \textit{ and is compatible with } I\},\end{aligned}$$

where, as usual, we let $\inf\emptyset = \mathrm{id}_{\mathcal{Q}(\mathcal{V})}$*, the identity rejection function that maps every option set of itself.*

Again, we can equivalently define the natural extension under indifference as a *choice function*—or a *choice relation* for that matter—instead of a rejection function, but that turns out to be notationally more involved. The translation to the other types of choice models of this notion and other results in this section, is straightforward.

To help link Definition 41 with a more constructive and explicit expression, consider the special rejection function $R_{\mathcal{B},I}$, defined by:

$$R_{\mathcal{B},I}(A) := \{u \in A : [u] \in R_{\mathcal{B}/I}(A/I)\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{V}), \tag{5.6}$$

where we let $\mathcal{B}/I := \{B/I : B \in \mathcal{B}\} \subseteq \mathcal{Q}_{[0]}(\mathcal{V}/I)$, being—loosely speaking—the assessment expressed in the quotient space $\mathcal{V}/I$. Recall that $R_{\mathcal{B}}$, as defined in Equation $(3.1)_{92}$, is relative to a given but otherwise arbitrary vector space $\mathcal{V}$.

Our special rejection function $R_{\mathcal{B},I}$ uses the version $R_{\mathcal{B}/I}$ on $\mathcal{V}/I$ instead of $\mathcal{V}$. Explicitly, it is given by

$$R_{\mathcal{B}/I}(\tilde{A}) = \Big\{\tilde{u} \in \tilde{A} : (\exists \tilde{A}' \in \mathcal{Q}(\mathcal{V}/I))\big(\tilde{A}' \supseteq \tilde{A} \text{ and } (\forall \tilde{v} \in \{\tilde{u}\} \cup (\tilde{A}' \setminus \tilde{A}))$$
$$\big((\tilde{A}' - \{\tilde{v}\}) \cap \mathcal{V}/I_{\succ[0]} \neq \varnothing \text{ or } (\exists \tilde{B} \in \mathcal{B}/I, \exists \mu \in \mathbb{R}_{>0})\{\tilde{v}\} + \mu\tilde{B} \lessdot \tilde{A}'\big)\big)\Big\}$$

for all $\tilde{A}$ in $\mathcal{Q}(\mathcal{V}/I)$.

It can be useful to have a more direct equivalent expression for $R_{\mathcal{B},I}$:

Lemma 139. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_{\overline{0}}(\mathcal{V})$ and any coherent set of indifferent options $I \subseteq \mathcal{V}$. Then $R_{\mathcal{B},I} = R'_{\mathcal{B},I}$, where we let*

$$R'_{\mathcal{B},I}(A) \coloneqq \Big\{u \in A : (\exists A' \in \mathcal{Q}(\mathcal{V}))\big(A' \supseteq A \textit{ and } (\forall v \in \{u\} \cup (A' \setminus A))$$
$$\big((A'/I - \{[v]\}) \cap \mathcal{V}/I_{\succ[0]} \neq \varnothing \textit{ or } (\exists \tilde{B} \in \mathcal{B}/I, \exists \mu \in \mathbb{R}_{>0})\{[v]\} + \mu\tilde{B} \lessdot A'/I\big)\big)\Big\} \tag{5.7}$$

for all A in $\mathcal{Q}(\mathcal{V})$.

Proof. By plugging in the expression for $R_{\mathcal{B}/I}$ into Equation (5.6), we find that

$$R_{\mathcal{B},I}(A) = \{u \in A : (\exists \tilde{A}' \in \mathcal{Q}(\mathcal{V}/I))\big(\tilde{A}' \supseteq A/I \text{ and } (\forall \tilde{v} \in \{[u]\} \cup (\tilde{A}' \setminus A/I))$$
$$\big((\tilde{A}' - \{\tilde{v}\}) \cap \mathcal{V}/I_{\succ[0]} \neq \varnothing \text{ or } (\exists \tilde{B} \in \mathcal{B}/I, \exists \mu \in \mathbb{R}_{>0})\{\tilde{v}\} + \mu\tilde{B} \lessdot \tilde{A}'\big)\big)\} \tag{5.8}$$

for all A in $\mathcal{Q}(\mathcal{V})$, and therefore also

$$R_{\mathcal{B},I}(A) = \{u \in A : (\exists A' \in \mathcal{Q}(\mathcal{V}))\big(A'/I \supseteq A/I \text{ and } (\forall \tilde{v} \in \{[u]\} \cup (A'/I \setminus A/I))$$
$$\big((A'/I - \{\tilde{v}\}) \cap \mathcal{V}/I_{\succ[0]} \neq \varnothing \text{ or } (\exists \tilde{B} \in \mathcal{B}/I, \exists \mu \in \mathbb{R}_{>0})\{\tilde{v}\} + \mu\tilde{B} \lessdot A'/I\big)\big)\} \tag{5.9}$$

for all A in $\mathcal{Q}(\mathcal{V})$. We will use this expression to prove that $R_{\mathcal{B},I} \sqsubseteq R'_{\mathcal{B},I}$ and $R'_{\mathcal{B},I} \sqsubseteq R_{\mathcal{B},I}$.

To prove that $R_{\mathcal{B},I} \sqsubseteq R'_{\mathcal{B},I}$, consider any A in $\mathcal{Q}(\mathcal{V})$ and any u in $R_{\mathcal{B},I}(A)$. We show that $u \in R'_{\mathcal{B},I}(A)$. By Equation (5.9), since $u \in R_{\mathcal{B},I}(A)$, there is some A' in $\mathcal{Q}(\mathcal{V})$ such that $A'/I \supseteq A/I$ and

$$(\forall \tilde{v} \in \{[u]\} \cup (A'/I \setminus A/I))$$
$$\big((A'/I - \{\tilde{v}\}) \cap \mathcal{V}/I_{\succ[0]} \neq \varnothing \text{ or } (\exists \tilde{B} \in \mathcal{B}/I, \exists \mu \in \mathbb{R}_{>0})\{\tilde{v}\} + \mu\tilde{B} \lessdot A'/I\big). \tag{5.10}$$

Let $A'' \coloneqq A \cup \{u \in A' : [u] \notin A/I\} \supseteq A$. We state that then $\{[u]\} \cup (A'/I \setminus A/I) = (\{u\} \cup (A'' \setminus A))/I$. To prove this, since $(\{u\} \cup (A'' \setminus A))/I = \{u\}/I \cup (A'' \setminus A)/I$ and $\{[u]\} = \{u\}/I$, it suffices to show that $(A'' \setminus A)/I = A'/I \setminus A/I$. So consider any $\tilde{u}$ in

$\mathcal{V}/I$, and infer that

$$\begin{aligned}\tilde{u} \in (A'' \setminus A)/I &\Leftrightarrow (\exists v \in \tilde{u}) v \in A'' \setminus A\\ &\Leftrightarrow (\exists v \in \tilde{u})(v \in A'' \text{ and } v \notin A)\\ &\Leftrightarrow (\exists v \in \tilde{u}) v \in \{w \in A' : [w] \notin A/I\}\\ &\Leftrightarrow (\exists v \in \tilde{u})(v \in A' \text{ and } [v] \notin A/I)\\ &\Leftrightarrow (\exists v \in \tilde{u}) v \in A' \text{ and } \tilde{u} \notin A/I \qquad \text{since } (\forall v \in \tilde{u})[v] = \tilde{u}\\ &\Leftrightarrow \tilde{u} \in A'/I \text{ and } \tilde{u} \notin A/I \Leftrightarrow \tilde{u} \in A'/I \setminus A/I,\end{aligned}$$

whence indeed $(A'' \setminus A)/I = A'/I \setminus A/I$. Therefore, by Equation (5.10)$_\curvearrowleft$,

$$\begin{aligned}&(\forall \tilde{v} \in (\{u\} \cup (A'' \setminus A))/I)\\ &\qquad\big((A'/I - \{\tilde{v}\}) \cap \mathcal{V}/I_{\succ[0]} \neq \emptyset \text{ or } (\exists \tilde{B} \in \mathcal{B}/I, \exists \mu \in \mathbb{R}_{>0})\{\tilde{v}\} + \mu\tilde{B} \preccurlyeq A'/I\big),\end{aligned}$$

whence

$$\begin{aligned}&(\forall v \in \{u\} \cup (A'' \setminus A))\\ &\qquad\big((A'/I - \{[v]\}) \cap \mathcal{V}/I_{\succ[0]} \neq \emptyset \text{ or } (\exists \tilde{B} \in \mathcal{B}/I, \exists \mu \in \mathbb{R}_{>0})\{[v]\} + \mu\tilde{B} \preccurlyeq A'/I\big).\end{aligned}$$

Remark that $A''/I = A/I \cup \{[u] : u \in A' \text{ and } [u] \notin A/I\} = A/I \cup (A'/I \setminus A/I) = A'/I$, and therefore

$$\begin{aligned}&(\forall v \in \{u\} \cup (A'' \setminus A))\\ &\qquad\big((A''/I - \{[v]\}) \cap \mathcal{V}/I_{\succ[0]} \neq \emptyset \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{[v]\} + \mu B/I \preccurlyeq A''/I\big).\end{aligned}$$

Since $A'' \supseteq A$, by Equation (5.7)$_\curvearrowleft$ then indeed $u \in R'_{\mathcal{B},I}(A)$.

Conversely, to prove that $R'_{\mathcal{B},I} \sqsubseteq R_{\mathcal{B},I}$, consider any A in $\mathcal{Q}(\mathcal{V})$ and any u in $R'_{\mathcal{B},I}(A)$. We will show that then $u \in R_{\mathcal{B},I}(A)$. By Equation (5.7)$_\curvearrowleft$, since $u \in R'_{\mathcal{B},I}(A)$, there is some $A' \supseteq A$ in $\mathcal{Q}(\mathcal{V})$ such that

$$\begin{aligned}&(\forall v \in \{u\} \cup (A' \setminus A))\\ &\qquad\big((A'/I - \{[v]\}) \cap \mathcal{V}/I_{\succ[0]} \neq \emptyset \text{ or } (\exists \tilde{B} \in \mathcal{B}/I, \exists \mu \in \mathbb{R}_{>0})\{[v]\} + \mu\tilde{B} \preccurlyeq A'/I\big),\end{aligned}$$

whence

$$\begin{aligned}&(\forall \tilde{v} \in (\{u\} \cup (A' \setminus A))/I)\\ &\qquad\big((A'/I - \{\tilde{v}\}) \cap \mathcal{V}/I_{\succ[0]} \neq \emptyset \text{ or } (\exists \tilde{B} \in \mathcal{B}/I, \exists \mu \in \mathbb{R}_{>0})\{\tilde{v}\} + \mu\tilde{B} \preccurlyeq A'/I\big). \quad (5.11)\end{aligned}$$

Let $\tilde{A}' \coloneqq A'/I \supseteq A/I$. We state that then $\{[u]\} \cup (\tilde{A}' \setminus A/I) \subseteq (\{u\} \cup (A' \setminus A))/I$. To prove this, since $(\{u\} \cup (A' \setminus A))/I = \{u\}/I \cup (A' \setminus A)/I$ and $\{[u]\} = \{u\}/I$, it suffices to show that $\tilde{A}' \setminus A/I \subseteq (A' \setminus A)/I$. So consider any $\tilde{u}$ in $\mathcal{V}/I$, and infer that

$$\begin{aligned}\tilde{u} \in \tilde{A}' \setminus A/I &\Rightarrow \tilde{u} \in \tilde{A}' \text{ and } \tilde{u} \notin A/I\\ &\Rightarrow ((\exists v \in \tilde{u}) v \in A') \text{ and } ((\forall w \in \tilde{u}) w \notin A)\\ &\Rightarrow (\exists v \in \tilde{u})(v \in A' \text{ and } v \notin A) \Rightarrow (\exists v \in \tilde{u}) v \in A' \setminus A \Rightarrow \tilde{u} \in (A' \setminus A)/I,\end{aligned}$$

whence indeed $\tilde{A}' \setminus A/I \subseteq (A' \setminus A)/I$. Therefore, by Equation (5.11) in particular

$$(\forall \tilde{v} \in \{[u]\} \cup (\tilde{A}' \setminus A/I))$$
$$\big((\tilde{A}' - \{\tilde{v}\}) \cap \mathcal{V}/I_{\succ[0]} \neq \emptyset \text{ or } (\exists \tilde{B} \in \mathcal{B}/I, \exists \mu \in \mathbb{R}_{>0})\{\tilde{v}\} + \mu\tilde{B} \lessdot \tilde{A}'\big).$$

Since $\tilde{A}' \supseteq A/I$, by Equation (5.8)$_{195}$ then indeed $u \in R_{\mathcal{B},I}(A)$. □

Similarly as in Chapter 3$_{89}$, the special rejection function $R_{\mathcal{B},I}$ satisfies a number of interesting properties:

Lemma 140. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{V})$ and any coherent set of indifferent options $I \subseteq \mathcal{V}$. Then $R_{\mathcal{B},I}$ extends $\mathcal{B}$, is compatible with I, and satisfies Axioms* R2$_{20}$–R4$_{20}$.

Proof. To show that $R_{\mathcal{B},I}$ extends $\mathcal{B}$, we need to prove that $0 \in R_{\mathcal{B},I}(B)$ for every B in $\mathcal{B}$. So consider any B in $\mathcal{B}$. By Equation (5.6)$_{194}$, $R_{\mathcal{B},I}(B) = \{u \in B : [u] \in R_{\mathcal{B}/I}(B/I)\}$, and therefore $0 \in R_{\mathcal{B},I}(B) \Leftrightarrow [0] \in R_{\mathcal{B}/I}(B/I)$. By Lemma 78$_{94}$, we know that $R_{\mathcal{B}/I}$ extends $\mathcal{B}/I$, and therefore $[0] \in R_{\mathcal{B}/I}(B/I)$. Hence indeed $0 \in R_{\mathcal{B},I}(B)$.

That $R_{\mathcal{B},I}$ is compatible with I, is immediate because it is represented by $R_{\mathcal{B}/I}$ on $\mathcal{V}/I$.

To show that $R_{\mathcal{B},I}$ satisfies Axioms R2$_{20}$–R4$_{20}$, first use Lemma 77$_{92}$ to infer that its representing rejection function $R_{\mathcal{B}/I}$ satisfies those axioms. Therefore, by Proposition 127$_{182}$, $R_{\mathcal{B},I}$ indeed satisfies Axioms R2$_{20}$–R4$_{20}$. □

Now we already know that $R_{\mathcal{B},I}$ satisfies the rationality Axioms R2$_{20}$–R4$_{20}$, extends $\mathcal{B}$, and is compatible with I, but if we want to use it as an expression for the natural extension, it will help us if we can prove that it is the *least informative* such rejection function.

Proposition 141. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{V})$ and any coherent set of indifferent options $I \subseteq \mathcal{V}$. Then $R_{\mathcal{B},I}$ is the least informative rejection function that satisfies Axioms* R2$_{20}$–R4$_{20}$, *extends $\mathcal{B}$, and is compatible with I.*

Proof. We already know from Lemma 140 that $R_{\mathcal{B},I}$ satisfies Axioms R2$_{20}$–R4$_{20}$, extends $\mathcal{B}$, and is compatible with I, so it suffices to show that $R_{\mathcal{B},I}$ is the least informative such rejection function. Consider any rejection function R' that satisfies Axioms R2$_{20}$–R4$_{20}$, extends $\mathcal{B}$ and is compatible with I. We will show that $R_{\mathcal{B},I} \sqsubseteq R'$, or, in other words, that $R_{\mathcal{B},I}(A) \subseteq R'(A)$ for all A in $\mathcal{Q}(\mathcal{V})$. Since both $R_{\mathcal{B},I}$ and R' satisfy Axiom R4b$_{20}$, it suffices to show that $0 \in R_{\mathcal{B},I}(A) \Rightarrow 0 \in R'(A)$ for all A in $\mathcal{Q}(\mathcal{V})$.

Since R' is compatible with I, by there is some representing choice function R'' on $\mathcal{V}/I$ such that $R'(A) = \{u \in A : [u] \in R''(A/I)\}$ for all A in $\mathcal{Q}(\mathcal{V})$. We state that R'' extends $\mathcal{B}/I$. To see this, consider any $\tilde{B}$ in $\mathcal{B}/I$, and we will show that $[0] \in R''(\tilde{B})$. Because $\tilde{B}$ belongs to $\mathcal{B}/I$, then $\tilde{B} = B/I$ for some B in $\mathcal{B}$. Since R' extends $\mathcal{B}$, therefore $0 \in R'(B) = \{u \in B : [u] \in R''(\tilde{B})\}$, whence $[0] \in R''(\tilde{B})$. Note that the choice of $\tilde{B}$ in $\mathcal{B}/I$ was arbitrary, so therefore R'' indeed extends $\mathcal{B}/I$.

Also, by Proposition 127$_{182}$, since R' satisfies Axioms R2$_{20}$–R4$_{20}$, R'' on $\mathcal{V}/I$ satisfies those axioms as well.

We now turn to showing that $0 \in R_{\mathcal{B},I}(A) \Rightarrow 0 \in R'(A)$ for all A in $\mathcal{Q}(\mathcal{V})$. Consider any A in $\mathcal{Q}(\mathcal{V})$ and assume that $0 \in R_{\mathcal{B},I}(A)$. Infer already that then $0 \in A$. By Lemma 139_{195} then $0 \in R'_{\mathcal{B},I}(A)$, so there is some $A' \supseteq A$ in $\mathcal{Q}(\mathcal{V})$ such that

$$(\forall v \in \{0\} \cup (A' \setminus A))$$
$$\big((A'/I - \{[v]\}) \cap \mathcal{V}/I_{\succ[0]} \neq \emptyset \text{ or } (\exists \tilde{B} \in \mathcal{B}/I, \exists \mu \in \mathbb{R}_{>0})\{[v]\} + \mu\tilde{B} \preccurlyeq A'/I\big)$$

Consider any v in $\{0\} \cup (A' \setminus A)$, then $(A'/I - \{[v]\}) \cap \mathcal{V}/I_{\succ[0]} \neq \emptyset$—and therefore $[v] \prec \tilde{u}$ for some $\tilde{u}$ in A'/I, whence by Axiom R2_{20}, $[v] \in R''(\{\tilde{u},[v]\})$, so by Axiom R3a_{20}, $[v] \in R''(A'/I)$—or $\{[v]\} + \mu\tilde{B} \preccurlyeq A'/I$ for some $\tilde{B}$ in $\mathcal{B}/I$ and μ in $\mathbb{R}_{>0}$—and therefore, since R'' extends $\mathcal{B}/I$, $[0] \in R''(\tilde{B})$, so by Axiom R4a_{20}, we have that $[0] \in R''(\mu\tilde{B})$, and using Axiom R4b_{20}, that $[v] = [0] + [v] \in R''(\{[v]\} + \mu\tilde{B})$, and therefore finally, using Proposition 34_{44}, we infer that $[v] \in R''(A'/I)$. So we have shown that $[v] \in R''(A'/I)$—and by compatibility therefore also that $v \in R'(A')$—for every v in $\{0\} \cup (A' \setminus A)$. Therefore, since R' satisfies Axiom R3b_{20}, using Axiom R3b_{20} [with $\tilde{A} \coloneqq A' \setminus A, \tilde{A}_1 \coloneqq \{0\} \cup (A' \setminus A)$ and $\tilde{A}_2 \coloneqq A'$; then $\tilde{A}_1 \setminus \tilde{A} = \{0\}$ since $0 \in A \subseteq A'$ and $\tilde{A}_2 \setminus \tilde{A} = A$ since $A \subseteq A'$] we find that then indeed $0 \in R'(A)$. □

5.9.2 *Assessments avoiding complete rejection under indifference*

Recall from our results on the (normal) natural extension that not every assessment is extendible to a coherent rejection function: this is only the case if the assessment avoids complete rejection. Here too, when we deal with the natural extension under indifference, something similar occurs.

Definition 42 (Avoiding complete rejection under indifference). *Given any assessment $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{V})$ and any coherent set of indifferent options $I \subseteq \mathcal{V}$, we say that $\mathcal{B}$* avoids complete rejection under *I when $R_{\mathcal{B},I}$ satisfies Axiom* R1_{20}.

Due to the extra indifference assessment, avoiding complete rejection under indifference is typically more difficult to fulfil than avoiding complete rejection, as illustrated in the following example.

Example 22. Consider some option u in $\mathcal{V}$ such that $0 \nless u \nless 0$,[3] the assessment $\mathcal{B} \coloneqq \{\{0,u\}\} \subseteq \mathcal{Q}_0(\mathcal{V})$ and the set of indifferent options $I \coloneqq \operatorname{span}\{u\} = \{\lambda u : \lambda \in \mathbb{R}\}$. Then by Corollary 88_{106}, $\mathcal{B}$ avoids complete rejection [to see this, consider for instance the coherent set of desirable options $D \coloneqq \operatorname{posi}(\mathcal{V}_{>0} \cup \{u\})$; then indeed $(\forall B \in \mathcal{B}) D \cap B \neq \emptyset$] and because I is a linear hull, and $\lambda u \notin \mathcal{V}_{>0} \cup \mathcal{V}_{<0}$, I is a coherent set of indifferent options. But $\mathcal{B}$ *does not avoid complete rejection under I*. Indeed, expressed in the quotient space $\mathcal{V}/I$, the assessment is $\mathcal{B}/I = \{\{[0],[u]\}\} = \{\{[0]\}\}$, since $[u] = [0]$. Therefore, since $R_{\mathcal{B}/I}$ extends $\mathcal{B}/I$ [see Lemma 78_{94}], $[0] \in R_{\mathcal{B}/I}(\{[0]\})$, so $R_{\mathcal{B}/I}$ does not satisfy Axiom R1. Consider the option set $A \coloneqq \{0,u\}$. Then $A/I = \{[0]\}$, whence by Equation $(5.6)_{194}$

[3] Or, in other words, such that $u \notin \mathcal{V}_{>0} \cup \mathcal{V}_{<0}$.

$R_{\mathcal{B},I}(A) = \{v \in A : [v] \in R_{\mathcal{B}/I}(A/I)\} = \{v \in \{0,u\} : [v] \in R_{\mathcal{B}/I}(\{[0]\})\} = \{0,u\} = A$, so $R_{\mathcal{B},I}$ does not satisfy Axiom R1$_{20}$ either. Therefore indeed $\mathcal{B}$ does not avoid complete rejection under I.

However, avoiding complete rejection under indifference is sufficient for avoiding complete rejection:

Proposition 142. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{V})$ and any coherent set of indifferent options $I \subseteq \mathcal{V}$. Then $\mathcal{B}$ avoids complete rejection under I if and only if $\mathcal{B}/I$ avoids complete rejection, and both those equivalent conditions imply that $\mathcal{B}$ avoids complete rejection.*

Proof. For the first statement, that $\mathcal{B}$ avoids complete rejection under I if and only if $\mathcal{B}/I$ avoids complete rejection, we first prove necessity. So assume that $\mathcal{B}$ avoids complete rejection under I. Then, by Definition 42, $R_{\mathcal{B},I}$ satisfies Axiom R1$_{20}$, whence $R_{\mathcal{B},I}(A) = \{u \in A : [u] \in R_{\mathcal{B}/I}(A/I)\} \neq A$ for all A in $\mathcal{Q}(\mathcal{V})$. Consider any A in $\mathcal{Q}(\mathcal{V})$. Since $A \neq R_{\mathcal{B},I}(A)$ there is some u in A such that $u \notin R_{\mathcal{B},I}(A)$ and by Equation (5.6)$_{194}$ therefore $[u] \notin R_{\mathcal{B}/I}(A/I)$, whence $R_{\mathcal{B}/I}(A/I) \neq A/I$. Since the choice of A was arbitrary, this means that $R_{\mathcal{B}/I}$ satisfies Axiom R1$_{20}$, and therefore indeed $\mathcal{B}/I$ avoids complete rejection.

For sufficiency, assume that $\mathcal{B}/I$ avoids complete rejection, then by Definition 32$_{95}$, $R_{\mathcal{B}/I}$ satisfies Axiom R1$_{20}$. Consider any A in $\mathcal{Q}(\mathcal{V})$. Then, since $R_{\mathcal{B}/I}$ satisfies Axiom R1$_{20}$, we have that $R_{\mathcal{B}/I}(A/I) \neq A/I$. So there is some u in A such that $[u] \notin R_{\mathcal{B}/I}(A/I)$, whence by Equation (5.6)$_{194}$ $u \notin R_{\mathcal{B},I}(A)$, whence $R_{\mathcal{B},I}(A) \neq A$. Since the choice of A was arbitrary, this means that $R_{\mathcal{B},I}$ satisfies Axiom R1$_{20}$, and therefore indeed $\mathcal{B}$ avoids complete rejection under I.

For the last statement, we will prove the contraposition. Assume that $\mathcal{B}$ does not avoid complete rejection. We know from Lemma 77$_{92}$ that $R_{\mathcal{B}}$ satisfies Axioms R2$_{20}$–R4$_{20}$. By Corollary 26$_{39}$ therefore $0 \in R_{\mathcal{B}}(\{0\})$, and by Equation (3.1)$_{92}$ then there is some $A \supseteq \{0\}$ in $\mathcal{Q}(\mathcal{V})$ such that

$$(\forall u \in A)\big((A - \{u\}) \cap \mathcal{V}_{>0} \neq \emptyset \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{u\} + \mu B \lessapprox A\big)$$

Consider any u in A. If $(A - \{u\}) \cap \mathcal{V}_{>0} \neq \emptyset$ then $u \prec v$ for some v in A, and therefore also $[u] \prec [v]$, so this implies that $(A/I - \{[u]\}) \cap \mathcal{V}/I_{\succ [0]} \neq \emptyset$. If $\{u\} + \mu B \lessapprox A$ for some B in $\mathcal{B}$ and μ in $\mathbb{R}_{>0}$, then, similarly, also $\{[u]\} + \mu B/I \lessapprox A/I$. Therefore

$$(\forall u \in A)\big((A/I - \{[u]\}) \cap \mathcal{V}/I_{\succ [0]} \neq \emptyset \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{[u]\} + \mu B/I \lessapprox A/I\big).$$

By Equation (5.7)$_{195}$ then $0 \in R'_{\mathcal{B},I}(\{0\})$, and by Lemma 139$_{195}$ therefore $0 \in R_{\mathcal{B},I}(\{0\})$. But then $R_{\mathcal{B},I}$ fails to satisfy Axiom R1$_{20}$, so $\mathcal{B}$ does not avoids complete rejection under I, a contradiction. □

As a corollary to Corollary 88$_{106}$, we find the following sufficient condition for avoiding complete rejection under indifference, that is easier to check:

Corollary 143. *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{V})$ and any coherent set of indifferent options $I \subseteq \mathcal{V}$. If $(\exists \tilde{D} \in \overline{\mathbf{D}}(\mathcal{V}/I))(\forall \tilde{B} \in \mathcal{B}/I)\tilde{B} \cap \tilde{D} \neq \emptyset$, then $\mathcal{B}$ avoids complete rejection under I.*

Proof. Assume that $(\exists\tilde{D} \in \overline{\mathbf{D}}(\mathcal{V}/I))(\forall\tilde{B} \in \mathcal{B}/I)\tilde{B} \cap \tilde{D} \neq \emptyset$. By using Corollary 88_{106} with the specific vector space $\mathcal{V}/I$, this implies that $\mathcal{B}/I$ avoids complete rejection. By Proposition $142_{\curvearrowleft}$, this indeed implies that $\mathcal{B}$ avoids complete rejection under I. □

5.9.3 *Natural extension under indifference*

We now formulate a counterpart to Theorem 81_{97} for natural extension under indifference:

Theorem 144 (Natural extension under indifference)**.** *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$ and any coherent set of indifferent options $I \subseteq \mathcal{V}$. Then the following statements are equivalent:*

(i) *$\mathcal{B}$ avoids complete rejection under I;*

(ii) *There is a coherent extension of $\mathcal{B}$ that is compatible with I:*

$$(\forall B \in \mathcal{B})0 \in R(B) \text{ and } (\forall A \in \mathcal{Q}(\mathcal{V}))R(A) = \{u \in A : [u] \in R(A)/I\}$$

for some R in $\overline{\mathbf{R}}(\mathcal{V})$;

(iii) *$\mathcal{E}_I(\mathcal{B}) \neq \mathrm{id}_{\mathcal{Q}(\mathcal{V})}$;*

(iv) *$\mathcal{E}_I(\mathcal{B}) \in \overline{\mathbf{R}}(\mathcal{V})$;*

(v) *$\mathcal{E}_I(\mathcal{B})$ is the least informative rejection function that is coherent, extends $\mathcal{B}$, and is compatible with I.*

When any (and hence all) of these equivalent statements hold, then $\mathcal{E}_I(\mathcal{B}) = R_{\mathcal{B},I}$.

Proof. This is a direct consequence of the Natural Extension Theorem 81_{97} and the representation result of coherent choice functions Proposition 132_{185}. □

5.9.4 *Purely binary assessments*

What happens in the case of binary assessments? Consider that, as in Section 3.5_{99}, a desirability assessment is a subset of $\mathcal{V}$, and consists of options that the agents find desirable (or strictly preferred to the zero option 0). Any desirability assessment $B \subseteq \mathcal{V}$ can be transformed into an assessment for rejection functions $\mathcal{B}_B \coloneqq \{\{0,u\} : u \in B\}$; conversely, given such an assessment $\mathcal{B}_B$, we can retrieve B as $B \coloneqq (\bigcup\mathcal{B}_B) \setminus \{0\}$.

The corresponding notion of avoiding complete rejection under indifference, is *avoiding non-positivity under indifference*, formulated as follows

Definition 43 (Avoiding non-positivity under indifference [31, Definition 4][4])**.**

[4] Actually, in Reference [31, Section 4], De Cooman and Quaeghebeur study the natural extension under *exchangeability*, which is a special indifference assessment. They do this in the context of desirability. However, their treatment is sufficiently general to immediately see—by simply replacing their set $\mathcal{D}_{\mathcal{U}_N}$ with an arbitrary set of indifferent options I—that they actually give the natural extension under indifference for desirability. Therefore, for more details about the natural extension under indifference for desirability, we refer to Reference [31, Section 4].

Given any assessment $B \subseteq \mathcal{V}$ and any coherent set of indifferent options $I \subseteq \mathcal{V}$, we say that B avoids non-positivity under I *when $I + (\mathcal{V}_{>0} \cup B)$ avoids non-positivity, or, in other words,*[5] *when*

$$\operatorname{posi}(I + (\mathcal{V}_{>0} \cup B)) \cap \mathcal{V}_{\leq 0} = \emptyset.$$

Theorem 144 is the counterpart for choice models of the natural extension theorem under indifference for desirability. To be able to compare the two, and to make this thesis more self-contained, we next state the natural extension theorem under indifference for desirability.

Theorem 145 (Natural extension under indifference for desirability [31, Theorem 13])**.** *Consider any desirability assessment $B \subseteq \mathcal{V}$ and any coherent set of indifferent options $I \subseteq \mathcal{V}$, and define its natural extension as*

$$\mathcal{E}_I^{\mathbf{D}}(B) \coloneqq \inf\{D \in \overline{\mathbf{D}} : B \subseteq D\}.^6$$

Then the following statements are equivalent:

(i) *B avoids non-positivity under I;*

(ii) *B is included in some coherent set of desirable options that is compatible with I: $(\exists D \in \overline{\mathbf{D}})(B \subseteq D$ and $D = \bigcup D/I)$;*

(iii) $\mathcal{E}_I^{\mathbf{D}}(B) \neq \mathcal{V}$;

(iv) $\mathcal{E}_I^{\mathbf{D}}(B) \in \overline{\mathbf{D}}(\mathcal{V})$;

(v) *$\mathcal{E}_I^{\mathbf{D}}(B)$ is the least informative set of desirable options that is coherent, includes B and is compatible with I.*

When any (and hence all) of these equivalent statements hold, then $\mathcal{E}_I^{\mathbf{D}}(B) = \operatorname{posi}(I + (\mathcal{V}_{>0} \cup B))$.

Let us go back to the Natural Extension Theorem 144 under indifference (for choice models) and consider a desirability assessment $B \subseteq \mathcal{V}$, its completely binary (choice models) assessment $\mathcal{B}_B$, and a coherent set of indifferent options $I \subseteq \mathcal{V}$. If B avoids non-positivity, then we wonder whether we can retrieve using Theorem 144 the formula $\mathcal{E}^{\mathbf{D}}(B) = \operatorname{posi}(I + (\mathcal{V}_{>0} \cup B))$, as Theorem 145 indicates.

Theorem 146. *Consider any desirability assessment $B \subseteq \mathcal{V}$ and any coherent set of indifferent options $I \subseteq \mathcal{V}$. Then B avoids non-positivity under I if and only if $\mathcal{B}_B$ avoids complete rejection under I, and if this is the case, then $\mathcal{E}_I(\mathcal{B}_B) = R_{\operatorname{posi}(I + (\mathcal{V}_{>0} \cup B))}$.*

Proof. We start with the first part, that B avoids non-positivity under I if and only if $\mathcal{B}_B$ avoids complete rejection under I. For necessity, since B avoids non-positivity under I,

[5] See Equation (3.7)$_{99}$.

[6] We let $\inf \emptyset = \mathcal{V}$.

by Theorem $145_{\curvearrowleft}$, we have that $B \subseteq D$ for some coherent set of desirable options D that is compatible with I. Consider the coherent rejection function R_D. By Theorem 86_{100} we know already that R_D extends $\mathcal{B}_B$, and by Proposition 136_{189} that R_D is compatible with I. Therefore, by Theorem 144_{200}, $\mathcal{B}_B$ indeed avoids complete rejection under I.

For sufficiency, since $\mathcal{B}_B$ avoids complete rejection under I, by Theorem 144_{200} $(\forall u \in B) 0 \in R(\{0,u\})$ for some coherent rejection function R on $\mathcal{V}$ that is compatible with I. Consider the coherent set of desirable options $D_R = \{u \in \mathcal{V} : 0 \in R(\{0,u\})\}$. By Theorem 86_{100} we know already that D_R extends B, and by Proposition 136_{189} that D_R is compatible with I. Therefore, by Theorem $145_{\curvearrowleft}$, B indeed avoids non-positivity under I.

We now show the second part, that $\mathcal{E}_I(\mathcal{B}_B) = R_{\operatorname{posi}(I+(\mathcal{V}_{>0}\cup B))}$ if B avoids non-positivity under I. But we have just shown that then $\mathcal{B}_B$ avoids complete rejection under I, so $\mathcal{E}_I(\mathcal{B}_B) = R_{\mathcal{B},I}$, and we are left to prove that $R_{\mathcal{B},I} = R_{\operatorname{posi}(I+(\mathcal{V}_{>0}\cup B))}$ Since both rejection functions are coherent, by Axiom R4b_{20} it suffices to prove that $0 \in R_{\mathcal{B}_B,I}(A) \Leftrightarrow 0 \in R_{\operatorname{posi}(I+(\mathcal{V}_{>0}\cup B))}(A)$ for all A in $\mathcal{Q}(\mathcal{V})$.

So consider any A in $\mathcal{Q}(\mathcal{V})$, and infer that

$$\begin{aligned}
0 \in R_{\mathcal{B}_B,I}(A) &\Leftrightarrow (0 \in A \text{ and } [0] \in R_{\mathcal{B}_B/I}(A/I))\\
&\Leftrightarrow (0 \in A \text{ and } [0] \in R_{\mathcal{B}_{B/I}}(A/I))\\
&\Leftrightarrow (0 \in A \text{ and } A/I \cap \operatorname{posi}(\mathcal{V}/I_{\succ[0]} \cup B/I) \neq \emptyset),
\end{aligned}$$

where the first equivalence is due to Equation $(5.6)_{194}$, the second is due to the observation that $\mathcal{B}_B/I = \{\{0,u\}/I : u \in B\} = \{\{[0],\tilde{u}\} : u \in B/I\} = \mathcal{B}_{B/I}$, and the third holds by Theorem 86_{100} for the vector space $\mathcal{V}/I$. So it suffices to show that $A/I \cap \operatorname{posi}(\mathcal{V}/I_{\succ[0]} \cup B/I) \neq \emptyset \Leftrightarrow A \cap \operatorname{posi}(I + (\mathcal{V}_{>0} \cup B)) \neq \emptyset$, because, by Proposition 55_{64} this together with $0 \in A$ is equivalent to $0 \in R_{\operatorname{posi}(I+(\mathcal{V}_{>0}\cup B))}(A)$. Infer that $A/I \cap \operatorname{posi}(\mathcal{V}/I_{\succ[0]} \cup B/I) \neq \emptyset$ is equivalent to $[u] \in \operatorname{posi}(\mathcal{V}/I_{\succ[0]} \cup B/I)$ for some u in A, and therefore, since $\mathcal{V}/I_{\succ[0]} \cup B/I = (\mathcal{V}_{>0} \cup B)/I$ and by Lemma 147, this is in turn equivalent to $u \in \operatorname{posi}(I + (\mathcal{V}_{>0} \cup B))$ for some u in A. In other words, this is indeed equivalent to $A \cap \operatorname{posi}(I + (\mathcal{V}_{>0} \cup B)) \neq \emptyset$. □

Lemma 147. *Consider any $A \subseteq \mathcal{V}$ and any coherent set of indifferent options $I \subseteq \mathcal{V}$. Then $[u] \in \operatorname{posi}(A/I) \Leftrightarrow u \in \operatorname{posi}(A + I)$ for all u in $\mathcal{V}$.*

Proof. Consider any u in $\mathcal{V}$. Observe that

$$\begin{aligned}
[u] \in \operatorname{posi}(A/I) &\Leftrightarrow (\exists n \in \mathbb{N}, \exists \lambda_1,\dots,\lambda_n \in \mathbb{R}_{>0}, \exists u_1,\dots,u_n \in A)[u] = \sum_{k=1}^{n} \lambda_k [u_k]\\
&\Leftrightarrow (\exists n \in \mathbb{N}, \exists \lambda_1,\dots,\lambda_n \in \mathbb{R}_{>0}, \exists u_1,\dots,u_n \in A)[u] = [\sum_{k=1}^{n} \lambda_k u_k]\\
&\Leftrightarrow (\exists n \in \mathbb{N}, \exists \lambda_1,\dots,\lambda_n \in \mathbb{R}_{>0}, \exists u_1,\dots,u_n \in A, \exists w \in I) u = w + \sum_{k=1}^{n} \lambda_k u_k\\
&\Leftrightarrow (\exists w \in I) u \in \{w\} + \operatorname{posi}(A)\\
&\Leftrightarrow u \in I + \operatorname{posi}(A).
\end{aligned}$$

Therefore, since $\operatorname{posi}(A + I) = \operatorname{posi}(A) + \operatorname{posi}(I) = \operatorname{posi}(A) + I$, this proves the desired statement. □

Similarly to its counterpart in Section 3.5$_{99}$, Theorem 146$_{201}$ consists of three remarkable statements: The first statement is that the natural extension under indifference (represented by a coherent set of indifferent options I) of a purely binary assessment $\mathcal{B}_B$, for some $B \subseteq \mathcal{V}$, is a rejection function that is purely binary itself; the second one is that its (binary) behaviour is exactly given by $\operatorname{posi}(I + (\mathcal{V}_{>0} \cup B))$; both statements are conditional on $\mathcal{B}_B$ avoiding complete rejection under I—which is furthermore, as a third statement, precisely equivalent to B avoiding non-positivity under I.

Focusing on the second statement, for any coherent set of indifferent options $I \subseteq \mathcal{V}$ and any desirability assessment $B \subseteq \mathcal{V}$ that avoids non-positivity, the natural extension $\mathcal{E}_I(\mathcal{B}_B)$ (for choice models) induces the binary choice $D_{\mathcal{E}_I(\mathcal{B}_B)}$ reflected by $\operatorname{posi}(I + (\mathcal{V}_{>0} \cup B))$. To see this, Theorem 146$_{201}$ guarantees that $\mathcal{E}_I(\mathcal{B}_B) = R_{\operatorname{posi}(I+(\mathcal{V}_{>0}\cup B))}$, where by Theorem 145$_{201}$, $\operatorname{posi}(I + (\mathcal{V}_{>0} \cup B))$ is a coherent set of desirable options and by Proposition 58$_{66}$, therefore indeed $D_{\mathcal{E}(\mathcal{B}_B)} = \operatorname{posi}(I + (\mathcal{V}_{>0} \cup B))$.

To summarise these statements, consider the following commuting diagram in Figure 5.1—which is the counterpart of Figure 3.1$_{104}$ for natural extension under indifference. We used the maps

$$
\begin{aligned}
&\mathcal{E}_I^{\mathbf{D}} : \mathcal{P}(\mathcal{V}) \to \mathbf{D} : B \mapsto \mathcal{E}_I^{\mathbf{D}}(B),\\
&\mathcal{B}_\bullet : \mathcal{P}(\mathcal{V}) \to \mathcal{Q}_0 : B \mapsto \mathcal{B}_B \coloneqq \{\{0,u\} : u \in B\},\\
&\mathcal{E}_I : \mathcal{P}(\mathcal{Q}_0) \to \mathbf{R} : \mathcal{B} \mapsto \mathcal{E}_I(\mathcal{B}),\\
&D_\bullet : \mathbf{R} \to \mathbf{D} : R \mapsto D_R \coloneqq \{u \in \mathcal{V} : 0 \in R(\{0,u\})\},\\
&R_\bullet : \mathbf{D} \to \mathbf{R} : D \mapsto R_D,
\end{aligned}
$$

with $\mathcal{E}_I^{\mathbf{D}}(B)$ defined in Theorem 145$_{201}$, $\mathcal{E}_I(\mathcal{B})$ in Theorem 144$_{200}$ and, as usual, R_D given by $R_D(A) = \{u \in A : (\forall v \in A) v - u \notin D\}$ for all A in $\mathcal{Q}$. The root of the diagram is any desirability assessment $B \subseteq \mathcal{V}$ that avoids non-positivity under indifference, captured by a coherent set of indifferent options I.

$$
\begin{tikzcd}
B \arrow[r, "\mathcal{E}_I^{\mathbf{D}}"] \arrow[d, "\mathcal{B}_\bullet"'] & \mathcal{E}_I^{\mathbf{D}}(B) = D_{\mathcal{E}_I(\mathcal{B}_B)} \arrow[d, shift right, "R_\bullet"'] \\
\mathcal{B}_B \arrow[r, "\mathcal{E}_I"] & \mathcal{E}_I(\mathcal{B}_B) = R_{\mathcal{E}_I^{\mathbf{D}}(B)} \arrow[u, shift right, "D_\bullet"']
\end{tikzcd}
$$

Figure 5.1: Commuting diagram for the natural extension under indifference for binary assessments

6

CONDITIONING

Consider a variable X that assumes values in a non-empty possibility space $\mathcal{X}$. Suppose that we have a belief model about X, be it a coherent choice function on $\mathcal{L}$, a coherent set of desirable gambles on $\mathcal{X}$, or—less general—a coherent lower prevision on $\mathcal{X}$, a set of mass functions on $\mathcal{X}$, or just a single mass function on $\mathcal{X}$. When new information becomes available, in the form of 'X assumes a value in some (non-empty) subset E of $\mathcal{X}$', we can take this into account by *conditioning* our belief model on E.

For some of these belief models, such as coherent lower previsions, and (sets of) mass functions, conditioning on events of probability zero can be problematic, because, roughly speaking, Bayes's Rule typically requires to divide by zero in these situations. However, working with sets of desirable gambles is one way of overcoming this problem. In this chapter, we will see why, and explain that choice functions do not suffer from this problem either.

We will work with the vector space $\mathcal{L}(\mathcal{X}\times\mathcal{R})$ of vector-valued gambles, in order to guarantee the connection, explained in Section 2.428, with the choice functions considered by Seidenfeld et al. [67]. The finite set $\mathcal{R}$ serves as a set of rewards, and is assumed to be fixed throughout.

We will first review how conditioning is done using desirability (see Reference [31] for more details). After that, we will introduce conditional choice functions, and study the connection with conditional sets of desirable gambles. We will let any event, except for the (trivially) impossible event $\varnothing$, serve as a conditioning event. We collect the allowed conditioning events in $\mathcal{P}_{\varnothing}(\mathcal{X}) \coloneqq \mathcal{P}(\mathcal{X}) \setminus \{\varnothing\} = \{E \subseteq \mathcal{X} : E \neq \varnothing\}$.

6.1 DESIRABILITY

For sets of desirable gambles conditioning is very elegant, as explained in detail in Reference [31]. We give an outline of the basic ideas here, and at the same time expand the treatment there to also deal with vector-valued, rather than real-valued, gambles. There are multiple equivalent definitions for conditioning on an event E in $\mathcal{P}_{\varnothing}(\mathcal{X})$. Starting from a coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X}\times\mathcal{R})$, the definitions used in References [13, 55, 82, 83] all result in a conditional set of desirable gambles $D|E$ that consists of *(vector-valued) gambles defined on the whole possibility space* $\mathcal{X}$. There are a number of different but equivalent definitions of such conditional sets of gambles on the whole possibility space; one of them, considered in Reference [31], is *mutatis mutandis*[1] given by

$$D|E \coloneqq \{f \in D : \mathbb{I}_{E\times\mathcal{R}} f = f\},$$

where $\mathbb{I}_{E\times\mathcal{R}}$ is the indicator of $E\times\mathcal{R}$, defined in Section 2.1.2$_{12}$. However, we will find it more useful and convenient that a conditional model is defined *on vector-valued gambles on E*—gambles on $E\times\mathcal{R}$—, rather than on $\mathcal{X}$, because, after getting to know that E occurs, the possibility space becomes effectively E. Therefore we work with a modified version $D\rfloor E$, first considered by De Cooman and Quaeghebeur in Reference [31]:

$$D\rfloor E \coloneqq \{f \in \mathcal{L}(E\times\mathcal{R}) : \mathbb{I}_E f \in D\}.$$

In this definition, we let[2] for any E in $\mathcal{P}_{\varnothing}(\mathcal{X})$ and any f in $\mathcal{L}(E\times\mathcal{R})$, $\mathbb{I}_E f$ be the gamble on $\mathcal{X}\times\mathcal{R}$ given by

$$\mathbb{I}_E f(x,r) \coloneqq \begin{cases} f(x,r) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \tag{6.1}$$

for all x in $\mathcal{X}$ and r in $\mathcal{R}$. Note that, for all E in $\mathcal{P}_{\varnothing}(\mathcal{X})$, and all f and g in $\mathcal{L}(E\times\mathcal{R})$, we have that $\mathbb{I}_{E\times\mathcal{R}}\mathbb{I}_E f = \mathbb{I}_E f$, that

$$\begin{aligned} f \neq g &\Leftrightarrow (\exists x \in E, r \in \mathcal{R})(f(x,r) \neq g(x,r)) \\ &\Leftrightarrow (\exists x \in E, r \in \mathcal{R})(\mathbb{I}_E f(x,r) \neq \mathbb{I}_E g(x,r)) \\ &\Leftrightarrow (\exists x \in \mathcal{X}, r \in \mathcal{R})(\mathbb{I}_E f(x,r) \neq \mathbb{I}_E g(x,r)) \Leftrightarrow \mathbb{I}_E f \neq \mathbb{I}_E g, \end{aligned} \tag{6.2}$$

[1] Taking into account that we use *vector-valued* gambles rather than (normal) gambles.

[2] $\mathbb{I}_E$ is the indicator (gamble) defined on $\mathcal{X}$, while f is a gamble defined on $E\times\mathcal{R}$: their domains differ, so their multiplication needs to be defined with some care.

and that

$$\begin{aligned} f < g &\Leftrightarrow (f \leq g \text{ and } f \neq g) \\ &\Leftrightarrow ((\forall x \in E, r \in \mathcal{R})(f(x,r) \leq g(x,r)) \text{ and } \mathbb{I}_E f \neq \mathbb{I}_E g) \\ &\Leftrightarrow ((\forall x \in E, r \in \mathcal{R})(\mathbb{I}_E f(x,r) \leq \mathbb{I}_E g(x,r)) \text{ and } \mathbb{I}_E f \neq \mathbb{I}_E g) \\ &\Leftrightarrow ((\forall x \in \mathcal{X}, r \in \mathcal{R})(\mathbb{I}_E f(x,r) \leq \mathbb{I}_E g(x,r)) \text{ and } \mathbb{I}_E f \neq \mathbb{I}_E g) \\ &\Leftrightarrow (\mathbb{I}_E f \leq \mathbb{I}_E g \text{ and } \mathbb{I}_E f \neq \mathbb{I}_E g) \Leftrightarrow \mathbb{I}_E f < \mathbb{I}_E g. \end{aligned} \tag{6.3}$$

Both definitions of conditioning are essentially equivalent, since $f \in D\rfloor E \Leftrightarrow \mathbb{I}_E f \in D \Leftrightarrow \mathbb{I}_E f \in D|E$ for all f in $\mathcal{L}(E \times \mathcal{R})$, and we will only consider the version $D\rfloor E$ in the remainder of this dissertation.

Proposition 148 ([31, Proposition 8])**.** *Consider any coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X} \times \mathcal{R})$ and any event E in $\mathcal{P}_\varnothing(\mathcal{X})$. Then $D\rfloor E$ is a coherent set of desirable gambles on $E \times \mathcal{R}$.*

Proof. Immediate adaptation of the proof in Reference [31, Proposition 8] to deal with vector-valued gambles. □

Equivalently, a preference relation $\prec$ on $\mathcal{L}(\mathcal{X} \times \mathcal{R})$ can be conditioned on any event E in $\mathcal{P}_\varnothing(\mathcal{X})$. There are again different definitions. Some versions yield preference relations on (a subset of) $\mathcal{L}(\mathcal{X} \times \mathcal{R})$, but here we will focus on a version that yields a preference relation on $\mathcal{L}(E \times \mathcal{R})$, because, again, we will find it more useful and convenient that conditional models are defined on vector-valued gambles on E rather than on $\mathcal{X}$. We let $\prec\rfloor E$ be the preference relation conditional on E, defined by

$$f \prec\rfloor E\, g \Leftrightarrow \mathbb{I}_E f \prec \mathbb{I}_E g, \text{ for all } f \text{ and } g \text{ in } \mathcal{L}(E \times \mathcal{R}).$$

It turns out that the definitions of conditioning for sets of desirable gambles and for preference relations are essentially equivalent:

Proposition 149. *Consider a coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X} \times \mathcal{R})$, a coherent preference relation $\prec$ on $\mathcal{L}(\mathcal{X} \times \mathcal{R})$ and a conditioning event E in $\mathcal{P}_\varnothing(\mathcal{X})$. Then $\prec_D\rfloor E = \prec_{D\rfloor E}$ and $D_\prec\rfloor E = D_{\prec\rfloor E}$. As a consequence, $\prec\rfloor E$ is coherent.*

Proof. To avoid notational overload and confusion, we will use the notations $\prec' := \prec\rfloor E$ and $\prec'_D := \prec_D\rfloor E$, and show that $\prec'_D = \prec_{D\rfloor E}$ and $D_\prec\rfloor E = D_{\prec'}$.

For the first statement, consider any f and g in $\mathcal{L}(E \times \mathcal{R})$, and infer, by the definition of conditional preference relations and the definition of $\prec_D$ (see Section 2.8.2$_{56}$), that $f \prec'_D g \Leftrightarrow \mathbb{I}_E f \prec_D \mathbb{I}_E g \Leftrightarrow \mathbb{I}_E(g - f) \in D$. By the definition of conditional sets of desirable gambles, this is equivalent to $g - f \in D\rfloor E$, and using the definition of $\prec_D$ again, therefore indeed also to $f \prec_{D\rfloor E} g$.

For the second statement, observe that indeed $D_\prec\rfloor E = \{f \in \mathcal{L}(E \times \mathcal{R}) : \mathbb{I}_E f \in D_\prec\} = \{f \in \mathcal{L}(E \times \mathcal{R}) : 0 \prec \mathbb{I}_E f\} = \{f \in \mathcal{L}(E \times \mathcal{R}) : 0 \prec' f\} = D_{\prec'}$, where the first equality is due to the definition of conditional sets of desirable gambles, the second and the fourth equalities result from the definition of $D_\prec$ (see Section $2.8.2_{56}$), and the third equality is due to the definition of conditional preference relations.

For the consequence, we know from the discussion in Section $2.8.2_{56}$ that the coherence of $\prec$ implies the coherence of $D_\prec$, and by Proposition $148_\curvearrowleft$, we have that $D_\prec\rfloor E$ is a coherent set of desirable gambles, and therefore $\prec_{D_\prec\rfloor E}$ is a coherent preference relation. We have just shown that $D_\prec\rfloor E = D_{\prec\rfloor E}$, so $\prec_{D_\prec\rfloor E} = \prec\rfloor E$, so $\prec\rfloor E$ is indeed a coherent preference relation. $\square$

This proposition is summarised in the commuting diagram of Figure 6.1, where we use the maps

$$D_\bullet : \overline{\mathbf{P}} \to \overline{\mathbf{D}} : \prec \mapsto D_\prec,$$
$$\prec_\bullet : \overline{\mathbf{D}} \to \overline{\mathbf{P}} : D \mapsto \prec_D,$$

and $\bullet\rfloor E$, to denote conditioning a set of desirable gambles D or a preference relation $\prec$.

$$\begin{tikzcd}
D \arrow[rr, shift left, "D_\bullet"] \arrow[d, "\bullet\rfloor E"'] & & \prec \arrow[ll, shift left, "\prec_\bullet"] \arrow[d, "\bullet\rfloor E"] \\
D\rfloor E \arrow[rr, shift left, "D_\bullet"] & & \prec\rfloor E \arrow[ll, shift left, "\prec_\bullet"]
\end{tikzcd}$$

Figure 6.1: Commuting diagram for conditioning desirability models: Start with a coherent set of desirable gambles D and its corresponding preference relation $\prec$ on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$. Conditioning them on E result in a conditional set of desirable gambles $D\rfloor E$ and a preference relation $\prec\rfloor E$ on $\mathcal{L}(E\times\mathcal{R})$ that again correspond.

6.2 CHOICE MODELS

For choice models, conditioning can defined using the same simple underlying ideas.

Definition 44 (Conditional choice function)**.** *Given any choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$ and any conditioning event E in $\mathcal{P}_\varnothing(\mathcal{X})$, we define the choice function $C\rfloor E$ on $\mathcal{L}(E\times\mathcal{R})$ as*

$$C\rfloor E(A) \coloneqq \{f \in A : \mathbb{I}_E f \in C(\{\mathbb{I}_E f : f \in A\})\} \textit{ for all } A \textit{ in } \mathcal{Q}(\mathcal{L}(E\times\mathcal{R})),$$

and will also call it a conditional choice function.

From here on, we will use the simplifying notational convention that $\mathbb{I}_E A \coloneqq \{\mathbb{I}_E f : f \in A\} \in \mathcal{Q}(\mathcal{L}(\mathcal{X}\times\mathcal{R}))$ for any A in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$, where, by Equation (6.1)$_{206}$, $\mathbb{I}_E f$ is the gamble on $\mathcal{X}\times\mathcal{R}$, that is equal to f on $E\times\mathcal{R}$ and to 0 on $E^c\times\mathcal{R}$. Using this notational convention,

$$C\rfloor E(A) = \{f \in A : \mathbb{I}_E f \in C(\mathbb{I}_E A)\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(E\times\mathcal{R})),$$

or equivalently,

$$f \in C\rfloor E(A) \Leftrightarrow \mathbb{I}_E f \in C(\mathbb{I}_E A), \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(E\times\mathcal{R})) \text{ and all } f \text{ in } A.$$

Proposition 150. *Consider any choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$, and any event E in $\mathcal{P}_\varnothing(\mathcal{X})$. Then, for any property* C$*$ *in*

$$\{\mathrm{C1}_{20}, \mathrm{C2}_{20}, \mathrm{C3a}_{20}, \mathrm{C3b}_{20}, \mathrm{C4a}_{20}, \mathrm{C4b}_{20}, \mathrm{C5}_{25}, \mathrm{C6}_{25}\},$$

if C satisfies C$*$*, then $C\rfloor E$ satisfies* C$*$*. As a consequence, if C is coherent, then so is $C\rfloor E$.*

Proof. For Axiom C1$_{20}$, consider any option set A of gambles on $E\times\mathcal{R}$. Since $C(\mathbb{I}_E A) \neq \varnothing$, indeed also $C\rfloor E(A) \neq \varnothing$.

For Axiom C2$_{20}$, consider any f and g in $\mathcal{L}(E\times\mathcal{R})$ for which $f < g$. Then $\mathbb{I}_E f < \mathbb{I}_E g$ by Equation (6.3)$_{207}$, so $\mathbb{I}_E f \notin C(\{\mathbb{I}_E f, \mathbb{I}_E g\})$, and therefore indeed $f \notin C\rfloor E(\{f,g\})$.

For Axiom C3a$_{20}$, consider any A, A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$ such that $C\rfloor E(A_2) \subseteq A_2 \setminus A_1$ and $A_1 \subseteq A_2 \subseteq A$. Then $C(\mathbb{I}_E A_2) \subseteq \mathbb{I}_E(A_2\setminus A_1) = \mathbb{I}_E A_2 \setminus \mathbb{I}_E A_1$—using Equation (6.2)$_{206}$ in the equality—and $\mathbb{I}_E A_1 \subseteq \mathbb{I}_E A_2 \subseteq \mathbb{I}_E A$, and therefore, $C(\mathbb{I}_E A) \subseteq \mathbb{I}_E A \setminus \mathbb{I}_E A_1$. But then indeed $C\rfloor E(A) \subseteq A \setminus A_1$.

For Axiom C3b$_{20}$, consider any A, A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$ such that $C\rfloor E(A_2) \subseteq A_2 \setminus A_1$ and $A \subseteq A_1$. Then $C(\mathbb{I}_E A_2) \subseteq \mathbb{I}_E(A_2\setminus A_1) = \mathbb{I}_E A_2 \setminus \mathbb{I}_E A_1$—using Equation (6.2)$_{206}$ in the equality—and $\mathbb{I}_E A \subseteq \mathbb{I}_E A_1$, and therefore, $C(\mathbb{I}_E(A_2\setminus A)) = C(\mathbb{I}_E A_2 \setminus \mathbb{I}_E A) \subseteq \mathbb{I}_E A_2 \setminus \mathbb{I}_E A_1 = \mathbb{I}_E(A_2\setminus A_1)$. But then indeed $C\rfloor E(A_2\setminus A) \subseteq A_2 \setminus A_1$.

For Axiom C4a$_{20}$, consider any A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$ and any λ in $\mathbb{R}_{>0}$ for which $A_1 \subseteq C\rfloor E(A_2)$. Then $\mathbb{I}_E A_1 \subseteq C(\mathbb{I}_E A_2)$, and therefore, $\mathbb{I}_E \lambda A_1 \subseteq C(\mathbb{I}_E \lambda A_2)$. But then indeed $\lambda A_1 \subseteq C\rfloor E(\lambda A_2)$.

For Axiom C4b$_{20}$, consider any A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$ and any f in $\mathcal{L}(E\times\mathcal{R})$ for which $A_1 \subseteq C\rfloor E(A_2)$. Then $\mathbb{I}_E A_1 \subseteq C(\mathbb{I}_E A_2)$, and therefore, $\mathbb{I}_E(A_1 + \{f\}) = \mathbb{I}_E A_1 + \{\mathbb{I}_E f\} \subseteq C(\mathbb{I}_E A_2 + \{\mathbb{I}_E f\}) = C(\mathbb{I}_E(A_2 + \{f\}))$. But then indeed $\lambda A_1 + \{f\} \subseteq C\rfloor E(\lambda A_2 + \{f\})$.

For Property C5$_{25}$, consider any A and A_1 in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$ such that $A \subseteq A_1 \subseteq \mathrm{conv}(A)$. Then $\mathbb{I}_E A \subseteq \mathbb{I}_E A_1 \subseteq \mathrm{conv}(\mathbb{I}_E A)$, whence $C(\mathbb{I}_E A) \subseteq C(\mathbb{I}_E A_1)$, and therefore indeed $C\rfloor E(A) \subseteq C\rfloor E(A_1)$.

For Property C6$_{25}$, consider any n in $\mathbb{N}$, any $\lambda_1, \ldots, \lambda_n$ in $\mathbb{R}_{>0}$ and any $f_1, \ldots, f_n$ in $\mathcal{L}(E\times\mathcal{R})$ such that $0 \in C\rfloor E(\{0, f_1, \ldots, f_n\})$. Then $0 \in C(\{0, \mathbb{I}_E f_1, \ldots, \mathbb{I}_E f_n\})$, and therefore $0 \in C(\{0, \mathbb{I}_E \lambda_1 f_1, \ldots, \mathbb{I}_E \lambda_n f_n\})$, whence indeed $0 \in C\rfloor E(\{0, f_1, \ldots, f_n\})$. $\square$

Remark that there are no constraints on the conditioning event E in Proposition $150_{\curvearrowleft}$: if C is coherent, then $C\rfloor E$ is coherent for *every* conditioning event E in $\mathcal{P}_{\emptyset}(\mathcal{X})$. This means that, using choice functions, we can condition on every event, even if this event has probability zero for some of the linear previsions in the set of linear previsions corresponding to the lower prevision $\underline{P}_C$ associated with the choice function C through Equation $(2.25)_{72}$.

For rejection sets, the definition of conditioning on an event E in $\mathcal{P}_{\emptyset}(\mathcal{X})$ is very similar. Since $R(A) = A \setminus C(A)$ for all A in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$, we find that

$$R\rfloor E(A) = A \setminus C\rfloor E(A) = A \setminus \{f \in A : \mathbb{I}_E f \in C(\mathbb{I}_E A)\} = \{f \in A : \mathbb{I}_E f \in R(\mathbb{I}_E A)\}$$

for all A in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$. Therefore, by Proposition $150_{\curvearrowleft}$, if R is a coherent rejection function on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$ and E a conditioning event in $\mathcal{P}_{\emptyset}(\mathcal{X})$, then $R\rfloor E$ is a coherent rejection function on $\mathcal{L}(E\times\mathcal{R})$.

Definition 45 (Conditional choice relation)**.** *Given any choice relation $\triangleleft$ on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$ and any conditioning event E in $\mathcal{P}_{\emptyset}(\mathcal{X})$, we define the choice relation $\triangleleft\rfloor E$ conditional on E as*

$$A_1 \triangleleft\rfloor E\, A_2 \Leftrightarrow \mathbb{I}_E A_1 \triangleleft \mathbb{I}_E A_2, \textit{ for all } A_1 \textit{ and } A_2 \textit{ in } \mathcal{Q}(\mathcal{L}(E\times\mathcal{R})).$$

Definition 45 is conceptually a reformulation of Definition 44_{208}: they are essentially identical, as the following proposition shows.

Proposition 151. *Consider any coherent choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$, any coherent choice relation $\triangleleft$ on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$ and any conditioning event E in $\mathcal{P}_{\emptyset}(\mathcal{X})$. Then $\triangleleft_C\rfloor E = \triangleleft_{C\rfloor E}$ and $C_{\triangleleft}\rfloor E = C_{\triangleleft\rfloor E}$. As a consequence, $\triangleleft\rfloor E$ is coherent.*

Proof. For the first statement, consider any A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$, and infer from Definition 45 and Equation $(2.19)_{61}$ that $A_1 \triangleleft_C\rfloor E\, A_2 \Leftrightarrow \mathbb{I}_E A_1 \triangleleft_C \mathbb{I}_E A_2 \Leftrightarrow C(\mathbb{I}_E A_1 \cup \mathbb{I}_E A_2) \subseteq \mathbb{I}_E A_2 \setminus \mathbb{I}_E A_1 = \mathbb{I}_E(A_2 \setminus A_1)$. By Definition 44_{208}, this is equivalent to $C\rfloor E(A_1 \cup A_2) \subseteq A_2 \setminus A_1$, and using Equation $(2.19)_{61}$ therefore indeed also $A_1 \triangleleft_{C\rfloor E} A_2$.

For the second statement, consider any A in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$, and infer that indeed $C_{\triangleleft}\rfloor E(A) = \{f \in A : \mathbb{I}_E f \in C_{\triangleleft}(\mathbb{I}_E A)\} = \{f \in A : \{\mathbb{I}_E f\} \ntriangleleft \mathbb{I}_E A\} = \{f \in A : \{f\} \ntriangleleft\rfloor E\, A\} = C_{\triangleleft\rfloor E}(A)$, where the first equality is due to Definition 44_{208}, the second and the fourth equalities are by Definition 4_{16}, and the third equality is due to Definition 45.

For the consequence, by Proposition 13_{22} we know that the coherence of $\triangleleft$ implies the coherence of $C_{\triangleleft}$, and by Proposition $150_{\curvearrowleft}$, we have that $C_{\triangleleft}\rfloor E$ is a coherent choice function on $\mathcal{L}(E\times\mathcal{R})$, and therefore $\triangleleft_{C_{\triangleleft}\rfloor E}$ is a coherent choice relation. We have just shown that $C_{\triangleleft}\rfloor E = C_{\triangleleft\rfloor E}$, so $\triangleleft_{C_{\triangleleft}\rfloor E} = \triangleleft\rfloor E$, whence $\triangleleft\rfloor E$ is indeed a coherent choice relation. $\square$

Conditioning preserves the ordering between choice functions.

Proposition 152. *Consider coherent choice functions C_1 and C_2 on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$ and a conditioning event E in $\mathcal{P}_{\emptyset}(\mathcal{X})$. If $C_1 \sqsubseteq C_2$ then $C_1\rfloor E \sqsubseteq C_2\rfloor E$.*

Proof. Consider any A in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$ and any f in $C_2\rfloor E(A)$. We will show that then $f\in C_1\rfloor E(A)$. Since $f\in C_2\rfloor E(A)$, by Definition 44_{208} we infer that $\mathbb{I}_E f\in C_2(\mathbb{I}_E A)$, and therefore also $\mathbb{I}_E f\in C_1(\mathbb{I}_E A)$, because $C_1\sqsubseteq C_2$. By Definition 44_{208}, we see that then indeed $f\in C_1\rfloor E(A)$. □

As a consequence, Propositions 151 and 152 together imply that $\lhd_1\rfloor E\sqsubseteq\lhd_2\rfloor E$ for all coherent choice relations $\lhd_1$ and $\lhd_2$ on $\mathcal{L}(E\times\mathcal{R})$ for which $\lhd_1\sqsubseteq\lhd_2$, and every conditioning event E in $\mathcal{P}_\varnothing(\mathcal{X})$.

6.2.1 *Relation with desirability*

Is Definition 44_{208}—or the equivalent version Definition 45—a suitable definition of conditioning? One of the useful properties our definition has, is that it preserves coherence, as shown in Proposition 150_{209}. But does it also generalise the definition in Section 6.1_{206} of conditional sets of desirable gambles, or in other words, do Definitions 44_{208} and 45 reduce to the definition of conditioning sets of desirable gambles in Section 6.1_{206} when only considering binary choice? Of course, to investigate this, we must keep in mind the connection between choice models and desirability, explained in detail in Section 2.8_{55}.

For our two conditioning rules—the one for desirability and the one for choice models—to be a match, there are definitively two conditions to be met: (i) the conditioning rule for choice functions should revert to the known conditioning rule for the corresponding sets of desirable gambles, and (ii) in the special case of purely binary choice, the conditioning for choice functions should coincide with the conditioning rule for desirability. We proceed to show that both these requirements are satisfied. Mathematically, (i) means that $D_C\rfloor E=D_{C\rfloor E}$ for every coherent choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$ and conditioning event E in $\mathcal{P}_\varnothing(\mathcal{X})$, and (ii) means that $C_D\rfloor E=C_{D\rfloor E}$ for every coherent set of desirable gambles $D\subseteq\mathcal{L}(\mathcal{X}\times\mathcal{R})$ and conditioning event E in $\mathcal{P}_\varnothing(\mathcal{X})$.

Proposition 153. *Consider any coherent choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$, any coherent set of desirable gambles $D\subseteq\mathcal{L}(\mathcal{X}\times\mathcal{R})$ and any conditioning event E in $\mathcal{P}_\varnothing(\mathcal{X})$. Then $D_C\rfloor E=D_{C\rfloor E}$ and $C_D\rfloor E=C_{D\rfloor E}$.*

Proof. For the first statement, derive the following equalities, from the definition of conditional sets of desirable gambles and Proposition 53_{61}:

$$D_C\rfloor E=\{f\in\mathcal{L}(E\times\mathcal{R}):\mathbb{I}_E f\in D_C\}=\{f\in\mathcal{L}(E\times\mathcal{R}):0\notin C(\{0,\mathbb{I}_E f\})\}.$$

But by Definition 44_{208} $0\notin C(\{0,\mathbb{I}_E f\})\Leftrightarrow 0\notin C\rfloor E(\{0,f\})$, and therefore indeed $D_C\rfloor E=\{f\in\mathcal{L}(E\times\mathcal{R}):0\notin C\rfloor E(\{0,f\})\}=D_{C\rfloor E}$, using Proposition 53_{61} for the last equality.

For the second statement, consider any A in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$, then

$$\begin{aligned} C_D\rfloor E(A) = \{f\in A : \mathbb{I}_E f\in C_D(\mathbb{I}_E A)\} &= \{f\in A : (\forall g\in\mathbb{I}_E A) g-\mathbb{I}_E f\notin D\} \\ &= \{f\in A : (\forall g\in A)\mathbb{I}_E g-\mathbb{I}_E f\notin D\} \\ &= \{f\in A : (\forall g\in A) g-f\notin D\rfloor E\} = C_{D\rfloor E}(A), \end{aligned}$$

where the first equality follows from Definition 44_{208}, the second and the fifth equalities are due to Proposition 54_{62}, and the fourth equality follows from the definition of conditional sets of desirable gambles. □

As a consequence, by Propositions 149_{207} and 151_{210}, also $D_{\lhd}\rfloor E = D_{\lhd\rfloor E}$ and $\prec_{\lhd}\rfloor E = \prec_{\lhd\rfloor E}$ for every coherent choice relation $\lhd$ on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$, and $\prec_C\rfloor E = \prec_{C\rfloor E}$ for every coherent choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$. Likewise, consequently also $C_D\rfloor E = C_{D\rfloor E}$ and $\lhd_D\rfloor E = \lhd_{D\rfloor E}$ for every coherent set of desirable gambles $D\subseteq\mathcal{L}(\mathcal{X}\times\mathcal{R})$, and $\lhd_{\prec}\rfloor E = \lhd_{\prec\rfloor E}$ for every coherent preference relation $\prec$ on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$.

The interplay between choice models and desirability leads to a similar result as in Corollary 59_{67}:

Corollary 154. *Consider any coherent set of desirable gambles $D\subseteq\mathcal{L}(\mathcal{X}\times\mathcal{R})$, any coherent choice function C on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$ and any conditioning event E in $\mathcal{P}_\varnothing(\mathcal{X})$. Then $D\rfloor E = D_{C_D\rfloor E}$ and $C\rfloor E \sqsupseteq C_{D_C\rfloor E}$.*

Proof. For the first statement, use Proposition $153_{\curvearrowleft}$ with $C \coloneqq C_D$, and Corollary 59_{67}.

For the second statement, use Proposition $153_{\curvearrowleft}$ with $D\coloneqq D_C$ to find that $C_{D_C\rfloor E} = C_{D_C}\rfloor E$, and infer that, using Corollary 59_{67} and Proposition 152_{210}, therefore indeed $C\rfloor E\sqsupseteq C_{D_C\rfloor E}$. □

We can obtain similar properties by substituting $\prec$ for D and $\lhd$ for C.

These statements are summarised in the commuting diagram of Figure 6.2, where we use the maps

$$\begin{aligned} D_\bullet &: \overline{\mathbf{C}}\to\overline{\mathbf{D}} : C\mapsto D_C, \\ C_\bullet &: \overline{\mathbf{D}}\to\overline{\mathbf{C}} : D\mapsto C_D, \end{aligned}$$

and $\bullet\rfloor E$, to denote conditioning a set of desirable gambles D or a choice function C.

6.3 INFIMA OF CHOICE MODELS

Let us now investigate how conditioning fits into the framework of different classes of choice functions, such as M-admissible and E-admissible choice functions. If we have a collection $\mathcal{C}$ of coherent choice functions on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$, by Proposition 40_{48} its infimum $\inf\mathcal{C}$ is again a coherent choice function on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$, and therefore so is $\inf\mathcal{C}\rfloor E$, for every E in $\mathcal{P}_\varnothing(\mathcal{X})$. But can

$$
\begin{tikzcd}
C \arrow[r, "\bullet\rfloor E"] \arrow[d, "D_\bullet"'] & C\rfloor E \arrow[d, "D_\bullet"'] & \sqsupseteq & C_{D_C\rfloor E} \\
D_C \arrow[r, "\bullet\rfloor E"] & D_C\rfloor E & = & D_{C\rfloor E} \arrow[u, dashed, "C_\bullet"']
\end{tikzcd}
$$

Figure 6.2: Commuting diagram for conditioning with choice models and desirability: Start with a coherent choice function C. Conditioning it on E directly leads to a more informative model than going to the corresponding set of desirable gambles, and perform the conditioning there, before going back to the corresponding choice function.

we retrieve $\inf\mathcal{C}\rfloor E$ also by conditioning every choice function in $\mathcal{C}$? By the next proposition, it turns out that we can: we obtain a similar result as in the discussion in Section 2.6_{46}.

Proposition 155. *Consider any collection $\mathcal{C}$ of coherent choice functions on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$, and any conditioning event E in $\mathcal{P}_\varnothing(\mathcal{X})$. Then* $\inf\mathcal{C}\rfloor E = \inf\{C\rfloor E : C\in\mathcal{C}\}$.

Proof. Consider any A in $\mathcal{Q}(\mathcal{L}(E\times\mathcal{R}))$ and any f in A. By Definition 44_{208}, we see that $f\in\inf\mathcal{C}\rfloor E(A)\Leftrightarrow \mathbb{I}_E f\in\inf\mathcal{C}(\mathbb{I}_E A) = \bigcup_{C\in\mathcal{C}} C(\mathbb{I}_E A)$, or, equivalently, $\mathbb{I}_E f\in C(\mathbb{I}_E A)$ for some C in $\mathcal{C}$. By Definition 44_{208}, we also have the following equivalences

$$
\begin{aligned}
(\exists C\in\mathcal{C})\mathbb{I}_E f\in C(\mathbb{I}_E A) &\Leftrightarrow (\exists C\in\mathcal{C}) f\in C\rfloor E(A)\\
&\Leftrightarrow f\in\bigcup_{C\in\mathcal{C}} C\rfloor E(A) = (\inf\{C\rfloor E : C\in\mathcal{C}\})(A),
\end{aligned}
$$

so $f\in\inf\mathcal{C}\rfloor E(A)\Leftrightarrow f\in(\inf\{C\rfloor E : C\in\mathcal{C}\})(A)$, and therefore indeed $\inf\mathcal{C}\rfloor E = \inf\{C\rfloor E : C\in\mathcal{C}\}$. □

Therefore, in particular, for the collection $\mathcal{C}_{\mathcal{D}} \coloneqq \{C_D : D\in\mathcal{D}\}$ of coherent choice functions based on a collection $\mathcal{D}\subseteq\overline{\mathbf{D}}$ of coherent sets of desirable (vector-valued) gambles, we have $\inf\mathcal{C}_{\mathcal{D}}\rfloor E = \inf\{C_D\rfloor E : D\in\mathcal{D}\}$ for every E in $\mathcal{P}_\varnothing(\mathcal{X})$. But due to Proposition 153_{211}, $\inf\mathcal{C}_{\mathcal{D}}\rfloor E$ is now easy to find: it is equal to $\inf\{C_{D\rfloor E} : D\in\mathcal{D}\}$, so it suffices to condition every set of desirable gambles in $\mathcal{D}$.

Let us take this one step further, and investigate an example of conditioning choice functions based on $\{D_p : p\in\mathcal{M}\}$, where $\mathcal{M}$ is an arbitrary subset of $\Sigma_{\mathcal{X}}$, where $\mathcal{X}$ is now assumed to be finite. Remember from Section 2.10_{81} that we call such choice functions $\inf\{C_{D_p} : p\in\mathcal{M}\}$ *E-admissible* with respect to $\mathcal{M}$, and that we denote them by $C^{\mathrm{E}}_{\mathcal{M}}$, or equivalently, by $C^{\mathrm{E}}_{\mathcal{K}}$ when we use, instead of $\mathcal{M}$, the corresponding set $\mathcal{K}$ of linear previsions on $\mathcal{X}$, introduced in Section 2.8_{55}.

Example 23. Consider a set $\mathcal{M}$ of mass functions on $\mathcal{X}$ and a set $\mathcal{U}$ of utilities on $\mathcal{R}$—real-valued maps on $\mathcal{R}$ that attach a specific utility to every r in $\mathcal{R}$. With each mass function p in $\mathcal{M}$ and with each utility u in $\mathcal{U}$, we let correspond an expectation operator—or linear prevision—$E_{p,u}$ on $\mathcal{L}(\mathcal{X}\times\mathcal{R})$:

$$E_{p,u}(f) \coloneqq \sum_{x\in\mathcal{X}}\sum_{r\in\mathcal{R}} p(x)u(r)f(x,r) \text{ for all } f \text{ in } \mathcal{L}(\mathcal{X}\times\mathcal{R}).$$

We collect these expectation operators in the set $\mathcal{K} \coloneqq \{E_{p,u} : p\in\mathcal{M}, u\in\mathcal{U}\}$, and we can associate with $\mathcal{K}$ an E-admissible choice function (see Section 2.10_{81}), in the usual way:

$$C^{\mathrm{E}}_{\mathcal{K}}(A) \coloneqq \bigcup_{E\in\mathcal{K}} C^{\mathrm{E}}_{\{E\}}(A) = \bigcup_{E\in\mathcal{K}} \{f\in A : (\forall g\in A)(E(g)\leq E(f) \text{ and } f \nless g)\} \quad (6.4)$$

for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}\times\mathcal{R}))$.

Consider any event B in $\mathcal{P}_{\emptyset}(\mathcal{X})$ and let us find the conditioned E-admissible choice function $C^{\mathrm{E}}_{\mathcal{K}}\rfloor B$. Consider any option set A in $\mathcal{Q}(\mathcal{L}(B\times\mathcal{R}))$ and any f in A. Infer using Definition 44_{208} that

$$f\in C^{\mathrm{E}}_{\mathcal{K}}\rfloor B(A) \Leftrightarrow \mathbb{I}_B f\in C^{\mathrm{E}}_{\mathcal{K}}(\mathbb{I}_B A) = \bigcup_{E\in\mathcal{K}} C^{\mathrm{E}}_{\{E\}}(\mathbb{I}_B A),$$

and, using Equation (6.4), also that

$$f\in C^{\mathrm{E}}_{\mathcal{K}}\rfloor B(A) \Leftrightarrow (\exists E\in\mathcal{K})(\forall g\in A)\big(E(\mathbb{I}_B g)\leq E(\mathbb{I}_B f) \text{ and } f\nless g\big).$$

We distinguish between two possibilities: (i) $E(\mathbb{I}_{B\times\mathcal{R}})>0$ for all E in $\mathcal{K}$, or (ii) $E(\mathbb{I}_{B\times\mathcal{R}})=0$ for some E in $\mathcal{K}$. If (i) $(\forall E\in\mathcal{K})E(\mathbb{I}_{B\times\mathcal{R}})>0$ then, since Bayes's Rule tells us that[3] $E(h\rfloor B)=\frac{E(\mathbb{I}_B h)}{E(\mathbb{I}_{B\times\mathcal{R}})}$ for every gamble h on $B\times\mathcal{R}$ and any E in $\mathcal{K}$,

$$f\in C^{\mathrm{E}}_{\mathcal{K}}\rfloor B(A) \Leftrightarrow (\exists E\in\mathcal{K})(\forall g\in A)\big(E(g\rfloor B)\leq E(f\rfloor B) \text{ and } f\nless g\big).$$

This means that we obtain the same result as if we were to condition every element of $\mathcal{K}$ directly by Bayes's Rule. To simplify the notation, we introduce $\mathcal{K}\rfloor B$ as a short-hand notation for $\{E(\bullet\rfloor B) : E\in\mathcal{K}\}$. Then

$$f\in C^{\mathrm{E}}_{\mathcal{K}}\rfloor B(A) \Leftrightarrow (\exists E\in\mathcal{K}\rfloor B)(\forall g\in A)\big(E(g)\leq E(f) \text{ and } f\nless g\big) \Leftrightarrow f\in C^{\mathrm{E}}_{\mathcal{K}\rfloor B}(A),$$

[3] For the expectation operator $E_{p,u}$ that corresponds to the mass function p and the utility u, we let its conditional variant $E_{p,u}(\bullet\rfloor B)$ be the one that corresponds to the conditional mass function $p(\bullet\rfloor B)$ and the utility u. By Bayes's Rule—the conditioning rule for precise probabilities—this conditional mass function $p(\bullet\rfloor B)$ is given by $\frac{p(x)}{\sum_{y\in B}p(y)}$ for every x in B. Since $\sum_{y\in B}p(y)=E_{p,u}(\mathbb{I}_{B\times\mathcal{R}})$ therefore $E_{p,u}(f\rfloor B)=\frac{1}{E_{p,u}(\mathbb{I}_{B\times\mathcal{R}})}\sum_{x\in B}\sum_{r\in\mathcal{R}}p(x)u(r)f(x,r)=\frac{1}{E_{p,u}(\mathbb{I}_{B\times\mathcal{R}})}\sum_{x\in\mathcal{X}}\sum_{r\in\mathcal{R}}p(x)u(r)(\mathbb{I}_B f)(x,r)=\frac{E_{p,u}(\mathbb{I}_B f)}{E_{p,u}(\mathbb{I}_{B\times\mathcal{R}})}$ for every f in $\mathcal{L}(B\times\mathcal{R})$.

so $C^{\mathrm{E}}_{\mathcal{K}}\rfloor B = C^{\mathrm{E}}_{\mathcal{K}\rfloor B}$ if $E(\mathbb{I}_{B\times\mathcal{R}}) > 0$ for all E in $\mathcal{K}$.

If (ii) $(\exists E \in \mathcal{K})E(\mathbb{I}_{B\times\mathcal{R}}) = 0$ then, for this E, by linearity we find $E(\mathbb{I}_B h) = E\big(\sum_{x\in B}\sum_{r\in\mathcal{R}} h(x,r)\mathbb{I}_{\{x\}\times\mathcal{R}}\big) = \sum_{x\in B}\sum_{r\in\mathcal{R}} h(x,r)E\big(\mathbb{I}_{\{x\}\times\mathcal{R}}\big) = 0$ for every gamble h on B, since $0 = \min \mathbb{I}_{\{x\}\times\mathcal{R}} \le E\big(\mathbb{I}_{\{x\}\times\mathcal{R}}\big) \le E(\mathbb{I}_{B\times\mathcal{R}}) = 0$ for every x in B, and therefore

$$f \in C^{\mathrm{E}}_{\mathcal{K}}\rfloor B(A) \Leftrightarrow (\forall g \in A) f \nless g \Leftrightarrow f \in \max A \Leftrightarrow f \in C_{\mathrm{v}}(A).$$

Therefore $C^{\mathrm{E}}_{\mathcal{K}}\rfloor B = C_{\mathrm{v}}$ is the vacuous choice function on $\mathcal{L}(B\times\mathcal{R})$. Translating this back to sets of linear previsions, since we have seen in Example 11_{86} that $C_{\mathrm{v}} \sqsubset C^{\mathrm{E}}_{\mathbb{P}_B}$, with $\mathbb{P}_B := \{E_{p,u} : p \in \Sigma_B, u \in \mathcal{U}\}$ the set of all expectation operators corresponding with $\mathcal{U}$, and therefore $C^{\mathrm{E}}_{\mathcal{K}}\rfloor B \sqsubset C^{\mathrm{E}}_{\mathbb{P}_B}$, this means that even considering the set of all linear previsions on B is not uninformative enough to describe $\mathcal{K}\rfloor B$. But $\mathbb{P}_B$ is the unique least informative set of linear previsions, so the best we can do, is let $\mathcal{K}\rfloor B$ correspond with $\mathbb{P}_B$.

In conclusion, we retrieve *natural extension* [8, 18, 82] as a conditioning rule:

$$\mathcal{K}\rfloor B = \begin{cases} \{E(\bullet\rfloor B) : E \in \mathcal{K}\} & \text{if } E(\mathbb{I}_{B\times\mathcal{R}}) > 0 \text{ for all } E \text{ in } \mathcal{K},\\ \mathbb{P}_B & \text{otherwise,}\end{cases}$$

and we find that $C^{\mathrm{E}}_{\mathcal{K}}\rfloor B = C^{\mathrm{E}}_{\mathcal{K}\rfloor B}$ if $E(\mathbb{I}_{B\times\mathcal{R}}) > 0$ for all E in $\mathcal{K}$, and $C^{\mathrm{E}}_{\mathcal{K}}\rfloor B = C_{\mathrm{v}} \sqsubset C^{\mathrm{E}}_{\mathcal{K}\rfloor B} = C^{\mathrm{E}}_{\mathbb{P}_{\mathcal{X}}}$ if $E(\mathbb{I}_{B\times\mathcal{R}}) = 0$ for some E in $\mathcal{K}$. If $\mathcal{K} = \{E\}$ with $E(\mathbb{I}_{B\times\mathcal{R}}) > 0$, then we recover Bayes's Rule as a special case of our conditioning rule for choice functions. ◊

6.4 THE COIN EXAMPLE REVISITED

We revisit the "coin example" (Example 10_{85}) to show that allowing for non-archimedeanity is important in a conditioning context. As in Example 10_{85}, we will work with real-valued gambles $\mathcal{L}(\mathcal{X})$, which can be identified with vector-valued gambles $\mathcal{L}(\mathcal{X}\times\mathcal{R})$ with $|\mathcal{R}| = 1$.

Let us consider again the situation of Example 10_{85} where we have a coin with two identical sides of unknown type: either both sides are heads (H), or both sides are tails (T). In order to be able to obtain non-trivial conditional uncertainty models, we need to extend the binary possibility space $\{\mathrm{H},\mathrm{T}\}$ to some ternary set, at least. We therefore consider the possibility that the coin lands on its side, indicated by S, yielding the possibility space $\mathcal{X} = \{\mathrm{H},\mathrm{T},\mathrm{S}\}$. We consider two events to condition on: $\{\mathrm{H},\mathrm{T}\}$, and $\{\mathrm{H},\mathrm{S}\}$.

As we explained in some detail in Example 10_{85}, desirability is not very well suited for modelling this situation. Instead, we will use the more powerful choice function languages.

As a first attempt, we consider E-admissible choice functions, based on the set $\mathcal{K} := \{E_{\mathrm{H}}, E_{\mathrm{T}}\}$ consisting of the two degenerate linear previsions $E_{\mathrm{H}}(f) =$

$f(\mathrm{H})$ and $E_{\mathrm{T}}(f) = f(\mathrm{T})$ for every gamble f on $\mathcal{X}$, expressing certainty about H and T, respectively. Letting $C_1 := C^{\mathrm{E}}_{\mathcal{K}}$ we find that for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}))$:

$$
\begin{aligned}
0 \in R_1(A) &\Leftrightarrow 0 \in R^{\mathrm{E}}_{\{E_{\mathrm{H}}\}}(A) \cap R^{\mathrm{E}}_{\{E_{\mathrm{T}}\}}(A) \\
&\Leftrightarrow A \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } \big((\exists f_1, f_2 \in A) 0 < f_1(\mathrm{H}) \text{ and } 0 < f_2(\mathrm{T})\big).
\end{aligned}
$$

Conditioning an E-admissible choice function on an event E for which $P(\mathbb{I}_E) > 0$ for all P in $\mathcal{K}$, corresponds by Example 23_{214} to conditioning the set of linear previsions it is derived from. To find $C_1 \rfloor E$, with E one of our two envisioned events $\{\mathrm{H},\mathrm{T}\}$ and $\{\mathrm{H},\mathrm{S}\}$, all we have to do is find $\mathcal{K}\rfloor E$. For $\mathcal{K}\rfloor\{\mathrm{H},\mathrm{T}\}$, both $E_{\mathrm{H}}(\{\mathrm{H},\mathrm{T}\}) = \mathbb{I}_{\{\mathrm{H},\mathrm{T}\}}(\mathrm{H}) = 1 > 0$ and $E_{\mathrm{T}}(\{\mathrm{H},\mathrm{T}\}) = 1 > 0$ are positive, and therefore $\mathcal{K}\rfloor\{\mathrm{H},\mathrm{T}\} = \{P(\bullet\rfloor\{\mathrm{H},\mathrm{T}\}) : P \in \mathcal{K}\}$. But $E_{\mathrm{H}}(f\rfloor\{\mathrm{H},\mathrm{T}\}) = f(\mathrm{H})$ and $E_{\mathrm{T}}(f\rfloor\{\mathrm{H},\mathrm{T}\}) = f(\mathrm{T})$ for all gambles f on $\{\mathrm{H},\mathrm{T}\}$, so we find that

$$
0 \in R_1\rfloor\{\mathrm{H},\mathrm{T}\}(A) \Leftrightarrow A \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } \big((\exists f_1, f_2 \in A) 0 < f_1(\mathrm{H}) \text{ and } 0 < f_2(\mathrm{T})\big)
$$

for all A in $\mathcal{Q}(\mathcal{L}(\{\mathrm{H},\mathrm{T}\}))$, corresponding to the E-admissible model for the original coin example (Example 10_{85}). On the other hand, for $\mathcal{K}\rfloor\{\mathrm{H},\mathrm{S}\}$, remark that $E_{\mathrm{T}}(\{\mathrm{H},\mathrm{S}\}) = \mathbb{I}_{\{\mathrm{H},\mathrm{S}\}}(\mathrm{T}) = 0$. But then, by Example 23_{214}, $C_1\rfloor\{\mathrm{H},\mathrm{S}\}$ is the *vacuous* choice function on $\mathcal{L}(\{\mathrm{H},\mathrm{S}\})$. So E-admissible choice functions seem too restrictive to model this situation: conditioning on $\{\mathrm{H},\mathrm{S}\}$ results in a vacuous model, even though the conditioning event does not contradict the assumptions of the example. The following table summarises what C_1 conditioned on $\{\mathrm{H},\mathrm{T}\}$ and $\{\mathrm{H},\mathrm{S}\}$ expresses:

	$\rfloor\{\mathrm{H},\mathrm{T}\}$	$\rfloor\{\mathrm{H},\mathrm{S}\}$
C_1	certainty about H or T	vacuous

E-admissible choice functions—based on linear previsions—therefore seem to be not ideally suited for this example, but we now show that M-admissible choice functions—based on maximal sets of desirable gambles—behave more appropriately. The coherent sets of desirable gambles $\mathcal{L}_{>0} \cup \{f \in \mathcal{L}(\mathcal{X}) : f(\mathrm{H}) > 0\}$ and $\mathcal{L}_{>0} \cup \{f \in \mathcal{L}(\mathcal{X}) : f(\mathrm{T}) > 0\}$ correspond to E_{H} and E_{T} respectively—and therefore express certainty about H or about T—in the sense that they are the least-committal sets of desirable gambles that induce E_{H} and E_{T}, respectively.[4] But to get to an M-admissible choice function, we need to consider two coherent *maximal* sets of desirable gambles $\hat{D}_{\mathrm{H}} \supseteq \mathcal{L}_{>0} \cup \{f \in \mathcal{L}(\mathcal{X}) : f(\mathrm{H}) > 0\}$ and $\hat{D}_{\mathrm{T}} \supseteq \mathcal{L}_{>0} \cup \{f \in \mathcal{L}(\mathcal{X}) : f(\mathrm{T}) > 0\}$, expressing maximal beliefs that induce the linear previsions E_{H} and E_{T} respec-

[4] In Section 2.8_{55} we explain in some detail the connection between desirability and probability.

tively. In this example, we consider for instance

$$\hat{D}_{\mathrm{H}} \coloneqq \mathcal{L}_{>0} \cup \{f \in \mathcal{L}(\mathcal{X}) : f(\mathrm{H}) > 0 \text{ or } f(\mathrm{H}) = 0 < f(\mathrm{T})\}, \tag{6.5}$$

$$\hat{D}_{\mathrm{T}} \coloneqq \mathcal{L}_{>0} \cup \{f \in \mathcal{L}(\mathcal{X}) : f(\mathrm{T}) > 0 \text{ or } f(\mathrm{T}) = 0 < f(\mathrm{H})\}. \tag{6.6}$$

Actually, $\hat{D}_{\mathrm{H}}$ and $\hat{D}_{\mathrm{T}}$ are *lexicographic* sets of desirable gambles. $\hat{D}_{\mathrm{H}}$ corresponds to the lexicographic probability system $(p_{\mathrm{H}}, p_{\mathrm{T}}, p)$ of the two degenerate probability mass functions $p_{\mathrm{H}} = \mathbb{I}_{\{\mathrm{H}\}}$ and $p_{\mathrm{T}} = \mathbb{I}_{\{\mathrm{T}\}}$ and the arbitrary probability mass function p for which $p(\mathrm{S}) > 0$—for instance $p = p_{\mathrm{S}} \coloneqq \mathbb{I}_{\{\mathrm{S}\}}$ is a good choice for p—, and $\hat{D}_{\mathrm{T}}$ to $(p_{\mathrm{T}}, p_{\mathrm{H}}, p)$.

Lemma 156. *The sets of desirable gambles $\hat{D}_{\mathrm{H}}$ and $\hat{D}_{\mathrm{T}}$ as defined in Equations* (6.5) *and* (6.6) *are coherent and maximal.*

Proof. By Proposition 103_{139}, it suffices to show that $\ker E_{\mathrm{H}} \cap \ker E_{\mathrm{T}} \cap \ker P = \{0\}$, where P is the linear prevision associated with p: $P(f) = p(\mathrm{H})f(\mathrm{H}) + p(\mathrm{T})f(\mathrm{T}) + p(\mathrm{S})f(\mathrm{S})$ for every gamble f. Since $\ker E_{\mathrm{H}} = \ker \mathbb{I}_{\{\mathrm{H}\}} = \{f \in \mathcal{L} : f(\mathrm{H}) = 0\}$ and, similarly $\ker E_{\mathrm{T}} = \{f \in \mathcal{L} : f(\mathrm{T}) = 0\}$, we have already that $\ker E_{\mathrm{H}} \cap \ker E_{\mathrm{T}} = \{f \in \mathcal{L} : f(\mathrm{H}) = f(\mathrm{T}) = 0\}$. Note that therefore $\ker E_{\mathrm{H}} \cap \ker E_{\mathrm{T}} \cap \ker P \neq \{0\}$ if and only if there is some gamble f such that $f(\mathrm{H}) = f(\mathrm{T}) = 0 \neq f(\mathrm{S})$ in $\ker P$, which contradicts that $p(\mathrm{S}) > 0$. Therefore indeed $\ker E_{\mathrm{H}} \cap \ker E_{\mathrm{T}} \cap \ker P = \{0\}$. □

The M-admissible choice function C_2 that models this situation is then given by $C_2 \coloneqq \inf\{C_{\hat{D}_{\mathrm{H}}}, C_{\hat{D}_{\mathrm{T}}}\}$. To find $C_2\rfloor E$, with E one of our two envisioned events $\{\mathrm{H},\mathrm{T}\}$ and $\{\mathrm{H},\mathrm{S}\}$, thanks to Propositions 153_{211} and 155_{213}, all we have to do is find $\hat{D}_{\mathrm{H}}\rfloor E$ and $\hat{D}_{\mathrm{T}}\rfloor E$. So let us calculate $\hat{D}_{\mathrm{H}}\rfloor E$ and $\hat{D}_{\mathrm{T}}\rfloor E$ for the first envisioned conditioning event $E = \{\mathrm{H},\mathrm{T}\}$. By definition, $\hat{D}_{\mathrm{H}}\rfloor\{\mathrm{H},\mathrm{T}\} = \{f \in \mathcal{L}(\{\mathrm{H},\mathrm{T}\}) : \mathbb{I}_{\{\mathrm{H},\mathrm{T}\}} f \in \hat{D}_{\mathrm{H}}\}$. Remark that $\mathbb{I}_{\{\mathrm{H},\mathrm{T}\}} f$ evaluated in S is 0, so $\mathbb{I}_{\{\mathrm{H},\mathrm{T}\}} f \in \hat{D}_{\mathrm{H}} \Leftrightarrow (f \in \mathcal{L}_{>0}$ or $f(\mathrm{H}) > 0$ or $f(\mathrm{H}) = 0 < f(\mathrm{T}))$. If $f(\mathrm{H}) = 0 < f(\mathrm{T})$ then $f \in \mathcal{L}_{>0}$, and therefore $\hat{D}_{\mathrm{H}}\rfloor\{\mathrm{H},\mathrm{T}\} = \mathcal{L}_{>0} \cup \{f \in \mathcal{L}(\{\mathrm{H},\mathrm{T}\}) : f(\mathrm{H}) > 0\}$, so conditioning $\hat{D}_{\mathrm{H}}$ on $\{\mathrm{H},\mathrm{T}\}$ leads to a maximal set of desirable gambles that expresses only commitment towards H: any gamble that has a positive value in H is considered desirable. Similarly, $\hat{D}_{\mathrm{T}}\rfloor\{\mathrm{H},\mathrm{T}\} = \mathcal{L}_{>0} \cup \{f \in \mathcal{L}(\{\mathrm{H},\mathrm{T}\}) : f(\mathrm{T}) > 0\}$. But on the binary possibility space $\{\mathrm{H},\mathrm{T}\}$ the only coherent sets of desirable gambles that induce the degenerate linear previsions E_{H} and E_{T} respectively, are $\mathcal{L}_{>0} \cup \{f \in \mathcal{L} : f(\mathrm{H}) > 0\}$ and $\mathcal{L}_{>0} \cup \{f \in \mathcal{L} : f(\mathrm{T}) > 0\}$, so $C_2\rfloor\{\mathrm{H},\mathrm{T}\} = C_1\rfloor\{\mathrm{H},\mathrm{T}\}$, and behaves therefore also as expected under conditioning on $\{\mathrm{H},\mathrm{T}\}$.

To condition C_2 on the other envisioned conditioning event $\{\mathrm{H},\mathrm{S}\}$—recall from the discussion above that conditioning C_1 on $\{\mathrm{H},\mathrm{S}\}$ leads to the vacuous choice function $C_1\rfloor\{\mathrm{H},\mathrm{S}\}$, which may be considered too uninformative—it suffices again to find $\hat{D}_{\mathrm{H}}\rfloor\{\mathrm{H},\mathrm{S}\}$ and $\hat{D}_{\mathrm{T}}\rfloor\{\mathrm{H},\mathrm{S}\}$. By definition, $\hat{D}_{\mathrm{H}}\rfloor\{\mathrm{H},\mathrm{S}\} = \{f \in \mathcal{L}(\{\mathrm{H},\mathrm{S}\}) : \mathbb{I}_{\{\mathrm{H},\mathrm{S}\}} f \in \hat{D}_{\mathrm{H}}\}$. The gamble $\mathbb{I}_{\{\mathrm{H},\mathrm{S}\}} f$ evaluated in T is 0, so $\mathbb{I}_{\{\mathrm{H},\mathrm{S}\}} f \in \hat{D}_{\mathrm{H}} \Leftrightarrow (f \in \mathcal{L}_{>0}$ or $f(\mathrm{H}) > 0$ or $f(\mathrm{H}) = 0 < f(\mathrm{T}))$. Therefore $\hat{D}_{\mathrm{H}}\rfloor\{\mathrm{H},\mathrm{S}\} = \mathcal{L}_{>0} \cup \{f \in \mathcal{L}(\{\mathrm{H},\mathrm{S}\}) : f(\mathrm{H}) > 0\}$ is very similar to $\hat{D}_{\mathrm{H}}\rfloor\{\mathrm{H},\mathrm{T}\}$: it is again a maximal set of desirable gambles that expresses only commitment towards H. For $\hat{D}_{\mathrm{T}}$, by definition $\hat{D}_{\mathrm{T}}\rfloor\{\mathrm{H},\mathrm{S}\} = \{f \in \mathcal{L}(\{\mathrm{H},\mathrm{S}\}) : \mathbb{I}_{\{\mathrm{H},\mathrm{S}\}} f \in \hat{D}_{\mathrm{T}}\}$. Note that $\mathbb{I}_{\{\mathrm{H},\mathrm{S}\}} f$ evaluated in T is 0, and therefore $\mathbb{I}_{\{\mathrm{H},\mathrm{S}\}} f \in \hat{D}_{\mathrm{T}} \Leftrightarrow (f \in \mathcal{L}_{>0}$ or $f(\mathrm{H}) > 0)$. Then $\hat{D}_{\mathrm{T}}\rfloor\{\mathrm{H},\mathrm{S}\} = \hat{D}_{\mathrm{H}}\rfloor\{\mathrm{H},\mathrm{S}\}$ again we obtain a maximal set of desirable gambles that expresses only commitment towards H. This is as expected because we know the lexicographic probability system $(p_{\mathrm{T}}, p_{\mathrm{H}}, p)$ it corresponds to: if the first layer—corresponding to p_{T}—has probability zero, the next thing to do is consider the second layer—corresponding to p_{H}. So $C_2\rfloor\{\mathrm{H},\mathrm{S}\}$ is the infimum of two equal choice functions $C_{\hat{D}_{\mathrm{H}}\rfloor\{\mathrm{H},\mathrm{S}\}}$ and $C_{\hat{D}_{\mathrm{T}}\rfloor\{\mathrm{H},\mathrm{S}\}}$. It is given by

$$C_2\rfloor\{\mathrm{H},\mathrm{S}\}(A) = \arg\max\{f(\mathrm{H}) : f \in A\} \cap C_{\mathrm{v}}(A) \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\{\mathrm{H},\mathrm{S}\})),$$

so it is a maximal choice function. Compare this with the, in our view unnecessarily, vacuous $C_1\rfloor\{\mathrm{H},\mathrm{S}\}$.

The following table summarises the behaviour C_2 conditioned on $\{\mathrm{H},\mathrm{T}\}$ and $\{\mathrm{H},\mathrm{S}\}$, and compares it with that of C_1.

	$\rfloor\{\mathrm{H},\mathrm{T}\}$	$\rfloor\{\mathrm{H},\mathrm{S}\}$
C_1	certainty about H or T	vacuous
C_2	certainty about H or T	certainty about H

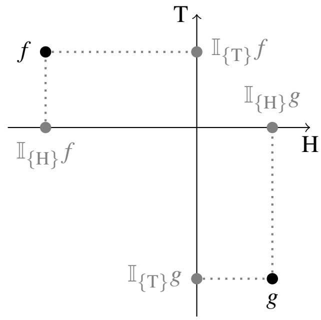
In analogy with the comparable situation for desirability, the non-Archimedean nature of the maximal choice functions here allows us to preserve non-trivial conditional preferences, and therefore represents non-vacuous conditional information. This is not the case for the Archimedean E-admissible choice function C_1.

6.5 CONCLUSION

We have defined how to condition choice models on an event. This is done using the same simple underlying ideas as conditioning sets of desirable gambles. We have established the useful result that the infimum of a collection $\mathcal{C}$ of coherent choice functions can be conditioned using the coherent choice functions in $\mathcal{C}$ directly. This implies that the infimum of a collection $\mathcal{C}_{\mathcal{D}}$ of coherent purely binary choice functions can be conditioned using the coherent sets of desirable vector-valued gambles in $\mathcal{D}$ directly. As we have shown in Example 23_{214}, this also shows that when using conditioning for choice models we retrieve the natural extension as a conditioning rule for sets of linear previsions.

The connection between conditioning for choice models and for desirability models is established in Proposition 153_{211}: it guarantees that a purely binary choice function can be conditioned using its defining set of desirable vector-valued gambles. Coherent sets of desirable gambles—or coherent preference relations for that matter—on $\mathcal{L}(\mathcal{X} \times \mathcal{R})$ have the property that, for any partition $\mathcal{T}$ of $\mathcal{X}$, if $f \in D\rfloor E$ for every E in $\mathcal{T}$, then also $f \in D$ unconditionally—this is a direct consequence of the fact that $f = \sum_{E\in\mathcal{T}} \mathbb{I}_E f$ and Axiom D4_{57}. In other words, if f is preferred to 0 conditionally on every E in $\mathcal{T}$, then f is preferred to 0 unconditionally. For choice models, however, one straight-forward generalisation of this—$((\forall E \in \mathcal{T})\{0\} \lhd\rfloor E\, A) \Rightarrow \{0\} \lhd A$—does not generally hold.

To see why, let $\mathcal{X} \coloneqq \{\mathrm{H},\mathrm{T}\}$ and $\mathcal{T} \coloneqq \{\{\mathrm{H}\},\{\mathrm{T}\}\}$ a partition of $\mathcal{X}$, and consider the following figure.



Collect the gambles f and g in A. Note that $\mathbb{I}_{\{\mathrm{T}\}}f > 0$, so $\{0\} \lhd\rfloor\{\mathrm{T}\}\, A$, and similarly, $\mathbb{I}_{\{\mathrm{H}\}}g > 0$, so $\{0\} \lhd\rfloor\{\mathrm{H}\}A$ for any coherent choice relation, while not necessarily $\{0\} \lhd A$: consider for instance the E-admissible choice function $C^{\mathrm{E}}_{\{E\}}$ with uniform prevision $E(h) \coloneqq \frac{1}{2}(h(\mathrm{H}) + h(\mathrm{T}))$. Imposing the additional axioms considered in Reference [67] does not help: indeed, $C^{\mathrm{E}}_{\{E\}} = C_{D_E}$ and D_E is a lexicographic set of desirable gambles, so by Proposition 91_{127}, $C^{\mathrm{E}}_{\{E\}}$ satisfies Property C5_{25}, and, moreover, it is an E-admissible choice function, so it is Archimedean as well (see Reference [45, Lemma 6]).

7

MULTIVARIATE CHOICE FUNCTIONS

In this chapter, we will generalise the concepts of marginalisation, weak (cylindrical) extension and irrelevant natural extension introduced by De Cooman and Miranda from sets of desirable gambles [29] to choice models. To avoid notational difficulties and to streamline the argument, we will work with vector-valued gambles. We will build on the previous chapters, where we have shown how to work with choice models on arbitrary vector spaces, and demonstrated how choice models on vector-valued gambles are a particular case of them.

We will provide the linear space of vector-valued gambles, on which we define our choice models, with a more complex structure: we will consider the vector space of all gambles whose state part[1] of its domain is a *Cartesian product* of a finite number of finite possibility spaces. More specifically, consider n in $\mathbb{N}$ variables $X_1, \dots, X_n$ that assume values in the *finite* possibility spaces $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. Belief models about these variables $X_1, \dots, X_n$—be they choice models or desirability models—will work with the vector-valued gambles on $\mathcal{X}_1, \dots, \mathcal{X}_n$: with elements of $\mathcal{L}(\mathcal{X}_1 \times \mathcal{R}), \dots, \mathcal{L}(\mathcal{X}_n \times \mathcal{R})$. The vector spaces $\mathcal{L}(\mathcal{X}_1 \times \mathcal{R}), \dots, \mathcal{L}(\mathcal{X}_n \times \mathcal{R})$ are ordered by the standard pointwise vector ordering $\leq$: for any k in $\{1, \dots, n\}$, and any f and g in $\mathcal{L}(\mathcal{X}_k \times \mathcal{R})$, we have that

$$f \leq g \Leftrightarrow (\forall x_k \in \mathcal{X}_k, r \in \mathcal{R}) f(x_k, r) \leq g(x_k, r) \Leftrightarrow (\forall x_k \in \mathcal{X}_k) f(x_k, \bullet) \leq g(x_k, \bullet).$$

We can define gambles also on the Cartesian product $(\times_{k=1}^{n} \mathcal{X}_k) \times \mathcal{R}$, giving rise

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[1] In the definition of vector-valued gambles, Definition 9_{28}, we have defined the state part of the domain as $\mathcal{X}$, and the reward part as $\mathcal{R}$. In this chapter, we assume that the reward pard $\mathcal{R}$ is fixed throughout.

to the particular $(\prod_{k=1}^{n}|\mathcal{X}_k|)|\mathcal{R}|$-dimensional linear space $\mathcal{L}((\times_{k=1}^{n}\mathcal{X}_k)\times\mathcal{R})$ of gambles on $(\times_{k=1}^{n}\mathcal{X}_k)\times\mathcal{R}$—or vector-valued gambles on $\times_{k=1}^{n}\mathcal{X}_k$.

As this chapter builds heavily on results of De Cooman and Miranda [29] for sets of desirable gambles, we follow the notation established there. Furthermore, the flow of the arguments presented here for choice models is roughly the same as that for desirability in their paper.

7.1 BASIC NOTATION AND CYLINDRICAL EXTENSION

For every non-empty subset $I\subseteq\{1,\dots,n\}$ of indices, we let X_I be the tuple of variables that takes values in $\mathcal{X}_I\coloneqq\times_{r\in I}\mathcal{X}_r$. This Cartesian product is the set of all maps x_I from I to $\bigcup_{i\in I}\mathcal{X}_i$ such that $x_i\coloneqq x_I(i)\in\mathcal{X}_i$ for all i in I. We will denote generic elements of $\mathcal{X}_I$ as x_I or z_I, with corresponding components $x_i\coloneqq x_I(i)$ or $z_i\coloneqq z_I(i)$, for all i in I. The set $\mathcal{L}(\mathcal{X}_I\times\mathcal{R})$ of all vector-valued gambles on $\mathcal{X}_I$ is a linear space. When $I=\{1,\dots,n\}$, we will use as a shorthand notation $X_{1:n}\coloneqq X_{\{1,\dots,n\}}$, taking values in $\mathcal{X}_{1:n}\coloneqq\mathcal{X}_{\{1,\dots,n\}}$ and whose generic elements are denoted by $x_{1:n}\coloneqq x_{\{1,\dots,n\}}=(x_1,\dots,x_n)$.

We assume that the variables $X_1,\dots,X_n$ are *logically independent*, meaning that for each non-empty subset I or $\{1,\dots,n\}$, x_I may assume every value in $\mathcal{X}_I$.

It will be useful for any vector-valued gamble f on $\mathcal{X}_{1:n}$, any non-empty proper subset I of $\{1,\dots,n\}$ and any x_I in $\mathcal{X}_I$, to interpret the partial map[2] $f(x_I,\bullet,\bullet)$ as a vector-valued gamble on $\mathcal{X}_{I^c}$, whose value in every x_{I^c} of $\mathcal{X}_{I^c}$ is given by $f(x_I,x_{I^c},\bullet)$, where $I^c\coloneqq\{1,\dots,n\}\setminus I$. Likewise, for any set A of gambles on $\mathcal{X}_{1:n}$, we let $A(x_I,\bullet)\coloneqq\{f(x_I,\bullet):f\in A\}$ be a corresponding set of vector-valued gambles on $\mathcal{X}_{I^c}$.

For every non-empty subset I of $\{1,\dots,n\}$, the linear space $\mathcal{L}(\mathcal{X}_I\times\mathcal{R})$ is also ordered by a vector ordering $\leq$. Given any two gambles f and g on $\mathcal{X}_I$, we let $f\leq g\Leftrightarrow(\forall x_I\in\mathcal{X}_I,r\in\mathcal{R})f(x_I,r)\leq g(x_I,r)$. As a particular case, if $I=\{1,\dots,i\}$ for some $i\leq n$, since $\mathcal{X}_I=\times_{i\in I}\mathcal{X}_i$, we denote this equivalently also as $(\forall x_1\in\mathcal{X}_1,\dots,x_i\in\mathcal{X}_i,r\in\mathcal{R})f(x_1,\dots,x_i,r)\leq g(x_1,\dots,x_i,r)$. Similarly, we have that $f<g\Leftrightarrow((\forall x_I\in\mathcal{X}_I,r\in\mathcal{R})f(x_I,r)\leq g(x_I,r)$ and $f\neq g)$. For any $O\subseteq I$, this is equivalent to $((\forall x_O\in\mathcal{X}_O)f(x_O,\bullet)\leq g(x_O,\bullet)$ and $(\exists z_O\in\mathcal{X}_O)f(z_O,\bullet)<g(z_O,\bullet))$.

We will need a way to relate gambles on different domains:

Definition 46 (Cylindrical extension)**.** *Given two disjoint and non-empty subsets I and I' of $\{1,\dots,n\}$ and any vector-valued gamble f on $\mathcal{X}_I$, we let its* cylindrical extension *f^* to $\mathcal{X}_{I\cup I'}$ be defined by*

$$f^*(x_I,x_{I'},r)\coloneqq f(x_I,r)\textit{ for all }x_I\textit{ in }\mathcal{X}_I,\ x_{I'}\textit{ in }\mathcal{X}_{I'},\textit{ and }r\textit{ in }\mathcal{R}.$$

[2]The rightmost placeholder refers to the reward part. When no confusion can arise, we will treat the reward part in the same way as the state part, and write $f(x_I,\bullet)$ to mean a vector-valued gamble on $\mathcal{X}_{I^c}$, or, in other words, a gamble on $\mathcal{X}_{I^c}\times\mathcal{R}$.

Similarly, given any set of gambles $A \subseteq \mathcal{L}(\mathcal{X}_I \times \mathcal{R})$*, we let its* cylindrical extension $A^* \subseteq \mathcal{L}(\mathcal{X}_{I\cup I'} \times \mathcal{R})$ *be defined as* $A^* \coloneqq \{f^* : f \in A\}$.

Formally, f^* belongs to $\mathcal{L}(\mathcal{X}_{I\cup I'} \times \mathcal{R})$ while f belongs to $\mathcal{L}(\mathcal{X}_I \times \mathcal{R})$. However, f^* is completely determined by f and *vice versa*: they clearly only depend on the value of $X_I \times \mathcal{R}$, and as such, they contain the same information and correspond to the same transaction. They are therefore indistinguishable from a behavioural point of view.

Remark 7.1. As is done in References [20,29], we will frequently use the simplifying device of *identifying* a vector-valued gamble f on $\mathcal{X}_I$ with its cylindrical extension f^* on $\mathcal{X}_{I\cup I'}$, for any disjoint and non-empty subsets I and I' of the index set $\{1,\dots,n\}$. This convention allows us for instance to identify $\mathcal{L}(\mathcal{X}_I \times \mathcal{R})$ with a subset of $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$, and, as another example, for any set $A \subseteq \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$, to regard $A \cap \mathcal{L}(\mathcal{X}_I \times \mathcal{R})$ as those vector-valued gambles in A that depend on the value of X_I only. Therefore, for any event $E \subseteq \mathcal{X}_I$ we can identify the gamble $\mathbb{I}_E$ with $\mathbb{I}_{E\times\mathcal{X}_{\{1,\dots,n\}\setminus I}}$, and hence also the event E with $E \times \mathcal{X}_{\{1,\dots,n\}\setminus I}$. This device for instance also allows us to write, for any f on $\mathcal{X}_I \times \mathcal{R}$ and f' on $\mathcal{X}_{I\cup I'} \times \mathcal{R}$ with I and I' disjoint and non-empty subsets of the index set $\{1,\dots,n\}$, that $f \leq f' \Leftrightarrow (\forall x_I \in \mathcal{X}_I, x_{I'} \in \mathcal{X}_{I'}, r \in \mathcal{R}) f(x_I,r) \leq f'(x_I,x_{I'},r)$, and more generally, for any $A \subseteq \mathcal{L}(\mathcal{X}_I \times \mathcal{R})$ and $A' \subseteq \mathcal{L}(\mathcal{X}_{I\cup I'} \times \mathcal{R})$, that

$$A \lessdot A' \Leftrightarrow (\forall f \in A)(\exists f' \in A')(\forall x_I \in \mathcal{X}_I, x_{I'} \in \mathcal{X}_{I'}, r \in \mathcal{R}) f(x_I,r) \leq f'(x_I,x_{I'},r). \Diamond$$

7.2 MARGINALISATION AND WEAK EXTENSION

Suppose we have a choice function C on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ modelling a subject's beliefs about the variable $X_{1:n}$. What is the information that C contains about X_O, where O is some non-empty subset of the index set $\{1,\dots,n\}$? Finding this information can be done through marginalisation.

Definition 47 (Marginalisation)**.** *Given any non-empty subset O of* $\{1,\dots,n\}$ *and any choice function C on* $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$*, its* marginal choice function $\mathrm{marg}_O C$ *is determined by*

$$\mathrm{marg}_O C(A) \coloneqq C(A) \textit{ for all } A \textit{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})).$$

Without using our simplifying device (see Remark 7.1) of identifying gambles with their cylindrical extension, and instead explicitly denoting the cylindrical extension by an asterisk as in Definition 46, marginalisation looks as follows

$$\mathrm{marg}_O C(A) = \{f \in A : f^* \in C(A^*)\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})).$$

As it will be always clear on which vector-valued gambles the joint choice function C is defined, the use of our simplifying devices cannot lead to confusion, and hence, in the remainder of this chapter, we will work with the notation used in Definition 47.

It follows at once from the definition that marginalisation preserves the 'at most as informative as' relation:

$$C_1 \sqsubseteq C_2 \Rightarrow \mathrm{marg}_O C_1 \sqsubseteq \mathrm{marg}_O C_2$$

for all choice functions C_1 and C_2 on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$.

Marginalisation can be defined for rejection functions and choice relations as well. We let the marginal rejection function $\mathrm{marg}_O R$ of some rejection function R on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ be given by $\mathrm{marg}_O R(A) \coloneqq R(A)$ for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$. Similarly, we let the marginal choice relation $\lhd' = \mathrm{marg}_O \lhd$ of some choice relation $\lhd$ on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ be given by $A_1 \lhd' A_2 \Leftrightarrow A_1 \lhd A_2$ for all A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$. These definitions are essentially equivalent:

Proposition 157. *Consider any non-empty subset O of $\{1,\dots,n\}$, any corresponding choice function C, rejection function R and choice relation $\lhd$ on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$, and any corresponding choice function C', rejection function R' and choice relation $\lhd'$ on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. Then the following three statements are equivalent:*
(i) $C' = \mathrm{marg}_O C$;
(ii) $R' = \mathrm{marg}_O R$;
(iii) $\lhd' = \mathrm{marg}_O \lhd$.
As a consequence, $\mathrm{marg}_O C = C_{\mathrm{marg}_O R} = C_{\mathrm{marg}_O \lhd}$, $\mathrm{marg}_O R = R_{\mathrm{marg}_O C} = R_{\mathrm{marg}_O \lhd}$ *and* $\mathrm{marg}_O \lhd = \lhd_{\mathrm{marg}_O C} = \lhd_{\mathrm{marg}_O R}$.

Proof. For the first part—that (i), (ii) and (iii) are equivalent—we will show that (i)⇔(ii) and (ii)⇔(iii). To show (i)⇔(ii), note that $R(A) = A \setminus C(A)$ and $R'(A) = A \setminus C'(A)$ for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$, and recall the following equivalences:

$$\begin{aligned} C' = \mathrm{marg}_O C &\Leftrightarrow (\forall A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})))C'(A) = C(A) \\ &\Leftrightarrow (\forall A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})))R'(A) = A \setminus C'(A) = A \setminus C(A) = R(A) \\ &\Leftrightarrow R' = \mathrm{marg}_O R. \end{aligned}$$

To show (ii)⇔(iii), note that $A_1 \lhd A_2 \Leftrightarrow A_1 \subseteq R(A_1 \cup A_2)$ and $A_1 \lhd' A_2 \Leftrightarrow A_1 \subseteq R'(A_1 \cup A_2)$ for all A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$. Recall the following equivalences:

$$\begin{aligned} R' = \mathrm{marg}_O R &\Leftrightarrow (\forall A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})))R'(A) = R(A) \\ &\Leftrightarrow (\forall A_1, A_2 \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})))(A_1 \subseteq R'(A_1 \cup A_2) \Leftrightarrow A_1 \subseteq R(A_1 \cup A_2)) \\ &\Leftrightarrow (\forall A_1, A_2 \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})))(A_1 \lhd' A_2 \Leftrightarrow A_1 \lhd A_2) \Leftrightarrow \lhd' = \mathrm{marg}_O \lhd. \end{aligned}$$

For the second part, because C, R and $\lhd$ are compatible, it suffices to prove only one of the three double identities. We will show the first one, that $\mathrm{marg}_O C = C_{\mathrm{marg}_O R} = C_{\mathrm{marg}_O \lhd}$. To this end, let the choice function $C' \coloneqq \mathrm{marg}_O C$ on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$, and therefore, using that $R' = R_{C'} = R_{\mathrm{marg}_O C}$ and $\lhd' = \lhd_{C'} = \lhd_{\mathrm{marg}_O C}$, and since we just have shown that in particular (i)⇒(ii) and (i)⇒(iii), this implies that $R_{\mathrm{marg}_O C} = \mathrm{marg}_O R$ and $\lhd_{\mathrm{marg}_O C} = \mathrm{marg}_O \lhd$. Because C, R and $\lhd$ are compatible therefore indeed $\mathrm{marg}_O C = C_{\mathrm{marg}_O R}$ and $\mathrm{marg}_O C = C_{\mathrm{marg}_O \lhd}$. □

Now that marginalisation has been defined for all types of choice models, and their connection is clear, we can focus on any one of them. In what follows, we will work with choice or rejection functions, and use them interchangeably. Observe that repeated marginalisation, with non-empty subsets O_1 and O_2 of $\{1,\dots,n\}$ such that $O_1 \cap O_2 \neq \emptyset$ is the same as marginalisation with $O_1 \cap O_2$:

$$\mathrm{marg}_{O_1} \circ \mathrm{marg}_{O_2} = \mathrm{marg}_{O_2} \circ \mathrm{marg}_{O_1} = \mathrm{marg}_{O_1 \cap O_2}.$$

Coherence is preserved under marginalisation:

Proposition 158. *Consider any choice function C on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$, and consider any non-empty subset O of $\{1,\dots,n\}$. Then, for any property* C$*$ *in* $\{\mathrm{C1}_{20}, \mathrm{C2}_{20}, \mathrm{C3a}_{20}, \mathrm{C3b}_{20}, \mathrm{C4a}_{20}, \mathrm{C4b}_{20}, \mathrm{C5}_{25}, \mathrm{C6}_{25}\}$, *if C satisfies* C$*$, *then* $\mathrm{marg}_O C$ *satisfies* C$*$. *As a consequence, if C is coherent, then so is* $\mathrm{marg}_O C$.

Proof. This proposition follows immediately, once we realise that $A_1 = \emptyset \Leftrightarrow A_1^* = \emptyset$, $f < g \Leftrightarrow f^* < g^*$, $A_1 \subseteq A_2 \Leftrightarrow A_1^* \subseteq A_2^*$, $f \in A_1 \Leftrightarrow \lambda f^* \in \lambda A_1^*$, $f \in \mathrm{conv}(A_1) \Leftrightarrow f^* \in \mathrm{conv}(A_1^*)$, and $f + g \in A_1 \Leftrightarrow f^* + g^* \in A_1^*$ for all f and g in $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$ whose cylindrical extensions are f^* and g^*, any A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ whose cylindrical extensions are $A_1^* = \{h^* : h \in A_1\}$ and $A_2^* = \{h^* : h \in A_2\}$, and any λ in $\mathbb{R}_{>0}$. □

Let us compare with desirability. We trivially generalise Reference [29] in defining, for any non-empty subset O of $\{1,\dots,n\}$ and any set of desirable (vector-valued) gambles $D \subseteq \mathcal{L}(\mathcal{X}_O \times \mathcal{R})$, its *marginal set of desirable (vector-valued) gambles* $\mathrm{marg}_O D$ as

$$f \in \mathrm{marg}_O D \Leftrightarrow f \in D, \text{ for all } f \text{ in } \mathcal{L}(\mathcal{X}_O \times \mathcal{R}).$$

Therefore, we can find $\mathrm{marg}_O D$ explicitly as

$$\mathrm{marg}_O D = \{f \in \mathcal{L}(\mathcal{X}_O \times \mathcal{R}) : f \in D\} = D \cap \mathcal{L}(\mathcal{X}_O \times \mathcal{R}). \tag{7.1}$$

Let us ascertain that the definition of marginalisation for choice models reduces, in the case of pairwise choice, to the one for desirability:

Proposition 159. *Consider any non-empty subset O of $\{1,\dots,n\}$, any choice function C on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$, and any set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$. Then* $\mathrm{marg}_O C_D = C_{\mathrm{marg}_O D}$ *and* $D_{\mathrm{marg}_O C} = \mathrm{marg}_O D_C$.

Proof. For the first statement, consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$, and observe that

$$\begin{aligned}\mathrm{marg}_O C_D(A) = C_D(A) &= \{f \in A : (\forall g \in A) g - f \notin D\}\\ &= \{f \in A : (\forall g \in A) g - f \notin D \cap \mathcal{L}(\mathcal{X}_O \times \mathcal{R})\}\\ &= \{f \in A : (\forall g \in A) g - f \notin \mathrm{marg}_O D\} = C_{\mathrm{marg}_O D}(A)\end{aligned}$$

where the first equality follows from Definition 47_{223}, the second and the last one from Proposition 54_{62}, the third one from the fact that every gamble in A is an element of $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$, and finally, the fourth one from Equation $(7.1)_{\curvearrowleft}$. Therefore indeed $\mathrm{marg}_O C_D = C_{\mathrm{marg}_O D}$.

For the second statement, observe that indeed

$$\begin{aligned} D_{\mathrm{marg}_O C} &= \{f \in \mathcal{L}(\mathcal{X}_O \times \mathcal{R}) : 0 \notin \mathrm{marg}_O C(\{0,f\})\} \\ &= \{f \in \mathcal{L}(\mathcal{X}_O \times \mathcal{R}) : 0 \notin C(\{0,f\})\} = \{f \in \mathcal{L}(\mathcal{X}_O \times \mathcal{R}) : f \in D_C\} = \mathrm{marg}_O D_C, \end{aligned}$$

where the first and third equalities follow from Proposition 53_{61}, the second one from Definition 47_{223}, and the fourth one from Equation $(7.1)_{\curvearrowleft}$.

□

Corollary 160. *Consider any non-empty subset O of $\{1,\dots,n\}$, any coherent choice function C on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$, and any coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$. Then $\mathrm{marg}_O D = D_{\mathrm{marg}_O C_D}$ and $\mathrm{marg}_O C \sqsupseteq C_{\mathrm{marg}_O D_C}$.*

Proof. For the first statement, use Proposition $159_{\curvearrowleft}$ with $C \coloneqq C_D$, and Corollary 59_{67}.

For the second statement, use Proposition $159_{\curvearrowleft}$ with $D \coloneqq D_C$ to find that $\mathrm{marg}_O C_{D_C} = C_{\mathrm{marg}_O D_C}$, and infer that, using that marginalisation preserves the 'at most as informative as' relation and Corollary 59_{67}, therefore indeed $\mathrm{marg}_O C \sqsupseteq C_{\mathrm{marg}_O D_C}$. □

Now that marginalisation is in place, and that we know that it coincides with the eponymous concept for desirability in the case of pairwise choice, we are ready to look for some kind of inverse operation to it. To fix the discussion, we will work with rejection functions, but the same ideas apply to the other types of choice models. Suppose we have a coherent rejection function R_O on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$ modelling a subject's beliefs about X_O, where O is a non-empty subset of $\{1,\dots,n\}$. We want to extend this to a coherent rejection function on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ that represents the same beliefs. So we are looking for a coherent rejection function R on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ such that $\mathrm{marg}_O R = R_O$ and that is *as uninformative as possible*. If it exists, then we call this least informative extension R the *weak extension* of R_O.[3]

We now study this notion of weak extension in detail. Given a non-empty subset O of $\{1,\dots,n\}$ and a rejection function R_O on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$, an assessment based on it that is important for the weak extension, is

$$\mathcal{B}_{R_O}^{1:n} \coloneqq \{A^* : A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})) \text{ and } 0 \in R_O(A)\} \subseteq \mathcal{Q}_0(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})). \quad (7.2)$$

To make clear that $\mathcal{B}_{R_O}^{1:n}$ is a collection of sets of gambles on $\mathcal{X}_{1:n}$, we have made the 'range of the' cylindrical extension explicit in its expression. Using

[3] De Cooman and Miranda [29] have called its counterpart for desirability *cylindrical extension*, thereby overloading the meaning of this term. Moral [55] calls the analogous notion in a very similar context—whose only difference is that 0 is considered desirable—weak extension. In order to avoid confusion with the cylindrical extension for gambles, we have decided in favour of the name 'weak extension' here.

our simplifying device of identifying gambles with their cylindrical extensions, we can equivalently write $\mathcal{B}^{1:n}_{R_O} = \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})) : 0 \in R_O(A)\}$—and we will do this throughout—, which we interpret as a subset of $\mathcal{Q}_0(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R}))$, and therefore as an assessment for choice models on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$.

It turns out that the weak extension always exists.

Proposition 161 (Weak extension)**.** *Consider any non-empty subset O of $\{1,\ldots,n\}$ and any coherent rejection function R_O on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. Then the least informative coherent rejection function on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ that marginalises to R_O is given by*

$$\mathrm{ext}_{1:n}(R_O) \coloneqq R_{\mathcal{B}^{1:n}_{R_O}},$$

and it satisfies

$$\mathrm{marg}_O(\mathrm{ext}_{1:n}(R_O)) = R_O.$$

Proof. Use Proposition 79$_{95}$ to find that $R_{\mathcal{B}^{1:n}_{R_O}}$ is the least informative rejection function that extends $\mathcal{B}^{1:n}_{R_O}$ and satisfies Axioms R2$_{20}$–R4$_{20}$.

We will first show that any coherent rejection function R' on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ that marginalises to R_O must be at least as informative as $\mathrm{ext}_{1:n}(R_O)$. To establish this, since R' marginalises to R_O, note that

$$R'(A) = R_O(A) \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})).$$

Since both R' and R_O satisfy Axiom R4b$_{20}$, this is equivalent to

$$0 \in R'(A) \Leftrightarrow 0 \in R_O(A), \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})).$$

Therefore, in particular, $0 \in R_O(A) \Rightarrow 0 \in R'(A)$ for every A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$, so by Definition 29$_{90}$, R' extends $\mathcal{B}^{1:n}_{R_O} = \{A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})) : 0 \in R_O(A)\}$, and since $\mathrm{ext}_{1:n}(R_O)$ is the least informative extension of $\mathcal{B}^{1:n}_{R_O}$ that satisfies Axioms R2$_{20}$–R4$_{20}$, therefore indeed $\mathrm{ext}_{1:n}(R_O) = \mathrm{ext}_{1:n}(R_O) \sqsubseteq R'$.

So we already know that any coherent rejection function that marginalises to R_O must be at least as informative as $\mathrm{ext}_{1:n}(R_O)$. It therefore suffices to prove that $\mathrm{ext}_{1:n}(R_O)$ is coherent and that it marginalises to R_O. To show that $\mathrm{ext}_{1:n}(R_O)$ is coherent, it suffices to show that $\mathcal{B}^{1:n}_{R_O}$ avoids complete rejection. Assume *ex absurdo* that this is not the case. By Lemma 80$_{96}$ there is some A' in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R}))$ such that $0 \in A' = \max A'$ and

$$(\forall g \in A')(\exists B \in \mathcal{B}^{1:n}_{R_O}, \exists \mu \in \mathbb{R}_{>0})\{g\} + \mu B \lessdot A',$$

or, in other words,

$$(\forall g \in A')(\exists B \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})), \exists \mu \in \mathbb{R}_{>0})(0 \in R_O(B) \text{ and } \{g\} + \mu B \lessdot A').$$

Since R_O is coherent, by Lemma 12$_{21}$ it satisfies $0 \in R_O(B) \Leftrightarrow 0 \in R_O(\mu B)$ for every B in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ and μ in $\mathbb{R}_{>0}$, and therefore

$$(\forall g \in A')(\exists B \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})))(0 \in R_O(B) \text{ and } \{g\} + B \lessdot A'),$$

or, using Proposition 33[43](v),

$$(\forall g \in A')(\exists B \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})))(0 \in R_O(B) \text{ and } B \lessdot A' - \{g\}).$$

Let $I := \{1,\dots,n\} \setminus O$ and fix any x_I in $\mathcal{X}_I$—so for every g in A', the partial map $g(x_I,\bullet)$ is a vector-valued gamble on $\mathcal{X}_O$ (a gamble on $\mathcal{X}_O \times \mathcal{R}$), and $A'(x_I,\bullet) = \{h(x_I,\bullet) : h \in A'\}$ is a set of vector-valued gambles on $\mathcal{X}_O$—then

$$(\forall g \in A')(\exists B \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})))(0 \in R_O(B) \text{ and } B \lessdot A'(x_I,\bullet) - \{g(x_I,\bullet)\}).$$

Use Proposition 34[44] to infer that then

$$(\forall g \in A')0 \in R_O(A'(x_I,\bullet) - \{g(x_I,\bullet)\}).$$

By the coherence of R_O—more specifically, by Axiom R4b[20]—we find that then

$$(\forall g \in A')g(x_I,\bullet) \in R_O(A'(x_I,\bullet)),$$

or, in other words, that

$$A'(x_I,\bullet) \subseteq R_O(A'(x_I,\bullet)),$$

which contradicts the coherence [Axiom R1[20]] of R_O. So $\mathcal{B}^{1:n}_{R_O}$ avoids complete rejection, and therefore $\mathrm{ext}_{1:n}(R_O)$ is indeed coherent by Theorem 81[97].

To complete the proof, we show that $\mathrm{ext}_{1:n}(R_O)$ marginalises to R_O—that $\mathrm{ext}_{1:n}(R_O)(A) = R_O(A)$ for every A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$. Because both $\mathrm{ext}_{1:n}(R_O)$ and R_O are coherent rejection functions, Axiom R4b[20] guarantees that it suffices to show that

$$0 \in \mathrm{ext}_{1:n}(R_O)(A) \Leftrightarrow 0 \in R_O(A), \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})).$$

For sufficiency, consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ such that $0 \in R_O(A)$. Then $A \in \mathcal{B}^{1:n}_{R_O}$, and since we already know that $\mathrm{ext}_{1:n}(R_O)$ extends $\mathcal{B}^{1:n}_{R_O}$, therefore indeed $0 \in \mathrm{ext}_{1:n}(R_O)(A)$.

For necessity, consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ such that $0 \in \mathrm{ext}_{1:n}(R_O)(A)$. If $A \cap \mathcal{L}(\mathcal{X}_O \times \mathcal{R})_{>0} \neq \emptyset$, then $0 \in R_O(A)$ by the coherence of R_O [more specifically, Axioms R2[20] and R3a[20]] and the proof is done, so assume that $A \cap \mathcal{L}(\mathcal{X}_O \times \mathcal{R})_{>0} = \emptyset$. Taking Equation (3.1)[92] and the reasoning above into account, there is now some $A' \supseteq A$ in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R}))$ such that

$$\begin{aligned}(\forall g \in \{0\} \cup (A' \setminus A))\big((A' - \{g\}) \cap \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \neq \emptyset \text{ or}\\ (\exists B \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})))(0 \in R_O(B) \text{ and } \{g\} + B \lessdot A')\big).\end{aligned} \tag{7.3}$$

Without loss of generality, let $A = \{0, f_1, \dots, f_m\}$ and $A' = A \cup \{g_1, \dots, g_\ell\}$ for some m and ℓ in $\mathbb{Z}_{\geq 0}$, $f_1, \dots, f_m$ gambles on $\mathcal{X}_O \times \mathcal{R}$, and $g_1, \dots, g_\ell$ gambles on $\mathcal{X}_{1:n} \times \mathcal{R}$, where we may assume without loss of generality that all the gambles involved differ from each other. As before, let $I := \{1,\dots,n\} \setminus O$, fix any x_I in $\mathcal{X}_I$ and let $A'_O := A'(x_I,\bullet) = A \cup \{g_1(x_I,\bullet),\dots,g_\ell(x_I,\bullet)\} \in \mathcal{Q}_0(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$. If we can prove that $\{0\} \cup (A'_O \setminus A) \subseteq R_O(A'_O)$, then Axiom R3b[20] will imply [with $\tilde{A} := A'_O \setminus A$, $\tilde{A}_1 := \{0\} \cup (A'_O \setminus A)$ and $\tilde{A}_2 := A'_O$; then $\tilde{A}_1 \setminus \tilde{A} = \{0\}$ since $0 \in A$ and hence $0 \notin \tilde{A}$, and $\tilde{A}_2 \setminus \tilde{A} = A'_O \cap ({A'_O}^c \cup A) = A'_O \cap A = A$] that $0 \in R_O(A)$, completing the proof.

To establish this, we will show that $g \in R_O(A'_O)$ for every g in $\{0\} \cup (A'_O \setminus A)$. Consider first $g = 0$. Use Equation (7.3) to infer that $A' \cap \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \neq \emptyset$ or $(\exists B \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})))(0 \in R_O(B)$ and $B \preccurlyeq A')$. In the latter case we find that, since $B \preccurlyeq A'$ implies that $B \preccurlyeq A'_O$, by Proposition 34_{44}, $0 \in R_O(A'_O)$ and the proof is done. In the former case, $A' \cap \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \neq \emptyset$, so $h > 0$ for some h in A'. From $A \cap \mathcal{L}(\mathcal{X}_O \times \mathcal{R})_{>0} = \emptyset$—and hence $A \cap \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} = \emptyset$ because $A \subseteq \mathcal{L}(\mathcal{X}_O \times \mathcal{R})$—we infer that $h \notin A$. Then $h \in A' \setminus A$, and we may assume without loss of generality that $h \in (\max A') \setminus A$: indeed, if this were not the case, then there would be some h' in $\max A'$ such that $0 < h < h'$, which therefore also is no element of A (since we have assumed that $A' \cap \mathcal{L}_{>0} = \emptyset$ and therefore also $A \cap \mathcal{L}_{>0} = \emptyset$). Since h belongs to $(\max A') \setminus A$, by Equation (7.3) then necessarily $0 \in R_O(B)$ and $\{h\} + B \preccurlyeq A'$ for some B in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$. Since $h > 0$, we have that $B \preccurlyeq \{h\} + B$, and by Proposition 33_{43}(ii) therefore $B \preccurlyeq A'$. This implies that $B \preccurlyeq A'_O$, so by Proposition 34_{44}, we find again that then $0 \in R_O(A'_O)$.

Next, we will consider any g in $A'_O \setminus A$ and show that $g \in R_O(A'_O)$. Now $g = g_i(x_I, \bullet)$ for some i in $\{1, \ldots, \ell\}$. If $(A'_O - \{g\}) \cap \mathcal{L}(\mathcal{X}_O \times \mathcal{R})_{>0} \neq \emptyset$, then $g \in R_O(A'_O)$ by the coherence [more specifically, by Axioms $\mathrm{R2}_{20}$ and $\mathrm{R3a}_{20}$] of R_O, and the proof is done. So assume that $(A'_O - \{g\}) \cap \mathcal{L}(\mathcal{X}_O \times \mathcal{R})_{>0} = \emptyset$, and therefore, since $A'_O - \{g\} \subseteq \mathcal{L}(\mathcal{X}_O \times \mathcal{R})$, also

$$(A'_O - \{g\}) \cap \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} = \emptyset. \tag{7.4}$$

Use Equation (7.3) to infer that $(A' - \{g_i\}) \cap \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \neq \emptyset$ or $(\exists B \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})))(0 \in R_O(B)$ and $\{g_i\} + B \preccurlyeq A')$. In the latter case, by a similar argument as before, we find that $g = g_i(x_I, \bullet) \in R_O(A \cup \{g_1(x_I, \bullet), \ldots, g_\ell(x_I, \bullet)\}) = R_O(A'_O)$ and the proof is done. In the former case, $(A' - \{g_i\}) \cap \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \neq \emptyset$, so $h > g_i$ for some h in A'. We may assume without loss of generality that h belongs to $\max A'$: indeed, if this were not the case, then there would be some h' in $\max A'$ such that $h < h'$, and hence also $g_i < h'$. By Equation (7.4), therefore $h \notin A'_O$, so in particular $h \notin A$ and therefore $h \in A' \setminus A$. Since $(A'_O - \{h\}) \cap \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} = \emptyset$, by Equation (7.3) then necessarily $0 \in R_O(B)$ and $\{h\} + B \preccurlyeq A'$ for some B in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$. Since $g_i < h$, therefore $\{g_i\} + B \preccurlyeq A'$, so a similar argument as before shows than then indeed $g = g_i(x_I, \bullet) = h(x_I, \bullet) \in R_O(B)$. □

Note that $\mathrm{ext}_{1:n}$ preserves the 'at most as informative as' relation:

Proposition 162. *Consider any non-empty subset O of $\{1, \ldots, n\}$, and two coherent rejection functions R_1 and R_2 on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. If $R_1 \sqsubseteq R_2$, then* $\mathrm{ext}_{1:n}(R_1) \sqsubseteq \mathrm{ext}_{1:n}(R_2)$.

Proof. We will first show that the assessment $\mathcal{B}^{1:n}_{R_2}$ is at least as strong as $\mathcal{B}^{1:n}_{R_1}$. By Definition 30_{91} it suffices to show that $\mathcal{B}^{1:n}_{R_1} \subseteq \mathcal{B}^{1:n}_{R_2}$. Consider any $B \in \mathcal{B}^{1:n}_{R_1}$, then $0 \in R_1(B)$ and, since $R_1 \sqsubseteq R_2$, therefore $0 \in R_2(B)$. Then $B \in \mathcal{B}^{1:n}_{R_2}$, so $\mathcal{B}^{1:n}_{R_1} \subseteq \mathcal{B}^{1:n}_{R_2}$. Now, use Corollary 76_{92} to infer that then indeed $\mathrm{ext}_{1:n}(R_1) = R_{\mathcal{B}^{1:n}_{R_1}} \sqsubseteq R_{\mathcal{B}^{1:n}_{R_2}} = \mathrm{ext}_{1:n}(R_2)$. □

Let us compare our conclusions with the ones obtained for desirability in the literature. De Cooman and Miranda [29, Proposition 7] show that, given any non-empty subset O of $\{1, \ldots, n\}$ and any coherent set of desirable gambles $D_O \subseteq \mathcal{L}(\mathcal{X}_O \times \mathcal{R})$, its *weak extension* $\mathrm{ext}^{\mathbf{D}}_{1:n}(D_O) \subseteq \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$—the least informative coherent set of desirable gambles on $\mathcal{X}_{1:n} \times \mathcal{R}$ that marginalises to D_O—exists and is given by

$$\mathrm{ext}^{\mathbf{D}}_{1:n}(D_O) \coloneqq \mathrm{posi}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \cup D_O). \tag{7.5}$$

Let us ascertain that we can retrieve the weak extension for desirability from Proposition 161_{227}:

Proposition 163. *Consider any non-empty subset O of $\{1,\dots,n\}$, any coherent choice function R on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$, and any coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. Then $\mathrm{ext}_{1:n}(R_D) = R_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)}$. As a consequence, $\mathrm{ext}^{\mathbf{D}}_{1:n}(D) = D_{\mathrm{ext}_{1:n}(R_D)}$. Moreover, $\mathrm{ext}^{\mathbf{D}}_{1:n}(D_R) \subseteq D_{\mathrm{ext}_{1:n}(R)}$.*

Proof. For the first statement—that $\mathrm{ext}_{1:n}(R_D) = R_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)}$—, we first show that $\mathrm{ext}_{1:n}(R_D) \sqsubseteq R_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)}$. As an intermediate result, we will first establish that the purely binary assessment $\mathcal{B}_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)} \coloneqq \{\{0,f\} : f \in \mathrm{ext}^{\mathbf{D}}_{1:n}(D)\} \subseteq \mathcal{Q}_0(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R}))$ is at least as strong as $\mathcal{B}^{1:n}_{R_D}$. Definition 30_{91} says we need to show that $(\forall B \in \mathcal{B}^{1:n}_{R_D})(\exists B' \in \mathcal{B}_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)})B' \preccurlyeq B$, so consider any B in $\mathcal{B}^{1:n}_{R_D}$—then we know already by Equation (7.2)$_{226}$ that $0 \in B$ and $0 \in R_D(B)$, whence, by Proposition 55_{64}, $B \cap D \neq \emptyset$, so $g \in D$—and therefore also $g \in \mathrm{ext}^{\mathbf{D}}_{1:n}(D)$—for some g in B. Hence $B' \coloneqq \{0,g\} \in \mathcal{B}_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)}$. Since also $0 \in B$, therefore $B' \subseteq B$, whence, by Proposition 33_{43}(i), $B' \preccurlyeq B$. We conclude that $\mathcal{B}_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)}$ is indeed at least as strong an assessment as $\mathcal{B}^{1:n}_{R_D}$, whence by Corollary 76_{92}, $\mathrm{ext}_{1:n}(R_D) = R_{\mathcal{B}^{1:n}_{R_D}} \sqsubseteq \mathcal{E}(\mathcal{B}_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)})$. Furthermore, since $\mathrm{ext}^{\mathbf{D}}_{1:n}(D)$ is coherent, Theorem 86_{100} implies that $\mathcal{E}(\mathcal{B}_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)}) = R_{\mathcal{E}^{\mathbf{D}}(\mathrm{ext}^{\mathbf{D}}_{1:n}(D))} = R_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)}$. We find that indeed $\mathrm{ext}_{1:n}(R_D) \sqsubseteq R_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)}$.

Let us establish the converse inequality—that $R_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)} \sqsubseteq \mathrm{ext}_{1:n}(R_D) = R_{\mathcal{B}^{1:n}_{R_D}}$. Since they are both coherent rejection functions, it suffices by Axiom R4b$_{20}$ to show that

$$0 \in R_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)}(A) \Rightarrow 0 \in R_{\mathcal{B}^{1:n}_{R_D}}(A), \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})).$$

So consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R}))$ such that $0 \in R_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)}(A)$. By Proposition 55_{64} then $A \cap \mathrm{ext}^{\mathbf{D}}_{1:n}(D) \neq \emptyset$, so $f \in \mathrm{ext}^{\mathbf{D}}_{1:n}(D) = \mathrm{posi}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \cup D)$ for some f in A. Use Lemma 1_{11} to infer that then $f \in \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \cup D \cup \big(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} + D\big)$. If $f \in \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0}$, then by the coherence of $R_{\mathcal{B}^{1:n}_{R_D}}$ [more specifically, by Axioms R2$_{20}$ and R3a$_{20}$], indeed $0 \in R_{\mathcal{B}^{1:n}_{R_D}}(A)$. If $f \in D \cup \big(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} + D\big) = \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{\geq 0} + D$, then $f \geq f_D$ for some f_D in D. Since f_D belongs to D, Proposition 55_{64} guarantees that $0 \in R_D(\{0, f_D\})$. Furthermore, by Proposition 161_{227}, $\mathrm{ext}_{1:n}(R_D) = R_{\mathcal{B}^{1:n}_{R_D}}$ marginalises to R_D, so we find that $0 \in R_{\mathcal{B}^{1:n}_{R_D}}(\{0, f_D\})$, and therefore, by Proposition 30_{41}(ii), that $0 \in R_{\mathcal{B}^{1:n}_{R_D}}(\{0, f\})$. Hence, by the coherence of $\mathrm{ext}_{1:n}(R_D)$ [more specifically Axiom R3a$_{20}$] we find that then indeed $0 \in R_{\mathcal{B}^{1:n}_{R_D}}(A)$.

For the second statement—that $\operatorname{ext}^{\mathbf{D}}_{1:n}(D) = D_{\operatorname{ext}_{1:n}(R_D)}$—, infer from the first one that $D_{\operatorname{ext}_{1:n}(R_D)} = D_{R_{\operatorname{ext}^{\mathbf{D}}_{1:n}(D)}}$. By Corollary 59_{67}, the latter is equal to $\operatorname{ext}^{\mathbf{D}}_{1:n}(D)$, and therefore indeed $\operatorname{ext}^{\mathbf{D}}_{1:n}(D) = D_{\operatorname{ext}_{1:n}(R_D)}$.

The final statement—that $\operatorname{ext}^{\mathbf{D}}_{1:n}(D_R) \subseteq D_{\operatorname{ext}_{1:n}(R)}$—can be obtained by plugging the specific set of desirable gambles D_R into the second statement: we find that then $\operatorname{ext}^{\mathbf{D}}_{1:n}(D_R) = D_{\operatorname{ext}_{1:n}(R_{D_R})}$. Now use Corollary 59_{67} and Proposition 162_{229} to infer that $\operatorname{ext}_{1:n}(R_{D_R}) \sqsubseteq \operatorname{ext}_{1:n}(R)$, whence by Proposition 60_{68}, $D_{\operatorname{ext}_{1:n}(R_{D_R})} \subseteq D_{\operatorname{ext}_{1:n}(R)}$. Therefore indeed $\operatorname{ext}^{\mathbf{D}}_{1:n}(D_R) \subseteq D_{\operatorname{ext}_{1:n}(R)}$. □

Proposition 163 implies that the weak extension of a purely binary rejection function R_D for some coherent set of desirable gambles D, is a rejection function that is itself purely binary. To summarise this, consider the following commuting diagram in Figure 7.1, where we have used the maps

$$
\begin{aligned}
&\operatorname{ext}^{\mathbf{D}}_{1:n}\colon \overline{\mathbf{D}}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})) \to \overline{\mathbf{D}}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R}))\colon D \mapsto \operatorname{ext}^{\mathbf{D}}_{1:n}(D),\\
&R_\bullet\colon \mathbf{D} \to \mathbf{R}\colon D \mapsto R_D,\\
&\operatorname{ext}_{1:n}\colon \overline{\mathbf{C}}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})) \to \overline{\mathbf{C}}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R}))\colon C \mapsto \operatorname{ext}_{1:n}(C),\\
&D_\bullet\colon \overline{\mathbf{R}}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})) \to \overline{\mathbf{D}}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R}))\colon R \mapsto D_R,
\end{aligned}
$$

with $\operatorname{ext}^{\mathbf{D}}_{1:n}(D)$ as defined in Equation (7.5), $\operatorname{ext}_{1:n}(C)$ in Proposition 161_{227}, $D_R = \{f \in \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R}) : 0 \in R(\{0, f\})\}$, and, as usual, R_D given by $R_D(A) = \{u \in A : (\forall v \in A) v - u \notin D\}$ for all A in $\mathcal{Q}$, whose domain $\mathcal{Q}$ we leave unspecified, since R_D works either on vector-valued gambles on $\mathcal{X}_{1:n}$ or $\mathcal{X}_O$, with O a non-empty subset of $\{1, \dots, n\}$. The root of the diagram is any coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. Taking the weak extension (for desirability) of it, commutes with performing the same operation in the language of choice models.

$$
\begin{tikzcd}
D \arrow[r, "\operatorname{ext}^{\mathbf{D}}_{1:n}"] \arrow[d, "R_\bullet"'] & \operatorname{ext}^{\mathbf{D}}_{1:n}(D) = D_{\operatorname{ext}_{1:n}(R_D)} \arrow[d, shift right, "R_\bullet"'] \\
R_D \arrow[r, "\operatorname{ext}_{1:n}"] & \operatorname{ext}_{1:n}(R_D) = R_{\operatorname{ext}^{\mathbf{D}}_{1:n}(D)} \arrow[u, shift right, "D_\bullet"']
\end{tikzcd}
$$

Figure 7.1: Commuting diagram for the weak extension

7.3 CONDITIONING ON VARIABLES

In Chapter 6_{205}, we have seen how we can condition choice functions on events. Here, we take a closer look at conditioning in a multivariate context.

Suppose we have a choice function C_n on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$, representing an agent's beliefs about the value of $X_{1:n}$. Assume now that we obtain the information that the I-tuple of variables X_I—where I is a non-empty subset of $\{1,\dots,n\}$—assumes a value in a certain non-empty subset E_I of $\mathcal{X}_I$—so E_I belongs to $\mathcal{P}_{\varnothing}(\mathcal{X}_I)$.[4] There is no new information about the other variables X_{I^c}, where $I^c \coloneqq \{1,\dots,n\} \setminus I$. How can we condition C_n using this new information?

This is a particular instance of Definition 44_{208}, with the following specifications:

$$\mathcal{X} = \mathcal{X}_{1:n} \quad \text{and} \quad E = E_I \times \mathcal{X}_{I^c}.$$

The indicator $\mathbb{I}_E$ of the conditioning event $E = E_I \times \mathcal{X}_{I^c}$ satisfies

$$\mathbb{I}_E(x_{1:n}) = \mathbb{I}_{E_I \times \mathcal{X}_{I^c}}(x_{1:n}) = \begin{cases} 1 & \text{if } x_I \in E_I \\ 0 & \text{if } x_I \notin E_I \end{cases}$$

for all $x_{1:n}$ in $\mathcal{X}_{1:n}$, and, taking Remark 7.1_{223} into account, therefore $\mathbb{I}_E = \mathbb{I}_{E_I \times \mathcal{X}_{I^c}} = \mathbb{I}_{E_I}$ and $E = E_I \times \mathcal{X}_{I^c} = E_I$. Equation $(6.1)_{206}$ defines the multiplication of a gamble f on $E_I \times \mathcal{X}_{I^c \times \mathcal{R}}$ with $\mathbb{I}_{E_I}$ to be a gamble $\mathbb{I}_{E_I} f$ on $\mathcal{X}_{1:n} \times \mathcal{R}$, given by

$$\mathbb{I}_{E_I} f(x_{1:n}, r) = \mathbb{I}_{E_I \times \mathcal{X}_{I^c}} f(x_{1:n}, r) = \begin{cases} f(x_{1:n}, r) & \text{if } x_{1:n} \in E_I \times \mathcal{X}_{I^c} \\ 0 & \text{if } x_{1:n} \notin E_I \times \mathcal{X}_{I^c} \end{cases}$$
$$= \begin{cases} f(x_{1:n}, r) & \text{if } x_I \in E_I \\ 0 & \text{if } x_I \notin E_I \end{cases} \tag{7.6}$$

for all $x_{1:n}$ in $\mathcal{X}_{1:n}$ and r in $\mathcal{R}$, and the multiplication of $\mathbb{I}_{E_I}$ with a set A of gambles on $E_I \times \mathcal{X}_{I^c} \times \mathcal{R}$ results in a set $\mathbb{I}_{E_I} A = \{\mathbb{I}_{E_I} f : f \in A\}$ of gambles on $\mathcal{X}_{1:n} \times \mathcal{R}$.

Now that we have instantiated all the relevant aspects of Definition 44_{208}, we are ready to find the conditional choice function $C_n \rfloor E_I$, given a joint choice function C_n on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$:

$$C_n \rfloor E_I(A) = \{f \in A : \mathbb{I}_{E_I} f \in C_n(\mathbb{I}_{E_I} A)\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(E_I \times \mathcal{X}_{I^c} \times \mathcal{R})),$$

and, equivalently,

$$f \in C_n \rfloor E_I(A) \Leftrightarrow \mathbb{I}_{E_I} f \in C_n(\mathbb{I}_{E_I} A), \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(E_I \times \mathcal{X}_{I^c} \times \mathcal{R})) \text{ and } f \text{ in } A.$$

[4]This is a more general type of information than what is usually considered in this context: for instance, in their treatment [29], De Cooman and Miranda only condition their models on *singletons*. The reasons why we need this more general approach will become clear later on.

The conditional choice function $C_n\rfloor E_I$ is defined on vector-valued gambles on $E_I \times \mathcal{X}_{I^c}$. However, usually—see, for instance, References [18, 29]—conditioning on information about X_I results in a model on vector-valued gambles on $\mathcal{X}_{I^c}$—being (identified with) a subset of $E_I \times \mathcal{X}_{I^c}$. We therefore consider

$$\mathrm{marg}_{I^c}(C_n\rfloor E_I)(A) = \{f \in A : \mathbb{I}_{E_I} f \in C_n(\mathbb{I}_{E_I} A)\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}_{I^c} \times \mathcal{R}))$$

as the choice function that represents the conditional beliefs about X_{I^c}, given $X_I \in E_I$. The multiplication $\mathbb{I}_{E_I} f$ of $\mathbb{I}_{E_I}$ and f in this context is defined through Equation (7.6):

$$\mathbb{I}_{E_I} f(x_{1:n}, r) = \begin{cases} f(x_{I^c}, r) & \text{if } x_I \in E_I \\ 0 & \text{if } x_I \notin E_I \end{cases}$$

for all $x_{1:n}$ in $\mathcal{X}_{1:n}$ and r in $\mathcal{R}$.

Note that, in the particular case of conditioning on a singleton—then, say, $E_I = \{x_I\}$ for some x_I in $\mathcal{X}_I$—the choice function[5] $C_n\rfloor x_I$ works on gambles f on $\{x_I\} \times \mathcal{X}_{I^c}$. Such f can be uniquely identified with a gamble $f(x_I, \bullet)$ on $\mathcal{X}_{I^c}$, and therefore $\{x_I\} \times \mathcal{X}_{I^c}$ can be identified with $\mathcal{X}_{I^c}$. So the resulting choice function $C_n\rfloor x_I$ can be identified with its marginal $\mathrm{marg}_{I^c}(C_n\rfloor x_I)$.

Proposition 164. *Consider any choice function C_n on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$, any non-empty subset I of $\{1, \dots, n\}$ and any E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$. Then, for any property* $\mathrm{C}{*}$ *in* $\{\mathrm{C1}_{20}, \mathrm{C2}_{20}, \mathrm{C3a}_{20}, \mathrm{C3a}_{20}, \mathrm{C4a}_{20}, \mathrm{C4b}_{20}, \mathrm{C5}_{25}, \mathrm{C6}_{25}\}$*, if C_n satisfies* $\mathrm{C}{*}$*, then* $\mathrm{marg}_{I^c}(C_n\rfloor E_I)$ *satisfies* $\mathrm{C}{*}$*. As a consequence, if C_n is coherent, then so is* $\mathrm{marg}_{I^c}(C_n\rfloor E_I)$.

Proof. It is a direct consequence of Proposition 150_{209}—after the instantiation of $\mathcal{X} = \mathcal{X}_{1:n}$ and $E = E_I = E_I \times \mathcal{X}_{I^c}$—that, if C_n satisfies a certain property in $\{\mathrm{C1}_{20}, \mathrm{C2}_{20}, \mathrm{C3a}_{20}, \mathrm{C3a}_{20}, \mathrm{C4a}_{20}, \mathrm{C4b}_{20}, \mathrm{C5}_{25}, \mathrm{C6}_{25}\}$, then so does $C_n\rfloor E_I$, and using Proposition 158_{225}, therefore indeed so does $\mathrm{marg}_O(C_n\rfloor E_I)$. ☐

As is the case for desirability (see Reference [29, Proposition 9]), the order of marginalisation and conditioning can be reversed, under some conditions:

Proposition 165. *Consider any coherent choice function C_n on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ and any disjoint and non-empty subsets I and O of $\{1, \dots, n\}$. Then*

$$\mathrm{marg}_O(C_n\rfloor E_I) = \mathrm{marg}_O((\mathrm{marg}_{I\cup O} C_n)\rfloor E_I) \textit{ for all } E_I \textit{ in } \mathcal{P}_\varnothing(\mathcal{X}_I).$$

[5] Actually, since the conditioning event is $\{x_I\}$, we should write $C_n\rfloor\{x_I\}$ rather than $C_n\rfloor x_I$, but since no confusion can arise, and for notational simplicity, we will use the latter notation. A similar choice has been made by De Cooman and Miranda in Reference [29].

Proof. Consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$, and consider the following chain of equalities:

$$\begin{aligned}\mathrm{marg}_O(C_n\rfloor E_I)(A) = C_n\rfloor E_I(A) &= \{f \in A : \mathbb{I}_{E_I} f \in C_n(\mathbb{I}_{E_I} A)\}\\ &= \{f \in A : \mathbb{I}_{E_I} f \in \mathrm{marg}_{I\cup O} C_n(\mathbb{I}_{E_I} A)\}\\ &= (\mathrm{marg}_{I\cup O} C_n)\rfloor E_I(A)\\ &= \mathrm{marg}_O((\mathrm{marg}_{I\cup O} C_n)\rfloor E_I)(A). \qquad \square\end{aligned}$$

7.4 IRRELEVANT NATURAL EXTENSION

Now that the basic operations of multivariate choice functions—marginalisation, weak extension and conditioning—are in place, we are ready to look at a simple type of *structural assessment*. In Section 5.9$_{193}$ we have already worked with a structural assessment, namely that C should be compatible with some coherent set of indifferent options. Here, the assessment that we will consider, is that of *epistemic irrelevance*. All the results in this section build on the corresponding results for desirability, as established by De Cooman and Miranda in Reference [29, Section 6].

Definition 48 (Epistemic (subset-)irrelevance)**.** *Let I and O be two disjoint and non-empty subsets of $\{1,\dots,n\}$. We say that X_I is* epistemically irrelevant *to X_O when learning about the value of X_I does not influence or change our subject's beliefs about X_O. A choice function C_n on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ is said to satisfy* epistemic irrelevance *of X_I to X_O when*

$$\mathrm{marg}_O(C_n\rfloor E_I) = \mathrm{marg}_O C_n \textit{ for all } E_I \textit{ in } \mathcal{P}_\varnothing(\mathcal{X}_I). \tag{7.7}$$

The idea behind this definition is that observing that X_I belongs to E_I turns C_n into the conditioned choice function $C_n\rfloor E_I$ on $\mathcal{L}(E_I \times \mathcal{X}_{I^c} \times \mathcal{R}) \supseteq \mathcal{L}(\mathcal{X}_O \times \mathcal{R})$, so requiring that learning that X_I belongs to E_I does not affect the subject's beliefs about X_O, amounts to requiring that the marginal models of C_n and $C_n\rfloor E_I$ should be equal.

Equivalently, Definition 48 can be expressed in terms of other choice models as well. For instance, we say that a rejection function R_n on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ satisfies epistemic irrelevance of X_I to X_O when $\mathrm{marg}_O(R_n\rfloor E_I) = \mathrm{marg}_O R_n$ for all E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$. Similarly, we say that a choice relation $\lhd_n$ on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ satisfies epistemic irrelevance of X_I to X_O when $\mathrm{marg}_O(\lhd_n\rfloor E_I) = \mathrm{marg}_O \lhd_n$ for all E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$.

This type of irrelevance is what De Bock [18] calls *epistemic subset-irrelevance*, since the epistemic irrelevance condition is imposed for every (non-empty) subset of $\mathcal{X}_I$. This contrasts the more conventional approach—which De Bock [18] refers to as *epistemic value-irrelevance*—of requiring that

$$\mathrm{marg}_O(C_n\rfloor x_I) = \mathrm{marg}_O C_n \text{ for all } x_I \text{ in } \mathcal{X}_I,$$

and we distinguish it from the epistemic (subset-)irrelevance from Definition 48.

The counterpart of Definition 48 for desirability (see Reference [18, Section 4.3.2]) is:

$$\mathrm{marg}_O(D_n \rfloor E_I) = \mathrm{marg}_O D_n \text{ for all } E_I \text{ in } \mathcal{P}_\varnothing(\mathcal{X}_I), \tag{7.8}$$

and this is *epistemic subset-irrelevance* for the set of desirable gambles $D_n \subseteq \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$. However, in the literature (see, for instance, References [14, 29, 82]), mostly epistemic *value*-irrelevance is considered: a set D_n of desirable gambles on $\mathcal{X}_{1:n} \times \mathcal{R}$ satisfies *epistemic value-irrelevance* of X_I to X_O when

$$\mathrm{marg}_O(D_n \rfloor x_I) = \mathrm{marg}_O D_n \text{ for all } x_I \text{ in } \mathcal{X}_I.$$

The eponymous concepts for choice models and desirability coincide:

Proposition 166. *Consider any two disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, any coherent choice function C_n on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$, and any coherent set of desirable gambles $D_n \subseteq \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$. If C_n satisfies epistemic subset-irrelevance of X_I to X_O, then so does D_{C_n}, and conversely, if D_n satisfies epistemic subset-irrelevance of X_I to X_O, then so does C_{D_n}. Moreover, if C_n satisfies epistemic value-irrelevance of X_I to X_O, then so does D_{C_n}, and conversely, if D_n satisfies epistemic value-irrelevance of X_I to X_O, then so does C_{D_n}.*

Proof. For the first statement—if C_n satisfies epistemic subset-irrelevance of X_I to X_O, then so does D_{C_n}—, it suffices by Equation (7.8) to prove that $\mathrm{marg}_O(D_{C_n} \rfloor E_I) = \mathrm{marg}_O D_{C_n}$ for all E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$. Consider any E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$ and recall the following chain of equalities:

$$\begin{aligned}
\mathrm{marg}_O(D_{C_n} \rfloor E_I) &= \mathrm{marg}_O(D_{C_n \rfloor E_I}) && \text{by Proposition } 153_{211}\\
&= D_{\mathrm{marg}_O(C_n \rfloor E_I)} && \text{by Proposition } 159_{225}\\
&= D_{\mathrm{marg}_O C_n} && \text{since } C_n \text{ satisfies epistemic subset-irrelevance}\\
&= \mathrm{marg}_O D_{C_n} && \text{by Proposition } 159_{225}.
\end{aligned}$$

For the second statement—if D_n satisfies epistemic subset-irrelevance of X_I to X_O, then so does C_{D_n}—, by Definition 48 it suffices to prove that $\mathrm{marg}_O(C_{D_n} \rfloor E_I) = \mathrm{marg}_O C_{D_n}$ for all E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$. Consider any E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$ and recall the following chain of equalities:

$$\begin{aligned}
\mathrm{marg}_O(C_{D_n} \rfloor E_I) &= \mathrm{marg}_O(C_{D_n \rfloor E_I}) && \text{by Proposition } 153_{211}\\
&= C_{\mathrm{marg}_O(D_n \rfloor E_I)} && \text{by Proposition } 159_{225}\\
&= C_{\mathrm{marg}_O D_n} && \text{since } D_n \text{ satisfies epistemic subset-irrelevance}\\
&= \mathrm{marg}_O C_{D_n} && \text{by Proposition } 159_{225}.
\end{aligned}$$

The final statements follow now at once since $\{x_I\} \in \mathcal{P}_\varnothing(\mathcal{X}_I)$ for every x_I in $\mathcal{X}_I$. □

Clearly, epistemic subset-irrelevance implies epistemic value-irrelevance. As shown by De Bock [18, Example 2] for desirability, the converse does not hold: epistemic subset-irrelevance is *strictly* stronger than epistemic value-irrelevance for desirability, and therefore, since choice models have desirability as a particular case, also for choice models.

Unlike epistemic value-irrelevance, epistemic subset-irrelevance requires *all* information about the value of X_I to be irrelevant for X_O, including partial information like $X_I \in E_I$, where E_I is a non-empty subset of $\mathcal{X}_I$. For instance, as shown for desirability by De Bock [18] in the same Example 2, if $|\mathcal{X}_I| \geq 3$, observing that X_I is not equal to some x_I in $\mathcal{X}_I$, can influence our beliefs about X_O even if we consider X_I to be epistemically value-irrelevant to X_O. Since choice models have desirability as a particular case, this therefore also holds for choice models. This would be impossible if X_I were epistemically subset-irrelevant to X_O. We therefore follow De Bock [18] in considering epistemic subset-irrelevance to be the more natural and compelling of the two concepts, and hence, in the remainder, we will only consider epistemic subset-irrelevance (Definition 48_{234}), and will simply call this condition *epistemic irrelevance*.

Epistemic irrelevance can be reformulated in an interesting and slightly different manner:

Proposition 167. *Consider a coherent rejection function R_n on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$, and any disjoint and non-empty subsets I and O of $\{1, \ldots, n\}$. Then the following statements are equivalent:*

(i) $\operatorname{marg}_O(R_n \rfloor E_I) = \operatorname{marg}_O R_n$ *for all E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$;*

(ii) $0 \in R_n(A) \Leftrightarrow 0 \in R_n(\mathbb{I}_{E_I} A)$ *for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ and E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$.*

Proof. To show that (i) implies (ii), consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ and any E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$, and recall the following equivalences:

$$\begin{aligned} 0 \in R_n(A) &\Leftrightarrow 0 \in \operatorname{marg}_O R_n(A) = \operatorname{marg}_O(R_n \rfloor E_I)(A) && \text{by Definition } 47_{223} \text{ and (i)} \\ &\Leftrightarrow 0 \in R_n \rfloor E_I(A) && \text{by Definition } 47_{223} \\ &\Leftrightarrow 0 \in R_n(\mathbb{I}_{E_I} A) && \text{by Definition } 44_{208}. \end{aligned}$$

To show that (ii) implies (i), consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ and any E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$, and recall the following equivalences:

$$\begin{aligned} 0 \in \operatorname{marg}_O(R_n \rfloor E_I)(A) &\Leftrightarrow 0 \in R_n \rfloor E_I(A) && \text{by Definition } 47_{223} \\ &\Leftrightarrow 0 \in R_n(\mathbb{I}_{E_I} A) && \text{by Definition } 44_{208} \\ &\Leftrightarrow 0 \in R_n(A) && \text{by (ii)} \\ &\Leftrightarrow 0 \in \operatorname{marg}_O R_n(A) && \text{by Definition } 47_{223}, \end{aligned}$$

since both $\operatorname{marg}_O(R_n \rfloor E_I)$ and $\operatorname{marg}_O R_n$ are coherent rejection functions, by Axiom R4b_{20} this implies that indeed $\operatorname{marg}_O(R_n \rfloor E_I) = \operatorname{marg}_O R_n$. □

Epistemic irrelevance assessments are useful in constructing rejection functions on larger domains from other ones on smaller domains, in a similar way as in Chapter 3_{89} for incomplete assessments. Suppose we have a rejection function R_O on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$, and an assessment that X_I is epistemically irrelevant to X_O, where I and O are disjoint and non-empty subsets of $\{1,\dots,n\}$. How can we combine R_O and this irrelevance assessment into a coherent rejection function on $\mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R})$, or more generally, on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$? We want this new rejection function furthermore to be as *least informative* as possible. The following assessment will play a crucial role:

$$\mathcal{B}_{R_O}^{I\to O} \coloneqq \{\mathbb{I}_{E_I} A : A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})) \text{ and } 0 \in R_O(A) \text{ and } E_I \in \mathcal{P}_\varnothing(\mathcal{X}_I)\}.$$

Observe that $\mathcal{B}_{R_O}^{I\to O} \subseteq \mathcal{Q}_0(\mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R}))$, so we interpret it as an assessment on $\mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R})$, based on a given coherent rejection function R_O on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$ and an epistemic irrelevance statement. The multiplication of a gamble f on $\mathcal{X}_O \times \mathcal{R}$ with $\mathbb{I}_{E_I}$ is defined through Equation $(7.6)_{232}$:

$$\mathbb{I}_{E_I} f(x_I, x_O, r) = \begin{cases} f(x_O, r) & \text{if } x_I \in E_I \\ 0 & \text{if } x_I \notin E_I \end{cases}$$

for all x_I in $\mathcal{X}_I$, x_O in $\mathcal{X}_O$ and r in $\mathcal{R}$. We first establish three basic facts about this assessment $\mathcal{B}_{R_O}^{I\to O}$, in the form of Lemmas 168–170_{239}.

Lemma 168. *Consider any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, and any coherent rejection function R_O on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. Then*

$$0 \in R_{\mathcal{B}_{R_O}^{I\to O}}(A) \Rightarrow 0 \in R_O\left(\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A\right\}\right)$$

for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R}))$.

Proof. Assume that $0 \in R_{\mathcal{B}_{R_O}^{I\to O}}(A)$. By Equation $(3.1)_{92}$, there is some $A' \supseteq A$ in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R}))$ such that

$$(\forall g \in \{0\} \cup (A' \setminus A))\big((A' - \{g\}) \cap \mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R})_{>0} \neq \varnothing \text{ or}$$
$$(\exists B \in \mathcal{B}_{R_O}^{I\to O}, \exists \mu \in \mathbb{R}_{>0})\{g\} + \mu B \lessapprox A'\big).$$

By Axiom R4b_{20}, $0 \in R_O(\mu B) \Leftrightarrow 0 \in R_O(B)$ for all μ in $\mathbb{R}_{>0}$, and therefore

$$(\forall g \in \{0\} \cup (A' \setminus A))\big((A' - \{g\}) \cap \mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R})_{>0} \neq \varnothing \text{ or}$$
$$(\exists B \in \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})), \exists E_I \in \mathcal{P}_\varnothing(\mathcal{X}_I))(0 \in R_O(B) \text{ and } \{g\} + \mathbb{I}_{E_I} B \lessapprox A')\big).$$

Consider any g in $\{0\} \cup (A' \setminus A)$. Then there are two possibilities: (i) $(A' - \{g\}) \cap \mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R})_{>0} \neq \varnothing$, and (ii) $0 \in R_O(B)$ and $\{g\} + \mathbb{I}_{E_I} B \lessapprox A'$ for some E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$ and B in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$.

In case (i), $(A' - \{g\}) \cap \mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R})_{>0} \neq \emptyset$, and then $g < h$ for some h in A'. Therefore $g(z_I, \bullet) < h(z_I, \bullet)$ for some z_I in $\mathcal{X}_I$, and $g(x_I, \bullet) \leq h(x_I, \bullet)$ for all x_I in $\mathcal{X}_I \setminus \{z_I\}$, whence $\sum_{x_I \in \mathcal{X}_I \setminus \{z_I\}} g(x_I, \bullet) \leq \sum_{x_I \in \mathcal{X}_I \setminus \{z_I\}} h(x_I, \bullet)$. Combining this with $g(z_I, \bullet) < h(z_I, \bullet)$, we find that $\sum_{x_I \in \mathcal{X}_I} g(x_I, \bullet) < \sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet)$. Since R_O is coherent, by Axiom R2$_{20}$ therefore $\sum_{x_I \in \mathcal{X}_I} g(x_I, \bullet) \in R_O(\{\sum_{x_I \in \mathcal{X}_I} g(x_I, \bullet), \sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet)\})$, whence, by Axiom R3a$_{20}$,

$$\sum_{x_I \in \mathcal{X}_I} g(x_I, \bullet) \in R_O\left(\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A'\right\}\right).$$

In case (ii), $0 \in R_O(B)$ and $\{g\} + \mathbb{I}_{E_I} B \lessdot A'$ for some B in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ and E_I in $\mathcal{P}_\emptyset(\mathcal{X}_I)$, and then for every f in B, there is some h in A' such that $g + \mathbb{I}_{E_I} f \leq h$, or, in other words, such that

$$(\forall x_I \in \mathcal{X}_I, x_O \in \mathcal{X}_O, r \in \mathcal{R}) g(x_I, x_O, r) + \mathbb{I}_{E_I} f(x_I, x_O, r) \leq h(x_I, x_O, r),$$

whence $f \leq h(x_I, \bullet) - g(x_I, \bullet)$ for all x_I in E_I, and $0 \leq h(z_I, \bullet) - g(z_I, \bullet)$ for every z_I in E_I^c. Therefore $|E_I| f \leq \sum_{x_I \in E_I} [h(x_I, \bullet) - g(x_I, \bullet)]$ and $0 \leq \sum_{z_I \in E_I^c} [h(z_I, \bullet) - g(z_I, \bullet)]$, whence $f \leq \frac{1}{|E_I|} \sum_{x_I \in \mathcal{X}_I} [h(x_I, \bullet) - g(x_I, \bullet)]$, so we find in particular that

$$\begin{aligned} B \lessdot \frac{1}{|E_I|} &\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) - \sum_{x_I \in \mathcal{X}_I} g(x_I, \bullet) : h \in A'\right\} \\ &= \frac{1}{|E_I|}\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A'\right\} - \frac{1}{|E_I|}\left\{\sum_{x_I \in \mathcal{X}_I} g(x_I, \bullet)\right\}. \end{aligned}$$

Use Proposition 34$_{44}$ to infer that, since $0 \in R_O(B)$,

$$0 \in R_O\left(\frac{1}{|E_I|}\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A'\right\} - \frac{1}{|E_I|}\left\{\sum_{x_I \in \mathcal{X}_I} g(x_I, \bullet)\right\}\right),$$

and since R_O is a coherent rejection function, by Axiom R4$_{20}$ here too

$$\sum_{x_I \in \mathcal{X}_I} g(x_I, \bullet) \in R_O\left(\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A'\right\}\right),$$

being the same conclusion as in (i).

So we have shown that

$$(\forall g \in \{0\} \cup (A' \setminus A)) \sum_{x_I \in \mathcal{X}_I} g(x_I, \bullet) \in R_O\left(\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A'\right\}\right).$$

Use Axiom R3b$_{20}$ with

$$\tilde{A} := \left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A' \setminus A \text{ and } \sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) \neq 0\right\},$$

$$\tilde{A}_1 := \left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in \{0\} \cup (A' \setminus A)\right\},$$

$$\tilde{A}_2 := \left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A'\right\},$$

[then $\tilde{A}_1 \subseteq R_O(\tilde{A}_2)$, $\tilde{A} \subseteq \tilde{A}_1$, $\tilde{A}_1 \setminus \tilde{A} = \{0\}$, and

$$\begin{aligned}\tilde{A}_2 \setminus \tilde{A} &= \left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A' \text{ and } \left(h \notin A' \setminus A \text{ or } \sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) = 0\right)\right\} \\ &= \left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A \text{ or } \left(h \in A' \text{ and } \sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) = 0\right)\right\} \\ &= \left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A\right\},\end{aligned}$$

where the second equality follows from the fact that $A \subseteq A'$ and the third one from the fact that $0 \in A$] to conclude that therefore indeed

$$0 \in R_O\left(\left\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A\right\}\right). \qquad \square$$

Lemma 169. *Consider any disjoint and non-empty subsets I and O of $\{1, \dots, n\}$, and any coherent rejection function R_O on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. Then $\mathcal{B}_{R_O}^{I \to O}$ avoids complete rejection. As a consequence, its natural extension $\mathcal{E}\big(\mathcal{B}_{R_O}^{I \to O}\big)$—a rejection function on $\mathcal{L}(\mathcal{X}_{I \cup O} \times \mathcal{R})$—is coherent and equal to $R_{\mathcal{B}_{R_O}^{I \to O}}$.*

Proof. We will provide a proof by contradiction. Assume *ex absurdo* that $R_{\mathcal{B}_{R_O}^{I \to O}}$ does not avoid complete rejection. By Lemma 80$_{96}$ then $0 \in R_{\mathcal{B}_{R_O}^{I \to O}}(\{0\})$. By Lemma 168$_{237}$ therefore $0 \in R_O(\{\sum_{x_I \in \mathcal{X}_I} 0\}) = R_O(\{0\})$, a contradiction with the coherence [Axiom R1$_{20}$] of R_O. So $R_{\mathcal{B}_{R_O}^{I \to O}}$ does satisfy Axiom R1$_{20}$, so $\mathcal{B}_{R_O}^{I \to O}$ indeed avoids complete rejection. That its natural extension is coherent and equal to $R_{\mathcal{B}_{R_O}^{I \to O}}$ then follows at once from Theorem 81$_{97}$. $\square$

Lemma 170. *Consider any disjoint and non-empty subsets I and O of $\{1, \dots, n\}$, and any coherent rejection function R_O on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. Then $\operatorname{marg}_O \mathcal{E}\big(\mathcal{B}_{R_O}^{I \to O}\big) = R_O$.*

Proof. By Lemma 169 we know that $\mathcal{E}\big(\mathcal{B}_{R_O}^{I \to O}\big)$ is coherent and equal to $R_{\mathcal{B}_{R_O}^{I \to O}}$. By Definition 47$_{223}$, we have to show that $\mathcal{E}\big(\mathcal{B}_{R_O}^{I \to O}\big)(A) = R_O(A)$ for every A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$. Since also R_O is coherent, it suffices to prove that

$$0 \in R_{\mathcal{B}_{R_O}^{I \to O}}(A) \Leftrightarrow 0 \in R_O(A), \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R})).$$

For necessity, consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ such that $0 \in R_{\mathcal{B}_{R_O}^{I \to O}}(A)$. By Lemma 168$_{237}$ therefore $0 \in R_O(\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A\})$. Since A is a set of gambles on $\mathcal{X}_O$, therefore $\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in A\} = \{\sum_{x_I \in \mathcal{X}_I} h : h \in A\} = \{|\mathcal{X}_I| h : h \in A\} = |\mathcal{X}_I| A$, whence $0 \in R_O(|\mathcal{X}_I| A)$. Since R_O is coherent, by Axiom R4a$_{20}$ therefore indeed $0 \in R_O(A)$.

For sufficiency, consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ such that $0 \in R_O(A)$. Then $A = \mathbb{I}_{\mathcal{X}_I} A$ and $\mathcal{X}_I \in \mathcal{P}_\varnothing(\mathcal{X}_I)$, so $A \in \mathcal{B}_{R_O}^{I \to O}$, and since we know by Proposition 79_{95} that $R_{\mathcal{B}_{R_O}^{I\to O}}$ extends $\mathcal{B}_{R_O}^{I \to O}$, therefore indeed $0 \in R_{\mathcal{B}_{R_O}^{I\to O}}(A)$. □

We are now ready to find the least informative coherent rejection function that marginalises to R_O and that satisfies epistemic irrelevance of X_I to X_O. We call it the *irrelevant natural extension*, just as its counterpart for desirability in Reference [29, Theorem 13].

Theorem 171 (Irrelevant natural extension)**.** *Consider any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, and any coherent rejection function R_O on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. The least informative coherent rejection function on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ that marginalises to R_O and that satisfies epistemic irrelevance of X_I to X_O is given by*

$$\mathrm{ext}_{1:n}\big(\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big)\big) = \mathrm{ext}_{1:n}\Big(R_{\mathcal{B}_{R_O}^{I\to O}}\Big).$$

Proof. Use Lemma $169_{\curvearrowleft}$ to find already that $\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big)$ is coherent, and Proposition 161_{227} to conclude that therefore $\mathrm{ext}_{1:n}\big(\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big)\big)$ is coherent as well.

We will first show that any coherent rejection function R' on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ that marginalises to R_O and that satisfies epistemic irrelevance of X_I to X_O must be at least as informative as $\mathrm{ext}_{1:n}\big(\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big)\big)$. Consider any B in $\mathcal{B}_{R_O}^{I\to O}$, then $B = \mathbb{I}_{E_I} A$ for some E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$ and A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ such that $0 \in R_O(A)$. Since R' marginalises to R_O, therefore also $0 \in R'(A)$. Furthermore, since R' satisfies epistemic irrelevance of X_I to X_O, by Proposition 167_{236} therefore also $0 \in R'(\mathbb{I}_{E_I} A) = R'(B)$. We conclude that $B \in \mathcal{B}_{R_O}^{I\to O} \Rightarrow 0 \in R'(B) \Leftrightarrow 0 \in \mathrm{marg}_{I\cup O} R'(B)$ for every B in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R}))$, so by Definition 29_{90}, $\mathrm{marg}_{I\cup O} R'$ extends the assessment $\mathcal{B}_{R_O}^{I\to O}$. Since by Proposition 79_{95}, $\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big)$ is the least informative rejection function on $\mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R})$ that extends $\mathcal{B}_{R_O}^{I\to O}$ and that satisfies axioms $\mathrm{R2}_{20}$–$\mathrm{R4}_{20}$, we find that $\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big) \sqsubseteq \mathrm{marg}_{I\cup O} R'$. Now, use Proposition 162_{229} to infer that then $\mathrm{ext}_{1:n}\big(\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big)\big) \sqsubseteq \mathrm{ext}_{1:n}(\mathrm{marg}_{I\cup O} R')$, and, since by Proposition 161_{227}, $\mathrm{ext}_{1:n}(\mathrm{marg}_{I\cup O} R')$ is the least informative coherent rejection function on $\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})$ that marginalises to $\mathrm{marg}_{I\cup O} R'$, therefore $\mathrm{ext}_{1:n}(\mathrm{marg}_{I\cup O} R') \sqsubseteq R'$. This implies that indeed $\mathrm{ext}_{1:n}\big(\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big)\big) \sqsubseteq R'$.

It therefore suffices to prove that $\mathrm{ext}_{1:n}\big(\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big)\big)$ (i) marginalises to R_O and (ii) satisfies epistemic irrelevance of X_I to X_O. To prove (i), consider the following chain of equalities:

$$\begin{aligned}
&\mathrm{marg}_O(\mathrm{ext}_{1:n}\big(\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big)\big)) \\
&\quad = \mathrm{marg}_O(\mathrm{marg}_{I\cup O}(\mathrm{ext}_{1:n}\big(\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big)\big))) \\
&\quad = \mathrm{marg}_O(\mathcal{E}\big(\mathcal{B}_{R_O}^{I\to O}\big)) && \text{by Proposition } 161_{227} \\
&\quad = R_O && \text{by Lemma } 170_{239}.
\end{aligned}$$

To prove (ii) observe that by Proposition 167_{236} it suffices to show that $0 \in \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}^{I\to O}_{R_O}))(A) \Leftrightarrow 0 \in \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}^{I\to O}_{R_O}))(\mathbb{I}_{E_I}A)$ for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ and E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$. For necessity, consider any E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$ and any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ such that $0 \in \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}^{I\to O}_{R_O}))(A)$. Since we just have shown that $\mathrm{marg}_O(\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}^{I\to O}_{R_O}))) = R_O$, therefore equivalently $0 \in R_O(A)$, whence $\mathbb{I}_{E_I}A \in \mathcal{B}^{I\to O}_{R_O}$. Since $\mathcal{E}(\mathcal{B}^{I\to O}_{R_O}) = \mathcal{E}(\mathcal{B}^{I\to O}_{R_O})$ extends $\mathcal{B}^{I\to O}_{R_O}$, therefore $0 \in \mathcal{E}(\mathcal{B}^{I\to O}_{R_O})(\mathbb{I}_{E_I}A)$, whence by Proposition 161_{227} indeed also $0 \in \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}^{I\to O}_{R_O}))(\mathbb{I}_{E_I}A)$.

For sufficiency, consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ and E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$ such that $0 \in \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}^{I\to O}_{R_O}))(\mathbb{I}_{E_I}A)$. Since by Proposition 161_{227}, $\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}^{I\to O}_{R_O}))$ marginalises to $\mathcal{E}(\mathcal{B}^{I\to O}_{R_O})$, therefore $0 \in \mathcal{E}(\mathcal{B}^{I\to O}_{R_O})(\mathbb{I}_{E_I}A)$. Use Lemma 168_{237} to infer that then $0 \in R_O(\{\sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) : h \in \mathbb{I}_{E_I}A\}) = R_O(\{\sum_{x_I \in \mathcal{X}_I} \mathbb{I}_{E_I} f(x_I, \bullet) : f \in A\}) = R_O(\{|E_I| f : f \in A\}) = R_O(|E_I|A)$. Since R_O is coherent, [Axiom $\mathrm{R4a}_{20}$] $0 \in R_O(A)$, and, since we already know that $\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}^{I\to O}_{R_O}))$ marginalises to R_O, this implies that indeed indeed $0 \in \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}^{I\to O}_{R_O}))(A)$. $\square$

As we did for the weak extension, let us compare our results here with the ones obtained for pairwise choice in the literature. De Cooman and Miranda [29, Theorem 13] show that, given any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$ and any coherent set D_O of desirable vector-valued gambles on $\mathcal{X}_O$, its *irrelevant natural extension from X_I to X_O*—the least informative coherent set of desirable vector-valued gambles on $\mathcal{X}_{1:n}$ that marginalises to D_O and that satisfies epistemic irrelevance of X_I to X_O—exists and is given by

$$\mathrm{ext}^{\mathbf{D}}_{1:n}(B^{I\to O}_{D_O}) = \mathrm{posi}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \cup B^{I\to O}_{D_O}) \tag{7.9}$$

where $B^{I\to O}_{D_O}$ is the set of desirable vector-valued gambles on $\mathcal{X}_{I\cup O}$ that is given by[6]

$$B^{I\to O}_{D_O} := \mathrm{posi}\{\mathbb{I}_{E_I} f : f \in D_O \text{ and } E_I \in \mathcal{P}_\varnothing(\mathcal{X}_I)\}.$$

Lemma 172. *Consider any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, and any coherent set of desirable gambles $D_O \subseteq \mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. Then $B^{I\to O}_{D_O}$ is a coherent set of desirable gambles on $\mathcal{X}_{I\cup O} \times \mathcal{R}$.*[7]

[6] Actually, since De Cooman and Miranda [29] deal with the weaker notion of epistemic *value*-irrelevance rather than epistemic *subset*-irrelevance—which we use here—their version $\tilde{B}^{I\to O}_{D_O}$ of $B^{I\to O}_{D_O}$ is given by $\tilde{B}^{I\to O}_{D_O} := \mathrm{posi}\{\mathbb{I}_{\{x_I\}} f : f \in D_O \text{ and } x_I \in \mathcal{X}_I\}$. What De Cooman and Miranda prove is that the value-irrelevant natural extension for desirability exists, and that it is given by $\mathrm{ext}^{\mathbf{D}}_{1:n}(\tilde{B}^{I\to O}_{D_O})$. We will show that, as a consequence of Theorem 171, the subset-irrelevant natural extension for desirability exists, and that it is given by $\mathrm{ext}^{\mathbf{D}}_{1:n}(B^{I\to O}_{D_O})$.

[7] In Reference [29, Lemma 11] it is shown that $\tilde{B}^{I\to O}_{D_O}$ is coherent, which does not immediately imply that $B^{I\to O}_{D_O}$ is coherent. Also, we are working in the slightly more general context of vector-valued gambles.

Proof. We will show that $B_{D_O}^{I\to O}$ satisfies all the rationality axioms D1$_{57}$–D4$_{57}$. For Axiom D1$_{57}$, assume *ex absurdo* that $0 \in B_{D_O}^{I\to O}$. Then $\sum_{k=1}^{m} \lambda_k \mathbb{I}_{E_k} f_k = 0$ for some m in $\mathbb{N}$, $f_1, \dots, f_m$ in D_O, $E_1, \dots, E_m$ in $\mathcal{P}_{\varnothing}(\mathcal{X}_I)$, and $\lambda_1, \dots, \lambda_m$ in $\mathbb{R}_{>0}$, and therefore also $\sum_{x_I \in \mathcal{X}_I} \sum_{k=1}^{m} \lambda_k \mathbb{I}_{E_k} f_k(x_I, \bullet) = \sum_{k=1}^{m} \sum_{x_I \in E_k} \lambda_k \mathbb{I}_{E_k} f_k(x_I, \bullet) = \sum_{k=1}^{m} \lambda_k |E_k| f_k = 0$. Since $|E_k| \geq 1$ for every k in $\{1, \dots, m\}$, this means that a positive linear combination of elements of D_O is equal to 0, which contradicts its coherence. Therefore indeed $0 \notin B_{D_O}^{I\to O}$.

For Axiom D2$_{57}$, consider any h in $\mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R})_{>0}$. Let $E_I \coloneqq \{z_I \in \mathcal{X}_I : h(z_I, \bullet) > 0\} \subseteq \mathcal{X}_I$. Observe that $E_I \neq \varnothing$, since $h > 0$. Then $h(z_I, \bullet) > 0$ for all z_I in E_I, and since $h \geq 0$, hence also $h(x_I, \bullet) = 0$ for all x_I in E_I^c. Therefore $f \coloneqq \sum_{z_I \in \mathcal{X}_I} h(z_I, \bullet) = \sum_{z_I \in E_I} h(z_I, \bullet) > 0$, and since D_O is coherent [Axiom D2$_{57}$] we find that $f \in D_O$. Since $\frac{1}{|E_I|} > 0$, Axiom D3$_{57}$ guarantees that $\frac{1}{|E_I|} f \in D_O$. Note that $h = \sum_{x_I \in \mathcal{X}_I} \mathbb{I}_{\{x_I\}} h(x_I, \bullet) = \sum_{x_I \in E_I} \mathbb{I}_{\{x_I\}} h(x_I, \bullet) = \frac{1}{|E_I|} \sum_{x_I \in E_I} \mathbb{I}_{E_I} h(x_I, \bullet) = \mathbb{I}_{E_I} \frac{1}{|E_I|} \sum_{x_I \in E_I} h(x_I, \bullet) = \mathbb{I}_{E_I} \frac{1}{|E_I|} f$, where the second equality follow from the fact that $h(x_I, \bullet) = 0$ for every x_I in E_I^c. Since $f \in D_O$ and $E_I \in \mathcal{P}_{\varnothing}(\mathcal{X}_I)$, indeed $h \in B_{D_O}^{I\to O}$.

That $B_{D_O}^{I\to O}$ satisfies Axioms D3$_{57}$ and D4$_{57}$ follows readily from posi $\circ$ posi = posi. □

Lemma 173. *Consider any disjoint and non-empty subsets I and O of $\{1, \dots, n\}$, and any coherent set of desirable gambles on $D_O \subseteq \mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. Then $\mathrm{marg}_O B_{D_O}^{I\to O} = D_O$.*[8]

Proof. That $D_O \subseteq \mathrm{marg}_O B_{D_O}^{I\to O}$ follows at once from Equation (7.9)$_{\curvearrowleft}$ with $E_I = \mathcal{X}_I$. It therefore suffices to show that $h \in B_{D_O}^{I\to O} \Rightarrow h \in D_O$ for every h in $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. So consider any h in $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$ such that $h \in B_{D_O}^{I\to O}$. Then $h = \sum_{k=1}^{m} \lambda_k \mathbb{I}_{E_k} f_k$ for some m in $\mathbb{N}$, $f_1, \dots, f_m$ in D_O, $E_1, \dots, E_m$ in $\mathcal{P}_{\varnothing}(\mathcal{X}_I)$, and $\lambda_1, \dots, \lambda_m$ in $\mathbb{R}_{>0}$. Therefore $|\mathcal{X}_I| h = \sum_{x_I \in \mathcal{X}_I} h = \sum_{x_I \in \mathcal{X}_I} h(x_I, \bullet) = \sum_{x_I \in \mathcal{X}_I} \sum_{k=1}^{m} \lambda_k \mathbb{I}_{E_k} f_k(x_I, \bullet) = \sum_{k=1}^{m} \lambda_k |E_k| f_k$, so h is a positive linear combination of elements of D_O. Because D_O is coherent [more specifically, by Axioms D3$_{57}$ and D4$_{57}$], indeed $h \in D_O$. □

Theorem 174. *Consider any disjoint and non-empty subsets I and O of $\{1, \dots, n\}$, any coherent rejection function R_O on $\mathcal{L}(\mathcal{X}_O \times \mathcal{R})$, and any coherent set of desirable gambles on $D \subseteq \mathcal{L}(\mathcal{X}_O \times \mathcal{R})$. Then $\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O})) = R_{\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O})}$. As a consequence, $\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O}) = D_{\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))}$. Moreover, $\mathrm{ext}_{1:n}^{\mathbf{D}}(B_{D_R}^{I\to O}) \subseteq D_{\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_R^{I\to O}))}$.*

[8] In Reference [29, Lemma 12] it is shown that $\mathrm{marg}_O \tilde{B}_{D_O}^{I\to O} = D_O$, which does not immediately imply that $\mathrm{marg}_O B_{D_O}^{I\to O} = D_O$. Also, we are working in the slightly more general context of vector-valued gambles.

Proof. We begin with the first statement—$\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O})) = R_{\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O})}$. To show that $\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O})) \sqsubseteq R_{\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O})}$, as an intermediate result, we will first establish that the purely binary assessment $\mathcal{B}_D^{I\to O} \coloneqq \{\{0,\mathbb{I}_{E_I}f\} : f \in D \text{ and } E_I \in \mathcal{P}_\varnothing(\mathcal{X}_I)\} \subseteq \mathcal{Q}_0(\mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R}))$ is at least as strong as $\mathcal{B}_{R_D}^{I\to O}$. To show this, by Definition 30_{91} we need to show that $(\forall B \in \mathcal{B}_{R_D}^{I\to O})(\exists B' \in \mathcal{B}_D^{I\to O})B' \preccurlyeq B$, so consider any B in $\mathcal{B}_{R_D}^{I\to O}$. Then $B = \mathbb{I}_{E_I}A$ for some E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$ and A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_O \times \mathcal{R}))$ such that $0 \in R_D(A)$, whence, by Proposition 55_{64}, $A \cap D \neq \varnothing$, so $g \in D$ for some g in A. Then $0 \in R_D(\{0,g\})$, so $\mathbb{I}_{E_I}g \in B$, and let $B' \coloneqq \{0,\mathbb{I}_{E_I}g\} \in \mathcal{B}_D^{I\to O}$. Since also 0 belongs to A and hence to B, $B' \subseteq B$, whence, by Proposition 33_{43}(i), $B' \preccurlyeq B$. So $\mathcal{B}_D^{I\to O}$ is indeed at least as strong an assessment as $\mathcal{B}_{R_D}^{I\to O}$, whence by Corollary 76_{92}, $\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}) \sqsubseteq \mathcal{E}(\mathcal{B}_D^{I\to O})$. Furthermore, since $\mathcal{B}_D^{I\to O} \subseteq \mathcal{B}_{B_D^{I\to O}} \coloneqq \{\{0,f\} : f \in B_D^{I\to O}\} = \{\{0,f\} : f \in \mathrm{posi}\{\mathbb{I}_{E_I}g : g \in D \text{ and } E_I \in \mathcal{P}_\varnothing(\mathcal{X}_I)\}\}$, $\mathcal{B}_{B_D^{I\to O}}$ is in turn at least as strong an assessment as $\mathcal{B}_D^{I\to O}$, whence, by Corollary 76_{92}, $\mathcal{E}(\mathcal{B}_D^{I\to O}) \sqsubseteq \mathcal{E}(\mathcal{B}_{B_D^{I\to O}})$. Therefore $\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}) \sqsubseteq \mathcal{E}(\mathcal{B}_{B_D^{I\to O}})$. But $\mathcal{B}_{B_D^{I\to O}}$ is the purely binary assessment based on $B_D^{I\to O}$, which is a coherent set of desirable gambles by Lemma 172_{241}. We can interpret the latter as a desirability assessment, so $\mathcal{E}^{\mathbf{D}}(B_D^{I\to O}) = B_D^{I\to O}$. Since it is coherent, it avoids non-positivity, and therefore Theorem 86_{100} implies that $\mathcal{E}(\mathcal{B}_{B_D^{I\to O}}) = R_{\mathcal{E}^{\mathbf{D}}(B_D^{I\to O})} = R_{B_D^{I\to O}}$. So we have that $\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}) \sqsubseteq R_{B_D^{I\to O}}$. By Proposition 162_{229}, then, $\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O})) \sqsubseteq \mathrm{ext}_{1:n}(R_{B_D^{I\to O}})$, and by Proposition 163_{230}, $\mathrm{ext}_{1:n}(R_{B_D^{I\to O}}) = R_{\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O})}$. Therefore indeed $\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O})) \sqsubseteq R_{\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O})}$.

To establish the converse inequality, namely that $R_{\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O})} \sqsubseteq \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))$, since they are both coherent rejection functions, it suffices by Axiom R4b to show that

$$0 \in R_{\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O})}(A) \Rightarrow 0 \in \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))(A), \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})).$$

So consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R}))$ such that $0 \in R_{\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O})}(A)$. By Proposition 55_{64} then $A \cap \mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O}) \neq \varnothing$, so $f \in \mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O}) = \mathrm{posi}(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \cup B_D^{I\to O})$ for some f in A. Use Lemma 1_{11} to infer that then $f \in \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \cup B_D^{I\to O} \cup \big(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} + B_D^{I\to O}\big) = \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0} \cup \big(\mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{\geq 0} + B_D^{I\to O}\big)$. If $f \in \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{>0}$, then by the coherence [more specifically, by Axioms $\mathrm{R2}_{20}$ and $\mathrm{R3a}_{20}$] of $\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))$, indeed $0 \in \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))(A)$. If $f \in \mathcal{L}(\mathcal{X}_{1:n} \times \mathcal{R})_{\geq 0} + B_D^{I\to O}$, then $f \geq f'$ for some f' in $B_D^{I\to O}$. Since f' belongs to $B_D^{I\to O}$, by Proposition 55_{64}, we find that $0 \in R_{B_D^{I\to O}}(\{0,f'\})$ Note that $B_D^{I\to O} = \mathcal{E}^{\mathbf{D}}(B)$ where we let $B \coloneqq \{\mathbb{I}_{E_I}f : f \in D \text{ and } E_I \in \mathcal{P}_\varnothing(\mathcal{X}_I)\} \subseteq \mathcal{L}(\mathcal{X}_{I\cup O} \times \mathcal{R})$, and by Theorem 85_{100} therefore $R_{B_D^{I\to O}} = \mathcal{E}(\mathcal{B}_B)$. Then $\mathcal{B}_B = \{\{0,\mathbb{I}_{E_I}h\} : h \in D \text{ and } E_I \in \mathcal{P}_\varnothing(\mathcal{X}_I)\} \subseteq \{\mathbb{I}_{E_I}A : 0 \in A \text{ and } A \cap D \neq \varnothing \text{ and } E_I \in \mathcal{P}_\varnothing(\mathcal{X}_I)\} = \{\mathbb{I}_{E_I}A : 0 \in R_D(A) \text{ and } E_I \in \mathcal{P}_\varnothing(\mathcal{X}_I)\} = \mathcal{B}_{R_D}^{I\to O}$, so $\mathcal{B}_{R_D}^{I\to O}$ is at least as strong an assessment as $\mathcal{B}_B$, whence by Corollary 76_{92}, $\mathcal{E}(\mathcal{B}_B) \sqsubseteq \mathcal{E}(\mathcal{B}_{R_D}^{I\to O})$, and therefore $R_{B_D^{I\to O}} \sqsubseteq \mathcal{E}(\mathcal{B}_{R_D}^{I\to O})$. Since we already know that $0 \in R_{B_D^{I\to O}}(\{0,f'\})$, we get that $0 \in \mathcal{E}(\mathcal{B}_{R_D}^{I\to O})(\{0, f'\})$. Since by Proposition 161_{227}, $\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))$ marginalises to $\mathcal{E}(\mathcal{B}_{R_D}^{I\to O})$, this tells us that $0 \in \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))(\{0, f'\})$. Since $f \geq f'$, by Proposition 30_{41}(ii), $0 \in \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))(\{0, f\})$, and since it is coherent, indeed $0 \in \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))(A)$.

For the second statement—$\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O}) = D_{\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))}$—infer from the first one that $D_{\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))} = D_{R_{\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O})}}$. By Corollary 59_{67}, the latter is equal to $\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O})$, and therefore indeed $\mathrm{ext}_{1:n}^{\mathbf{D}}(B_D^{I\to O}) = D_{\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_D}^{I\to O}))}$.

The final statement—$\mathrm{ext}_{1:n}^{\mathbf{D}}(B_{D_R}^{I\to O}) \subseteq D_{\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_R^{I\to O}))}$—can be obtained by plugging the specific set of desirable gambles D_R into the second statement: we find that then $\mathrm{ext}_{1:n}^{\mathbf{D}}(B_{D_R}^{I\to O}) = D_{\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_{D_R}}^{I\to O}))}$. Use Corollary 59_{67} to infer that $R_{D_R} \sqsubseteq R$ and therefore $\mathcal{B}_{R_{D_R}}^{I\to O} \subseteq \mathcal{B}_R^{I\to O}$. So $\mathcal{B}_R^{I\to O}$ is at least as strong an assessment as $\mathcal{B}_{R_{D_R}}^{I\to O}$, and therefore $\mathcal{E}(\mathcal{B}_{R_{D_R}}^{I\to O}) \sqsubseteq \mathcal{E}(\mathcal{B}_R^{I\to O})$. By Proposition 162_{229} therefore $\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_{D_R}}^{I\to O})) \sqsubseteq \mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_R^{I\to O}))$, so by Proposition 60_{68}, $D_{\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_{R_{D_R}}^{I\to O}))} \subseteq D_{\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_R^{I\to O}))}$. Therefore indeed $\mathrm{ext}_{1:n}^{\mathbf{D}}(B_{D_R}^{I\to O}) \subseteq D_{\mathrm{ext}_{1:n}(\mathcal{E}(\mathcal{B}_R^{I\to O}))}$. □

Theorem 174_{242} implies that the irrelevant natural extension of a purely binary rejection function R_D is a rejection function that is purely binary itself. This claim is by no means trivial: it implies that, when working in the framework of choice models, epistemic (subset-)irrelevance assessments preserve pairwise behaviour, and—even stronger—preserves the property of being purely binary.

7.5 CONCLUSION

In this chapter we have introduced multivariate choice functions and their basic operations: marginalisation, weak extension and conditioning. Furthermore, we have shown how to model an epistemic irrelevance assessment, and found the natural extension of such an assessment.

Credal networks (see, for instances, References [15, 18, 26] for a good introduction) have been—and still are—the subject of intense research. A credal network is a global uncertainty model for a finite number of variables. This global model is obtained by combining local uncertainty models using independence statements, such as, for instance, epistemic irrelevance. We can use this global uncertainty model to make specific inferences about some variables. The material in this chapter is the first necessary stepping stone to be able to work—at least on a theoretical level—with simple credal networks using choice functions as local models, and epistemic irrelevance as the associated independence notion. As a simple example, forward irrelevance[9] combines a number of epistemic irrelevance assessments, and, in principle, all the tools needed to study this type of assessment with choice models are in place.

However, studies of more general but still quite basic credal networks—such as networks whose graphs are trees—are not (yet) in the range of what is currently feasible with multivariate choice functions. Such networks mostly have *symmetric* irrelevances embedded in their structure: an irrelevance statement from one variable to another, and *vice versa*. Such symmetric statements are called *epistemic independence* statements. Therefore, an expression for—and the guarantee of the existence of—the independent natural extension is necessary to find the joint choice function of a general credal network whose local models are choice functions. In Reference [29], De Cooman and Miranda have found this natural extension for desirability. For choice models, however, the independent natural extension has received no attention thus far.

Let us go into a bit more detail. The variables X_1, …, X_n are called independent when learning about the values of any number of them does not influence or change our beliefs about the remaining ones:[10] for any two disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, X_I is epistemically irrelevant to X_O. A coherent choice function C on $\mathcal{L}(\mathcal{X}_{1:n}\times\mathcal{R})$ is called epistemically (subset-)independent[11] if

$$\mathrm{marg}_O(C\rfloor E_I) = \mathrm{marg}_O C$$

for *all* disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, and E_I in $\mathcal{P}_\varnothing(\mathcal{X}_I)$.

Suppose we have n coherent rejection functions R_k on $\mathcal{L}(\mathcal{X}_k\times\mathcal{R})$, one for every k in $\{1,\dots,n\}$, and an assessment that the variables X_1, …, X_n are epistemically independent. The independent natural extension is the least informative coherent rejection R function that is independent and such that $\mathrm{marg}_{\{k\}}R = R_k$ for every k in $\{1,\dots,n\}$. The results in Reference [29] lead me to suspect that the independent natural extension is the natural extension of the following assessment:

$$\mathcal{B} \coloneqq \bigcup_{k\in\{1,\dots,n\}} \{\mathbb{I}_{E_{k^c}}A : A\in\mathcal{Q}(\mathcal{L}(\mathcal{X}_k\times\mathcal{R})) \text{ and } 0\in R_k(A) \text{ and } E_{k^c}\in\mathcal{P}_\varnothing(\mathcal{X}_{k^c})\},$$

and that this assessment avoids complete rejection. However, to prove that $\mathcal{E}(\mathcal{B})$ is even coherent—let alone that it marginalises to R_k or is independent—a more profound study of the properties of the rejection function $R_{\mathcal{B}}$, as defined in Equation (3.1)$_{92}$, shall be needed. This is beyond the scope of this dissertation.

[9] See Reference [28] for a treatment of forward irrelevance with coherent lower previsions.

[10] This is called *many-to-many independence*, as opposed to the less stringent *many-to-one independence*; for more information, see Reference [30].

[11] The epistemically value-independence requirement would be $\mathrm{marg}_O(C\rfloor x_I) = \mathrm{marg}_O C$ for all disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, and x_I in $\mathcal{X}_I$.

8

EXCHANGEABILITY

In this chapter, we study how to model exchangeability, using choice functions. This work builds on earlier results by De Cooman et al. [31, 32] for sets of desirable gambles.

Exchangeability is a structural assessment on a sequence of variables that is important for inference purposes. Loosely speaking, making a judgement of exchangeability means that the order in which the variables are observed, is considered irrelevant. This irrelevancy is typically modelled through an invariance or indifference assessment. The first detailed study of exchangeability was given by de Finetti [34]. We refer to the paper by De Cooman and Quaeghebeur [31, Section 1] for a brief historical overview.

In Section 8.1, we derive de Finetti-like Representation Theorems for a finite sequence that is exchangeable. We take this one step further in Section 8.2263, where we consider countable exchangeable sequences and derive a representation theorem for them as well. To compare with earlier work by De Cooman and Quaeghebeur [31], we also provide corresponding representation theorems for sets of desirable gambles.

8.1 FINITE EXCHANGEABILITY

Consider n in $\mathbb{N}$ variables $X_1, \dots, X_n$ taking values in a non-empty *finite* set $\mathcal{X}$. The possibility space of the uncertain sequence $X = (X_1, \dots, X_n)$ is $\mathcal{X}^n$.

We denote by $x = (x_1, \dots, x_n)$ an arbitrary element of $\mathcal{X}^n$. For any n in $\mathbb{N}$ we let $\mathcal{P}_n$ be the group of all permutations of the index set $\{1, \dots, n\}$. There are $|\mathcal{P}_n| = n!$ such permutations. With any such permutation π, we associate a permutation of $\mathcal{X}^n$, also denoted by π, and defined by $(\pi x)_k \coloneqq x_{\pi(k)}$ for every k in $\{1, \dots, n\}$, or in other words, $\pi(x_1, \dots, x_n) = (x_{\pi(1)}, \dots, x_{\pi(n)})$: πx is obtained from x by permuting the indices of its components. Similarly, we lift π to a permutation π^t on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ by letting $(\pi^t f)(x,r) := f(\pi x, r)$ for all x in $\mathcal{X}^n$ and r in $\mathcal{R}$. Observe that π^t is a linear permutation of the vector space $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ of all vector-valued gambles on $\mathcal{X}^n$.

If a subject assesses that the sequence of variables X in $\mathcal{X}^n$ is exchangeable, this means that he is indifferent between any gamble f in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ and its permuted variant $\pi^t f$, for all π in $\mathcal{P}_n$. This leads to the following set of indifferent gambles:

$$I_{\mathcal{P}_n} := \operatorname{span}\{f - \pi^t f : f \in \mathcal{L}(\mathcal{X}^n \times \mathcal{R}) \text{ and } \pi \in \mathcal{P}_n\}. \tag{8.1}$$

Definition 49 (Finite exchangeability). *A choice function C on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ is called* (finitely) exchangeable *if it is compatible with $I_{\mathcal{P}_n}$. Similarly, a set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ is called* (finitely) exchangeable *if it is compatible with $I_{\mathcal{P}_n}$.*

Of course, so far, we do not yet know whether this notion of exchangeability is well-defined: indeed, we do not know yet whether $I_{\mathcal{P}_n}$ is a *coherent* set of indifferent gambles, in the sense of Definition 38_{176}. In the next section, we will show that this is indeed the case. But once we have established that $I_{\mathcal{P}_n}$ is a coherent set of indifferent gambles, exchangeability is nothing more fancy than compatibility with $I_{\mathcal{P}_n}$. The notion of compatibility of choice functions and sets of desirable gambles with a set of indifferent gambles was studied in some detail in Chapter 5_{175}. Proposition 124_{180} there gives a representation result in terms of equivalence classes of (vector-valued) gambles on $\mathcal{X}^n$. What we will do below, is use this general representation result to obtain a particular equivalent representation result for exchangeable choice functions, in terms of (vector-valued) gambles on count vectors.

8.1.1 *Count vectors*

Let us now provide the tools necessary to prove that $I_{\mathcal{P}_n}$ is a coherent set of indifferent gambles, as introduced in Definition 38_{176}.

The *permutation invariant atoms* $[x] := \{\pi x : x \in \mathcal{X}^n\}$, with x in $\mathcal{X}^n$, are the smallest permutation invariant subsets of $\mathcal{X}^n$. We consider the *counting map*

$$T : \mathcal{X}^n \to \mathcal{N}^n : x \mapsto T(x)$$

where $T(x)$ is called the *count vector* of x. It is the $\mathcal{X}$-tuple with components $T_z(x) := |\{k \in \{1,\ldots,n\} : x_k = z\}|$ for all z in $\mathcal{X}$, so $T_z(x)$ is the number of times that z occurs in the sequence $x_1,\ldots,x_n$. The range of T—the set $\mathcal{N}^n$—is called the set of possible count vectors and is given by

$$\mathcal{N}^n := \Big\{m \in \mathbb{Z}_{\geq 0}^{\mathcal{X}} : \sum_{x \in \mathcal{X}} m_x = n\Big\}.$$

Applying any permutation to x leaves its result under the counting map unchanged:

$$T(x) = T(\pi x) \text{ for al } x \text{ in } \mathcal{X}^n \text{ and } \pi \text{ in } \mathcal{P}_n.$$

For any x in $\mathcal{X}^n$, if $m = T(x)$ then $[x] = \{y \in \mathcal{X}^n : T(y) = m\}$, so the permutation invariant atom $[x]$ is completely determined by the count vector m of all its elements, and is therefore also denoted by $[T(x)] = [m]$. Remark that $\{[m] : m \in \mathcal{N}^n\}$ partitions $\mathcal{X}^n$ into disjoint parts with constant count vectors, and that $|[m]| = \binom{n}{m} := \frac{n!}{\prod_{z \in \mathcal{X}} m_z!}$.

In order to extend the application of the count vectors for use with gambles, let us define the *set of all permutation invariant vector-valued gambles* as

$$\mathcal{L}_{\mathcal{P}_n}(\mathcal{X}^n \times \mathcal{R}) := \{f \in \mathcal{L}(\mathcal{X}^n \times \mathcal{R}) : (\forall \pi \in \mathcal{P}_n)\pi^t f = f\} \subseteq \mathcal{L}(\mathcal{X}^n \times \mathcal{R}),$$

and a special transformation $\mathrm{inv}_{\mathcal{P}_n}$ of the linear space $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ given by

$$\mathrm{inv}_{\mathcal{P}_n} : \mathcal{L}(\mathcal{X}^n \times \mathcal{R}) \to \mathcal{L}(\mathcal{X}^n \times \mathcal{R}) : f \mapsto \mathrm{inv}_{\mathcal{P}_n}(f) := \frac{1}{n!} \sum_{\pi \in \mathcal{P}_n} \pi^t f,$$

which, as we will see in the following proposition, is closely linked with $\mathcal{L}_{\mathcal{P}_n}(\mathcal{X}^n \times \mathcal{R})$ (see also References [31,78]). To see how this comes about, note that $\mathrm{inv}_{\mathcal{P}_n}$ is a special case of the transformation $\mathrm{inv}_{\mathcal{P}}$ defined in Section 5.8_{191}. The following result is a direct consequence of Proposition 138_{192}:

Proposition 175. $\mathrm{inv}_{\mathcal{P}_n}$ *is a linear transformation of* $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$*, and*
(i) $\mathrm{inv}_{\mathcal{P}_n} \circ \pi^t = \mathrm{inv}_{\mathcal{P}_n} = \pi^t \circ \mathrm{inv}_{\mathcal{P}_n}$ *for all* π *in* $\mathcal{P}$*;*
(ii) $\mathrm{inv}_{\mathcal{P}_n} \circ \mathrm{inv}_{\mathcal{P}_n} = \mathrm{inv}_{\mathcal{P}_n}$*;*
(iii) $\ker(\mathrm{inv}_{\mathcal{P}_n}) = I_{\mathcal{P}_n}$*;*
(iv) $\mathrm{rng}(\mathrm{inv}_{\mathcal{P}_n}) = \mathcal{L}_{\mathcal{P}_n}(\mathcal{X}^n \times \mathcal{R})$.
Moreover, for any f *and* g *in* $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$*, we have that* $g \in f / I_{\mathcal{P}_n} \Leftrightarrow \mathrm{inv}_{\mathcal{P}_n} g = \mathrm{inv}_{\mathcal{P}_n} f$.

So we see that $\mathrm{inv}_{\mathcal{P}_n}$ is a linear projection operator that renders a vector-valued gamble insensitive to permutation (or permutation invariant) by replacing it with the uniform average of all its permutations. As a result, it assumes the same value for all vector-valued gambles that can be related to each other through some permutation: $\mathrm{inv}_{\mathcal{P}_n}(f) = \mathrm{inv}_{\mathcal{P}_n}(g)$ if $f = \pi^t g$ for some π in $\mathcal{P}_n$, for all f and g in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$. Furthermore, for any f in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$, its transformation $\mathrm{inv}_{\mathcal{P}_n}(f)$ is permutation invariant and therefore constant on the permutation invariant atoms $[m]$: $(\mathrm{inv}_{\mathcal{P}_n}(f))(x,r) = (\mathrm{inv}_{\mathcal{P}_n}(f))(y,r)$ if $[x] = [y]$, for all x and y in $\mathcal{X}^n$, and r in $\mathcal{R}$. We can use the properties of $\mathrm{inv}_{\mathcal{P}_n}$ to prove that $I_{\mathcal{P}_n}$ is coherent and therefore well suited for our definition of exchangeability.

Proposition 176. *For any* n *in* $\mathbb{N}$*, the set* $I_{\mathcal{P}_n}$*, defined in Equation* (8.1)*, is a coherent set of indifferent vector-valued gambles.*

Proof. For Axiom $\mathrm{I1}_{176}$, since $I_{\mathcal{P}_n}$ is a linear hull by its Definition $(8.1)_{248}$, 0 is included in $I_{\mathcal{P}_n}$. For Axiom $\mathrm{I2}_{176}$, consider any f in $I_{\mathcal{P}_n}$ and assume *ex absurdo* that $f \in \mathcal{L}(\mathcal{X}^n \times \mathcal{R})_{>0} \cup \mathcal{L}(\mathcal{X}^n \times \mathcal{R})_{<0}$. If $f \in \mathcal{L}(\mathcal{X}^n \times \mathcal{R})_{>0}$ then $\pi^t f \in \mathcal{L}(\mathcal{X}^n \times \mathcal{R})_{>0}$ for all π in $\mathcal{P}_n$, and therefore $\mathrm{inv}_{\mathcal{P}_n}(f) > 0$, a contradiction with Proposition 175↶(iii). If $f \in \mathcal{L}(\mathcal{X}^n \times \mathcal{R})_{<0}$ then, similarly $\mathrm{inv}_{\mathcal{P}_n}(f) < 0$, again a contradiction with Proposition 175↶(iii). Axioms $\mathrm{I3}_{176}$ and $\mathrm{I4}_{176}$ are satisfied because $I_{\mathcal{P}_n}$ is a linear hull. □

Since $I_{\mathcal{P}_n}$ is coherent, exchangeability is well-defined: a choice function C_n on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ and a set of desirable vector-valued gambles $D_n \subseteq \mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ are exchangeable if they are compatible with the *coherent* set of indifferent vector-valued gambles $I_{\mathcal{P}_n}$. By Definition 40_{179}, C_n is therefore represented by a choice function C' on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$, and similarly, by Definition 39_{176} D_n is represented by a set of desirable gambles $D' \subseteq \mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$. So we can focus on the quotient space and its elements: equivalence classes of mutually indifferent vector-valued gambles.

But before we do that in the next section, it will pay to further explore the notions we have introduced thus far. Consider any f in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$. What is the constant value that $\mathrm{inv}_{\mathcal{P}_n}(f)$ assumes on a permutation invariant atom $[m]$? To answer this question, consider any x in $[m]$, then

$$\begin{aligned}(\mathrm{inv}_{\mathcal{P}_n}(f))(x,\bullet) &= \frac{1}{n!}\sum_{\pi\in\mathcal{P}_n} f(\pi x,\bullet) = \frac{1}{n!}\frac{|\mathcal{P}_n|}{|[m]|}\sum_{y\in\{\pi x:\pi\in\mathcal{P}_n\}} f(y,\bullet)\\ &= \frac{1}{\binom{n}{m}}\sum_{y\in[x]} f(y,\bullet) = \frac{1}{\binom{n}{m}}\sum_{y\in[m]} f(y,\bullet)\end{aligned}$$

where we used the fact that $|\mathcal{P}_n| = n!$ and $|[m]| = \binom{n}{m}$. Therefore, for all r in $\mathcal{R}$,

$$(\mathrm{inv}_{\mathcal{P}_n}(f))(\bullet,r) = \sum_{m\in\mathcal{N}^n} \mathrm{H}_n(f(\bullet,r)|m)\mathbb{I}_{[m]}, \tag{8.2}$$

where $\mathrm{H}_n(\bullet|m)$ is the linear expectation operator associated with the uniform distribution on the invariant atom $[m]$:

$$\mathrm{H}_n(g|m) \coloneqq \frac{1}{\binom{n}{m}}\sum_{y\in[m]} g(y) \text{ for all } g \text{ in } \mathcal{L}(\mathcal{X}^n) \text{ and } m \text{ in } \mathcal{N}^n. \tag{8.3}$$

It characterises a (multivariate) hyper-geometric distribution [44], associated with random sampling without replacement from an urn with n balls of types $\mathcal{X}$, whose composition is characterised by the count vector m.

The result of subjecting a gamble f on $\mathcal{X}^n \times \mathcal{R}$ to the map

$$\begin{aligned}&\mathrm{H}_n : \mathcal{L}(\mathcal{X}^n \times \mathcal{R}) \to \mathcal{L}(\mathcal{N}^n \times \mathcal{R}) : f \mapsto \mathrm{H}_n(f)\\ &\qquad \text{with } (\mathrm{H}_n(f))(m,r) \coloneqq \mathrm{H}_n(f(\bullet,r)|m) \text{ for all } m \text{ in } \mathcal{N} \text{ and } r \text{ in } \mathcal{R},\end{aligned} \tag{8.4}$$

is the gamble $\mathrm{H}_n(f)$ on $\mathcal{N}^n \times \mathcal{R}$ that assumes the value $\frac{1}{\binom{n}{m}}\sum_{y\in[m]} f(y,r)$ in every m in $\mathcal{N}^n$ and r in $\mathcal{R}$.

8.1.2 *Exchangeable equivalence classes of gambles*

We already know that exchangeable choice functions are represented by choice functions on the quotient space $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$, and similarly for sets of desirable gambles. In the quest for an elegant representation theorem, we thus need to focus on the quotient space $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ and its elements, which are 'exchangeable' equivalence classes of vector-valued gambles.

In this section we investigate how the representation of permutation invariant gambles helps us find a representation result for exchangeable choice functions. This representation will use equivalence classes $[f] := f/I_{\mathcal{P}_n} = \{f\} + I_{\mathcal{P}_n}$ of gambles, for any f in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$. Recall that the quotient space $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n} := \{[f] : f \in \mathcal{L}(\mathcal{X}^n \times \mathcal{R})\}$ is a linear space itself, with additive identity $[0] = I_{\mathcal{P}_n}$, and therefore any element $\tilde{f}$ of $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ is invariant under addition of $I_{\mathcal{P}_n}$: $\tilde{f} + I_{\mathcal{P}_n} = \tilde{f}$. Elements of $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ will be generically denoted by $\tilde{f}$ or $\tilde{g}$.

Proposition 177. *Consider any f and g in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$. Then $[f] = [g]$ if and only if $\mathrm{H}_n(f) = \mathrm{H}_n(g)$.*

Proof. By Proposition 175249 we have that $[f] = [g] \Leftrightarrow \mathrm{inv}_{\mathcal{P}_n} f = \mathrm{inv}_{\mathcal{P}_n} g$, so it suffices to show that $\mathrm{inv}_{\mathcal{P}_n} f = \mathrm{inv}_{\mathcal{P}_n} g \Leftrightarrow \mathrm{H}_n(f) = \mathrm{H}_n(g)$. Observe that

$$\begin{aligned}
\mathrm{inv}_{\mathcal{P}_n} f = \mathrm{inv}_{\mathcal{P}_n} g &\Leftrightarrow (\forall r \in \mathcal{R})\mathrm{inv}_{\mathcal{P}_n}(f)(\bullet, r) = \mathrm{inv}_{\mathcal{P}_n}(g)(\bullet, r) \\
&\Leftrightarrow (\forall r \in \mathcal{R}) \sum_{m \in \mathcal{N}^n} \mathrm{H}_n(f(\bullet, r)|m)\mathbb{I}_{[m]} = \sum_{m \in \mathcal{N}^n} \mathrm{H}_n(g(\bullet, r)|m)\mathbb{I}_{[m]} \\
&\Leftrightarrow (\forall m \in \mathcal{N}^n, r \in \mathcal{R})\mathrm{H}_n(f(\bullet, r)|m) = \mathrm{H}_n(g(\bullet, r)|m) \\
&\Leftrightarrow (\forall m \in \mathcal{N}^n, r \in \mathcal{R})\mathrm{H}_n(f)(m, r) = \mathrm{H}_n(g)(m, r) \\
&\Leftrightarrow \mathrm{H}_n(f) = \mathrm{H}_n(g),
\end{aligned}$$

where the second equivalence follows from Equation (8.2) and the fourth one from Equation (8.4). □

Therefore, H_n is constant on the exchangeable equivalence classes of vector-valued gambles: for any $\tilde{f}$ in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$, we have that $\mathrm{H}_n(f) = \mathrm{H}_n(g)$ for all f and g in $\tilde{f}$. This means that the map $\tilde{\mathrm{H}}_n$, defined by:

$$\tilde{\mathrm{H}}_n : \mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n} \to \mathcal{L}(\mathcal{N}^n \times \mathcal{R}) : \tilde{f} \mapsto \mathrm{H}_n(f) \text{ for any } f \text{ in } \tilde{f}. \tag{8.5}$$

is well defined. Then Proposition 177 guarantees that elements of $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ can also be characterised using $\tilde{\mathrm{H}}_n$, in the sense that $\tilde{f} = \{f \in \mathcal{L}(\mathcal{X}^n \times \mathcal{R}) : \mathrm{H}_n(f) = \tilde{\mathrm{H}}_n(\tilde{f})\}$ for all $\tilde{f}$ in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$.

The map $\tilde{\mathrm{H}}_n$ takes as an argument any equivalence class of gambles, and maps it to some representing gamble on the count vectors. It will be useful later on to consider the map $\tilde{\mathrm{H}}_n^{-1}$:

$$\tilde{\mathrm{H}}_n^{-1} : \mathcal{L}(\mathcal{N}^n \times \mathcal{R}) \to \mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n} : h \mapsto \Big[\sum_{m \in \mathcal{N}^n} h(m, \bullet)\mathbb{I}_{[m]}\Big]. \tag{8.6}$$

The notation of $\tilde{\mathrm{H}}_n^{-1}$ is suggestive: as it turns out, $\tilde{\mathrm{H}}_n$ and $\tilde{\mathrm{H}}_n^{-1}$ indeed are each other's inverses, and are therefore bijective.

Proposition 178. *The maps* $\tilde{\mathrm{H}}_n$ *as defined in Equation* (8.5)$_{\curvearrowleft}$ *and* $\tilde{\mathrm{H}}_n^{-1}$ *as defined in Equation* (8.6)$_{\curvearrowleft}$ *are each other's inverses.*

Proof. This proof is structured as follows: we show that (i) $\tilde{\mathrm{H}}_n^{-1} \circ \tilde{\mathrm{H}}_n = \mathrm{id}_{\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}}$, and (ii) $\tilde{\mathrm{H}}_n \circ \tilde{\mathrm{H}}_n^{-1} = \mathrm{id}_{\mathcal{L}(\mathcal{N}^n \times \mathcal{R})}$, together implying that $\tilde{\mathrm{H}}_n$ and $\tilde{\mathrm{H}}_n^{-1}$ are each other's inverses.

For (i), consider any $\tilde{f}$ in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$. We need to show that then $\tilde{\mathrm{H}}_n^{-1}(\tilde{\mathrm{H}}_n(\tilde{f})) = \tilde{f}$. Let h be an arbitrary element of $\tilde{f}$, and $f \coloneqq \mathrm{inv}_{\mathcal{P}_n}(h)$. By Proposition 175_{249}(ii) then $\mathrm{inv}_{\mathcal{P}_n}(f) = \mathrm{inv}_{\mathcal{P}_n}(h)$, so f belongs to $\tilde{f}$ as well. Therefore $\tilde{\mathrm{H}}_n(\tilde{f})$ assumes the value $\mathrm{H}_n(f)(m,\bullet) = \frac{1}{\binom{n}{m}} \sum_{y \in [m]} f(y,\bullet)$ on every m in $\mathcal{N}^n$. But f is constant on every permutation invariant atom $[m]$, so $\mathrm{H}_n(f)(m,\bullet) = \frac{1}{\binom{n}{m}} |[m]| f(x,\bullet) = f(x,\bullet)$ for every x in $[m]$, and therefore

$$f = \sum_{m \in \mathcal{N}^n} \mathrm{H}_n(f)(m,\bullet)\mathbb{I}_{[m]} = \sum_{m \in \mathcal{N}^n} \tilde{\mathrm{H}}_n(\tilde{f})(m,\bullet)\mathbb{I}_{[m]}, \tag{8.7}$$

where the second equality holds by Equation (8.5)$_{\curvearrowleft}$ and since $f \in \tilde{f}$. Then indeed $\tilde{\mathrm{H}}_n^{-1}(\tilde{\mathrm{H}}_n(\tilde{f})) = [\sum_{m \in \mathcal{N}^n} \tilde{\mathrm{H}}_n(\tilde{f})(m,\bullet)\mathbb{I}_{[m]}] = [f] = \tilde{f}$, where the first equality follows from Equation (8.6)$_{\curvearrowleft}$, the second one from Equation (8.7), and the last one from the fact that $f \in \tilde{f}$ and therefore $[f] = \tilde{f}$.

For (ii), consider any h in $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$. We need to show that then $\tilde{\mathrm{H}}_n(\tilde{\mathrm{H}}_n^{-1}(h)) = h$. Let $f \coloneqq \sum_{m \in \mathcal{N}^n} h(m,\bullet)\mathbb{I}_{[m]}$, a gamble on $\mathcal{X}^n \times \mathcal{R}$. Then $\tilde{\mathrm{H}}_n^{-1}(h) = [f]$ by Equation (8.6)$_{\curvearrowleft}$, so $\tilde{\mathrm{H}}_n(\tilde{\mathrm{H}}_n^{-1}(h)) = \tilde{\mathrm{H}}_n([f])$, and since $f \in [f]$, we find using Equation (8.5)$_{\curvearrowleft}$ that $\tilde{\mathrm{H}}_n([f]) = \mathrm{H}_n(f)$ and therefore $\tilde{\mathrm{H}}_n(\tilde{\mathrm{H}}_n^{-1}(h)) = \mathrm{H}_n(f)$. The proof is finished if we can show that $\mathrm{H}_n(f) = h$. Consider any m' in $\mathcal{N}^n$, and observe that

$$\begin{aligned}
\mathrm{H}_n(f)(m',\bullet) = \frac{1}{\binom{n}{m'}} \sum_{y \in [m']} f(y,\bullet) &= \frac{1}{\binom{n}{m'}} \sum_{y \in [m']} \sum_{m \in \mathcal{N}^n} h(m,\bullet)\mathbb{I}_{[m]}(y) \\
&= \sum_{m \in \mathcal{N}^n} h(m,\bullet) \frac{1}{\binom{n}{m'}} \sum_{y \in [m']} \mathbb{I}_{[m]}(y) \\
&= \sum_{m \in \mathcal{N}^n} h(m,\bullet)\mathbb{I}_{\{m\}}(m') = h(m',\bullet),
\end{aligned}$$

where the first equality follows from Equation $(8.3)_{250}$ and the penultimate one from the facts that $\{[m] : m \in \mathcal{N}^n\}$ partitions $\mathcal{X}^n$ and $|[m']| = \binom{n}{m'}$, so

$$\frac{1}{\binom{n}{m'}} \sum_{y \in [m']} \mathbb{I}_{[m]}(y) = \begin{cases} 1 & \text{if } m' = m \\ 0 & \text{otherwise} \end{cases} = \mathbb{I}_{\{m\}}(m').$$

Therefore indeed $h = \mathrm{H}_n(f) = \tilde{\mathrm{H}}_n(\tilde{\mathrm{H}}_n^{-1}(h))$. □

The importance of Proposition 178 lies in the fact that now, $\tilde{\mathrm{H}}_n$ is a bijection between $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ and $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$, and therefore, exchangeable

$$
\begin{tikzcd}
\mathcal{L}(\mathcal{X}^n \times \mathcal{R}) \arrow[rr, "\mathrm{H}_n"] \arrow[d, "{[\bullet]}"'] & & \mathcal{L}(\mathcal{N}^n \times \mathcal{R}) \\
\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n} \arrow[urr, leftrightarrow, "\tilde{\mathrm{H}}_n"'] & &
\end{tikzcd}
$$

Figure 8.1: Commuting diagram for H_n and $\tilde{\mathrm{H}}_n$

equivalence classes of gambles are in a one-to-one correspondence with gambles on count vectors.

The commuting diagram of Figure 8.1 above illustrates the surjections $[\bullet]: \mathcal{L}(\mathcal{X}^n \times \mathcal{R}) \to \mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n} : f \mapsto [f]$ and H_n (indicated with a single arrow), and the bijection $\tilde{\mathrm{H}}_n$ (indicated with a double arrow). Since the representing choice function C' is defined from C_n through $[\bullet]$—working point-wise on sets—this already suggests that C' can be transformed into a choice function on $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$. To prove that they preserve coherence, there is only one missing link: the discussion in Section 2.7$_{54}$ shows that it suffices that the map $\tilde{\mathrm{H}}_n$ is linear and preserves the ordering between $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ and $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$.

Therefore, to define the ordering $\leq$ on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$, as usual, we let $\leq$ be inherited by the ordering $\leq$ on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$:

$$\tilde{f} \leq \tilde{g} \Leftrightarrow (\exists f \in \tilde{f}, \exists g \in \tilde{g}) f \leq g \tag{8.8}$$

for all $\tilde{f}$ and $\tilde{g}$ in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$, turning $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ into an ordered linear space. It turns out that this vector ordering on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ also can be represented elegantly using the map $\tilde{\mathrm{H}}_n$:

Proposition 179. *Consider any $\tilde{f}$ and $\tilde{g}$ in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$, then $\tilde{f} \leq \tilde{g}$ if and only if $\tilde{\mathrm{H}}_n(\tilde{f}) \leq \tilde{\mathrm{H}}_n(\tilde{g})$.*

Proof. For necessity, assume that $\tilde{f} \leq \tilde{g}$. Then, by Equation (8.8), $f \leq g$ for some f in $\tilde{f}$ and g in $\tilde{g}$. Consider any m in $\mathcal{N}^n$, and infer that $\mathrm{H}_n(f)(m, \bullet) = \frac{1}{\binom{n}{m}} \sum_{y \in [m]} f(y, \bullet) \leq \frac{1}{\binom{n}{m}} \sum_{y \in [m]} g(y, \bullet) = \mathrm{H}_n(g)(m, \bullet)$. Then $\mathrm{H}_n(f) \leq \mathrm{H}_n(g)$, and therefore, by Equation (8.5)$_{251}$, indeed $\tilde{\mathrm{H}}_n(\tilde{f}) \leq \tilde{\mathrm{H}}_n(\tilde{g})$.

For sufficiency, assume that $\tilde{\mathrm{H}}_n(\tilde{f}) \leq \tilde{\mathrm{H}}_n(\tilde{g})$. Then, by Equation (8.5)$_{251}$ and Proposition 177$_{251}$, $\mathrm{H}_n(f) \leq \mathrm{H}_n(g)$ for all f in $\tilde{f}$ and g in $\tilde{g}$. Consider any f in $\tilde{f}$ and g in $\tilde{g}$ and let $f' \coloneqq \mathrm{inv}_{\mathcal{P}_n}(f)$ and $g' \coloneqq \mathrm{inv}_{\mathcal{P}_n}(g)$. Then $\mathrm{inv}_{\mathcal{P}_n}(f') = \mathrm{inv}_{\mathcal{P}_n}(f)$ and $\mathrm{inv}_{\mathcal{P}_n}(g') = \mathrm{inv}_{\mathcal{P}_n}(g)$ by Proposition 175$_{249}$(ii), so Proposition 177$_{251}$ implies that $f' \in \tilde{f}$ and $g' \in \tilde{f}$, and therefore $\mathrm{H}_n(f') \leq \mathrm{H}_n(g')$. Then, by Equations (8.4)$_{250}$ and (8.3)$_{250}$, $\frac{1}{\binom{n}{m}} \sum_{y \in [m]} f'(y, \bullet) \leq \frac{1}{\binom{n}{m}} \sum_{y \in [m]} g'(y, \bullet)$ for every m in $\mathcal{N}^n$. But f' and g' are constant on every $[m]$, so $f'(y, \bullet) \leq g'(y, \bullet)$ for every y in $[m]$ and every m in $\mathcal{N}^n$. Then $f' \leq g'$, and therefore indeed $\tilde{f} \leq \tilde{g}$. $\square$

Propositions 178_{252} and $179_{\curvearrowleft}$ imply that $\tilde{\mathrm{H}}_n$ is a linear order isomorphism between $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ and $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$, and therefore, by the discussion in Section 2.7_{54}, every coherent choice function on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ can be identified with a coherent choice function on $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$. We will use this interesting conclusion in the next section.

8.1.3 *A representation theorem*

Now that the preparatory work is done, we are ready to establish representation for coherent and exchangeable choice functions.

Theorem 180 (Finite representation)**.** *Consider any choice function C_n on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$. Then C_n is exchangeable if and only if there is a unique representing choice function $\tilde{C}$ on $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$ such that*

$$C_n(A) = \{f \in A : \mathrm{H}_n(f) \in \tilde{C}(\mathrm{H}_n(A))\} \textit{ for all } A \textit{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})).$$

Furthermore, in that case, $\tilde{C}$ is given by $\tilde{C}(\mathrm{H}_n(A)) = \mathrm{H}_n(C_n(A))$ for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$. Finally, C_n is coherent if and only if $\tilde{C}$ is. We call $\tilde{C}$ the count representation *of C_n.*

Similarly, consider any set of desirable gambles $D_n \subseteq \mathcal{L}(\mathcal{X}^n \times \mathcal{R})$. Then D_n is exchangeable if and only if there is a unique representing set of desirable gambles $\tilde{D} \subseteq \mathcal{L}(\mathcal{N}^n \times \mathcal{R})$ such that $D_n = \bigcup \tilde{\mathrm{H}}_n^{-1}(\tilde{D})$. Furthermore, in that case, $\tilde{D}$ is given by $\tilde{D} = \mathrm{H}_n(D_n)$. Finally, D_n is coherent if and only if $\tilde{D}$ is. We call $\tilde{D}$ the count representation *of D_n.*

Proof. We begin with the representation of choice functions. For the first statement, note that C_n is exchangeable is equivalent to C_n is compatible with $I_{\mathcal{P}_n}$, by Definition 49_{248}. Equivalently, by Definition 40_{179}, there is some representing choice function C' on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ such that $C_n(A) = \{f \in A : [f] \in C'(A/I_{\mathcal{P}_n})\}$ for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$. We use the linear order isomorphism $\tilde{\mathrm{H}}_n$ to define a choice function $\tilde{C}$ on $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$: we let $[f] \in C'(A/I_{\mathcal{P}_n}) \Leftrightarrow \tilde{\mathrm{H}}_n([f]) \in \tilde{C}(\tilde{\mathrm{H}}_n(A/I_{\mathcal{P}_n}))$ for all f in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ and A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$. Since $f \in [f]$, use Proposition 177_{251} and Equation $(8.5)_{251}$ to infer that $\tilde{\mathrm{H}}_n([f]) = \mathrm{H}_n(f)$. Similarly, infer that $\tilde{\mathrm{H}}_n(A/I_{\mathcal{P}_n}) = \{\tilde{\mathrm{H}}_n([g]) : g \in A\} = \{\mathrm{H}_n(g) : g \in A\} = \mathrm{H}_n(A)$, so $[f] \in C'(A/I_{\mathcal{P}_n}) \Leftrightarrow \mathrm{H}_n(f) \in \tilde{C}(\mathrm{H}_n(A))$. Then indeed

$$C_n(A) = \{f \in A : \mathrm{H}_n(f) \in \tilde{C}(\mathrm{H}_n(A))\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})).$$

To show that $\tilde{C}$ is unique, use that C' is unique and $\tilde{\mathrm{H}}_n$ is a bijection.

For the second statement, consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$ and infer, using the definition of $\tilde{C}$, that $\tilde{C}(\tilde{\mathrm{H}}_n(A/I_{\mathcal{P}_n})) = \tilde{\mathrm{H}}_n(C'(A/I_{\mathcal{P}_n}))$, and therefore $\tilde{C}(\mathrm{H}_n(A)) = \tilde{\mathrm{H}}_n(C'(A/I_{\mathcal{P}_n}))$ by Equation $(8.5)_{251}$. By compatibility [Proposition 125_{181}], we have $C'(A/I_{\mathcal{P}_n}) = C_n(A)/I_{\mathcal{P}_n}$, so we find $\tilde{\mathrm{H}}_n(C'(A/I_{\mathcal{P}_n})) = \tilde{\mathrm{H}}_n(C_n(A)/I_{\mathcal{P}_n}) = \mathrm{H}_n(C_n(A))$, by Equation $(8.5)_{251}$, and therefore indeed $\tilde{C}(\mathrm{H}_n(A)) = \mathrm{H}_n(C_n(A))$.

For the third statement, the compatibility with $I_{\mathcal{P}_n}$ [Proposition 127_{182}] guarantees that C_n is coherent if and only if its representing choice function C' on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$

is coherent. But since $\tilde{C}$ is defined from C' using the linear order isomorphism H_n, we have immediately that $\tilde{C}$ is coherent if and only if C' is coherent.

The representation of sets of desirable gambles is a trivial extension to *vector-valued* gambles of the proof given in Reference [31, Theorem 17]. □

The number of occurrences of any outcome in a sequence $(x_1,\ldots,x_n)$ is fixed by its count vector m in $\mathcal{N}^n$. If we impose an exchangeability assessment on it, then we see, using Theorem 180, that the joint model on $\mathcal{X}^n$ is characterised by a model on $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$. So an exchangeable choice function C_n essentially represents preferences between gambles on the unknown composition m of an urn with n balls of types $\mathcal{X}$: the choice $C_n(A)$ between the gambles in A is based on m.

This representation result immediately translates to rejection functions and choice relations. A rejection function R on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ is exchangeable if and only if there is a unique representing rejection function $\tilde{R}$ on $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$, given by $\tilde{R}(\mathrm{H}_n(A)) = \mathrm{H}_n(R(A))$ for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$, that represents R, and that is coherent if and only if R is. Similarly, a choice relation $\lhd$ on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ is exchangeable if and only if there is a unique representing choice relation $\tilde{\lhd}$ on $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$, given by $\mathrm{H}_n(A_1) \mathrel{\tilde{\lhd}} \mathrm{H}_n(A_2) \Leftrightarrow A_1 \lhd A_2$ for all A_1 and A_2 in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$, that represents $\lhd$, and that is coherent if and only if $\lhd$ is.

8.1.4 *Exchangeable natural extension*

We use the same notations and ideas as in Reference [31, Sections 4.4&4.6] for desirability, and generalise it to choice models.

Suppose our subject has an assessment $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$. What is, if it exists, the least informative coherent and exchangeable choice function that extends $\mathcal{B}$, or, in other words, what is

$$\mathcal{E}^n_{\mathrm{ex}}(\mathcal{B}) := \inf\{R \in \overline{\mathbf{R}}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})) : R \text{ is exchangeable and extends } \mathcal{B}\}? \quad (8.9)$$

Because, as we have seen, exchangeability is a special indifference assessment, this coincides with the natural extension of $\mathcal{B}$ under $I_{\mathcal{P}_n}$, and the following result is an immediate consequence of Theorem 144_{200}.

Theorem 181 (Exchangeable natural extension)**.** *Consider any assessment $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$. Then the following statements are equivalent:*

(i) *$\mathcal{B}$ avoids complete rejection under $I_{\mathcal{P}_n}$;*

(ii) *There is a coherent and exchangeable extension of $\mathcal{B}$:*

$$(\forall B \in \mathcal{B})0 \in R(B) \text{ and}$$
$$(\forall A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})))R(A) = \{f \in A : [f] \in R(A)/I_{\mathcal{P}_n}\}$$

for some R in $\overline{\mathbf{R}}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$;

(iii) *$\mathcal{E}^n_{\mathrm{ex}}(\mathcal{B}) \neq \mathrm{id}_{\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))}$;*

(iv) $\mathcal{E}^n_{\mathrm{ex}}(\mathcal{B}) \in \overline{\mathbf{R}}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$;
(v) $\mathcal{E}^n_{\mathrm{ex}}(\mathcal{B})$ *is the least informative rejection function that is coherent, exchangeable and extends* $\mathcal{B}$.

When any (and hence all) of these equivalent statements hold, then $\mathcal{E}^n_{\mathrm{ex}}(\mathcal{B}) = R_{\mathcal{B},I_{\mathcal{P}_n}}$, *defined in Equation* (5.6)$_{194}$ *for generic sets of indifferent options I.*

For desirability, the exchangeable natural extension is easy to calculate in terms of the count representation, as shown by De Cooman and Quaeghebeur [31, Theorem 19]. For choice functions, we have the following simple result that has important consequences for practical implications of reasoning and inference under exchangeability:

Theorem 182. *Consider any assessment* $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$. *Then*
(i) $\mathcal{B}$ *avoids complete rejection under* $I_{\mathcal{P}_n}$ *if and only if* $\mathrm{H}_n(\mathcal{B}) := \{\mathrm{H}_n(B) : B \in \mathcal{B}\}$ *avoids complete rejection;*
(ii) $\mathrm{H}_n \circ \mathcal{E}^n_{\mathrm{ex}}(\mathcal{B}) = \mathcal{E}(\mathrm{H}_n(\mathcal{B})) \circ \mathrm{H}_n$.

Proof. For the first statement, observe using Proposition 79$_{95}$ that $R_{\mathrm{H}_n(\mathcal{B})}$ satisfies Axioms R2$_{20}$–R4$_{20}$, and using Proposition 141$_{197}$ that $R_{\mathcal{B},I_{\mathcal{P}_n}}$, too, satisfies Axioms R2$_{20}$–R4$_{20}$. By Definition 32$_{95}$ and Corollary 26$_{39}$, $\mathrm{H}_n(\mathcal{B})$ avoids complete rejection if and only if $0 \notin R_{\mathrm{H}_n(\mathcal{B})}(\{0\})$, and by Definition 42$_{198}$, $\mathcal{B}$ avoids complete rejection under $I_{\mathcal{P}_n}$ if and only if $0 \notin R_{\mathcal{B},I_{\mathcal{P}_n}}(\{0\})$. So it suffices to show that $0 \notin R_{\mathrm{H}_n(\mathcal{B})}(\{0\}) \Leftrightarrow 0 \notin R_{\mathcal{B},I_{\mathcal{P}_n}}(\{0\})$. Use the fact that $\tilde{\mathrm{H}}_n$ is a linear order isomorphism [Propositions 178$_{252}$ and 179$_{253}$] to infer that $0 \notin R_{\mathrm{H}_n(\mathcal{B})}(\{0\}) \Leftrightarrow [0] \notin R_{\mathcal{B}/I_{\mathcal{P}_n}}(\{[0]\})$, and Equation (5.6)$_{194}$ to infer that $0 \notin R_{\mathcal{B},I_{\mathcal{P}_n}}(\{0\}) \Leftrightarrow 0 \notin \{f \in \{0\} : [f] \in R_{\mathcal{B}/I_{\mathcal{P}_n}}(\{[0]\})\} \Leftrightarrow [0] \notin R_{\mathcal{B}/I_{\mathcal{P}_n}}(\{[0]\})$, whence indeed $0 \notin R_{\mathrm{H}_n(\mathcal{B})}(\{0\}) \Leftrightarrow 0 \notin R_{\mathcal{B},I_{\mathcal{P}_n}}(\{0\})$.

For the second statement, we show that $\mathrm{H}_n(\mathcal{E}^n_{\mathrm{ex}}(\mathcal{B})(A)) = \mathcal{E}(\mathrm{H}_n(\mathcal{B}))(\mathrm{H}_n(A))$ for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$. There are two possibilities: either $\mathcal{B}$ avoids complete rejection under $I_{\mathcal{P}_n}$ (and then, by the first statement, $\mathrm{H}_n(\mathcal{B})$ avoids complete rejection), or $\mathcal{B}$ does not avoid complete rejection under $I_{\mathcal{P}_n}$ (and then, by the first statement, $\mathrm{H}_n(\mathcal{B})$ does not avoid complete rejection). If $\mathcal{B}$ avoids complete rejection under $I_{\mathcal{P}_n}$, by Theorems 181$_{\curvearrowleft}$ and 81$_{97}$, it suffices to show that

$$0 \in \mathrm{H}_n(R_{\mathcal{B},I_{\mathcal{P}_n}}(A)) \Leftrightarrow 0 \in R_{\mathrm{H}_n(\mathcal{B})}(\mathrm{H}_n(A)) \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})),$$

taking the coherence of $R_{\mathcal{B},I_{\mathcal{P}_n}}$ and $R_{\mathrm{H}_n(\mathcal{B})}$ into account. Consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$ and observe the following equivalences:

$$\begin{aligned} 0 \in \mathrm{H}_n(R_{\mathcal{B},I_{\mathcal{P}_n}}(A)) &\Leftrightarrow [0] \in R_{\mathcal{B},I_{\mathcal{P}_n}}(A)/I_{\mathcal{P}_n} && \tilde{\mathrm{H}}_n \text{ is a linear order isomorphism} \\ &\Leftrightarrow [0] \in R_{\mathcal{B}/I_{\mathcal{P}_n}}(A/I_{\mathcal{P}_n}) && \text{Equation (5.6)}_{194} \\ &\Leftrightarrow 0 \in R_{\mathrm{H}_n(\mathcal{B})}(\mathrm{H}_n(A)) && \tilde{\mathrm{H}}_n \text{ is a linear order isomorphism.} \end{aligned}$$

If $\mathcal{B}$ does not avoid complete rejection under $I_{\mathcal{P}_n}$, use Theorems 181$_{\curvearrowleft}$ and 81$_{97}$ to infer that then indeed $\mathrm{H}_n(\mathcal{E}^n_{\mathrm{ex}}(\mathcal{B})(A)) = \mathrm{H}_n(A) = \mathcal{E}(\mathrm{H}_n(\mathcal{B}))(\mathrm{H}_n(A))$ for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$. □

8.1.5 *Conditioning exchangeable models*

We use the same notation and ideas as in Reference [31, Section 4.4] for desirability, and generalise it to choice models.

Consider an exchangeable and coherent choice function C on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$, and assume that we have observed the values $\check{x} := (\check{x}_1, \dots, \check{x}_{\check{n}})$ of the first $\check{n}$ variables $X_1, \dots, X_{\check{n}}$, and that we want to make inferences about the remaining $\hat{n} := n - \check{n}$ variables. To do this, we simply condition the choice function C on the singleton $\{\check{x}\}$:

$$f \in C\rfloor\check{x}(A) \Leftrightarrow \mathbb{I}_{\{\check{x}\}} f \in C(\mathbb{I}_{\{\check{x}\}} A), \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})) \text{ and } f \text{ in } A,$$

following the general discussion in Section 7.3_{232}.

Proposition 183. *Consider any $\check{x}$ in $\mathcal{X}^{\check{n}}$ and any coherent and exchangeable choice function C on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$. Then $C\rfloor\check{x}$ is a coherent and exchangeable choice function on $\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})$.*

Proof. That $C\rfloor\check{x}$ is coherent, follows from Proposition 164_{233}. It therefore suffices to show that $C\rfloor\check{x}$ is exchangeable. By Definition 49_{248}, we need to show that $C\rfloor\check{x}$ is compatible with $I_{\mathcal{P}_{\hat{n}}}$. By Proposition 134_{187}, this is equivalent to

$$(\forall g \in I_{\mathcal{P}_{\hat{n}}})(\forall A \in \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})))\big(\{0, g\} \subseteq A \Rightarrow (0 \in C\rfloor\check{x}(A) \Leftrightarrow g \in C\rfloor\check{x}(A))\big).$$

So consider any g in $I_{\mathcal{P}_{\hat{n}}}$ and any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R}))$ such that $\{0, g\} \subseteq A$. It suffices to show that then $0 \in C(\mathbb{I}_{\{\check{x}\}} A) \Leftrightarrow \mathbb{I}_{\{\check{x}\}} g \in C(\mathbb{I}_{\{\check{x}\}} A)$. Since C is exchangeable—compatible with $I_{\mathcal{P}_n}$—and again using Proposition 134_{187}, this will be the case if we can prove that $\mathbb{I}_{\{\check{x}\}} g$ belongs to $I_{\mathcal{P}_n}$. To show this, since $g \in I_{\mathcal{P}_{\hat{n}}}$, observe by Equation $(8.1)_{248}$ that $g = \sum_{i=1}^m \lambda_i (f_i - \hat{\pi}_i^t f_i)$ for some m in $\mathbb{N}$, $\lambda_1, \dots, \lambda_m$ in $\mathbb{R}$, $f_1, \dots, f_m$ in $\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})$, and $\hat{\pi}_1, \dots, \hat{\pi}_m$ in $\mathcal{P}_{\hat{n}}$. For every i in $\{1, \dots, m\}$, let $\pi_i \in \mathcal{P}_n$ be defined as

$$\pi_i(k) := \begin{cases} k & \text{if } k \in \{1, \dots, \check{n}\} \\ \check{n} + \hat{\pi}_i(k - \check{n}) & \text{if } k \in \{\check{n}+1, \dots, n\} \end{cases} \text{ for all } k \text{ in } \{1, \dots, n\}.$$

Then $\mathbb{I}_{\{\check{x}\}} \hat{\pi}_i^t f_i = \pi_i^t (\mathbb{I}_{\{\check{x}\}} f_i)$ for every i in $\{1, \dots, m\}$, whence

$$\mathbb{I}_{\{\check{x}\}} g = \sum_{i=1}^m \lambda_i (\mathbb{I}_{\{\check{x}\}} f_i - \mathbb{I}_{\{\check{x}\}} \hat{\pi}_i^t f_i) = \sum_{i=1}^m \lambda_i (\mathbb{I}_{\{\check{x}\}} f_i - \pi_i^t (\mathbb{I}_{\{\check{x}\}} f_i)),$$

so $\mathbb{I}_{\{\check{x}\}} g$ indeed belongs to $I_{\mathcal{P}_n}$. □

We also introduce another type of conditioning, where we observe a count vector $\check{m}$ in $\mathcal{N}^{\check{n}}$, and we condition the choice function C on all the possible sequences $[\check{m}]$ with these counts, to obtain $C\rfloor[\check{m}]$. The domain of this conditional choice function is $\mathcal{Q}(\mathcal{L}([\check{m}] \times \mathcal{X}^{\hat{n}} \times \mathcal{R}))$, and we are interested in its marginal $\operatorname{marg}_{\hat{n}}(C\rfloor[\check{m}])$, given by

$$\operatorname{marg}_{\hat{n}}(C\rfloor[\check{m}])(A) := C\rfloor[\check{m}](A) \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R}))$$

about the $\hat{n}$ remaining variables $X_{\check{n}+1},\dots,X_n$. Therefore

$$f \in (\mathrm{marg}_{\hat{n}}(C\rfloor[\check{m}]))(A) \Leftrightarrow \mathbb{I}_{[\check{m}]}f \in C(\mathbb{I}_{[\check{m}]}A),$$
$$\text{for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}}\times\mathcal{R})) \text{ and } f \text{ in } A.$$

To simplify the notation, we will let $C\rfloor\check{m} \coloneqq \mathrm{marg}_{\hat{n}}(C\rfloor[\check{m}])$. Interestingly, the count vector $\check{m}$ for an observed sample $\check{x}$ is a *sufficient statistic* in the sense that it extracts from $\check{x}$ all the information that is needed to characterise the conditional model:

Proposition 184 (Sufficiency of the observed count vector)**.** *Consider any coherent and exchangeable choice function C on $\mathcal{L}(\mathcal{X}^n\times\mathcal{R})$, and any $\check{x}$ and $\check{y}$ in $\mathcal{X}^{\check{n}}$. If $[\check{x}]=[\check{y}]$, or in other words, if $T(\check{x})=T(\check{y}) \eqqcolon \check{m}$, then $C\rfloor\check{x}=C\rfloor\check{y}=C\rfloor\check{m}$.*

Proof. From $[\check{x}]=[\check{y}]$, infer that $\check{y}=\check{\pi}\check{x}$ for some $\check{\pi}$ in $\mathcal{P}_{\check{n}}$, and from $T(\check{x})=T(\check{y})=\check{m}$, infer that $\check{x}\in[\check{m}]$ and $\check{y}\in[\check{m}]$. We will first show the intermediate result that then

$$\mathrm{H}_n(\mathbb{I}_{\{\check{x}\}}A)=\mathrm{H}_n(\mathbb{I}_{\{\check{y}\}}A)=\frac{1}{|[\check{m}]|}\mathrm{H}_n(\mathbb{I}_{[\check{m}]}A) \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}}\times\mathcal{R})).$$

Consider any A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}}\times\mathcal{R}))$. Define the permutation π in $\mathcal{P}_n$ as

$$\pi(k) \coloneqq \begin{cases}\check{\pi}^{-1}(k) & \text{if } k\in\{1,\dots,\check{n}\}\\ k & \text{if } k\in\{\check{n}+1,\dots,n\}\end{cases} \quad \text{for all } k \text{ in } \{1,\dots,n\}.$$

We claim that then $\pi^t(\mathbb{I}_{\{\check{x}\}}f)=\mathbb{I}_{\{\check{y}\}}f$ for all f in A. To establish this, consider any $\check{z}=(\check{z}_1,\dots,\check{z}_n)$ in $\mathcal{X}^n$ and any r in $\mathcal{R}$, and observe that indeed

$$\begin{aligned}(\pi^t(\mathbb{I}_{\{\check{x}\}}f))(\check{z},r)=\mathbb{I}_{\{\check{x}\}}f(\pi(\check{z}),r)&=\begin{cases}f(\pi(\check{z}_{\check{n}+1},\dots,\check{z}_n),r) & \text{if } \pi(\check{z}_1,\dots,\check{z}_{\check{n}})=\check{x}\\ 0 & \text{if } \pi(\check{z}_1,\dots,\check{z}_{\check{n}})\neq\check{x}\end{cases}\\ &=\begin{cases}f(\check{z}_{\check{n}+1},\dots,\check{z}_n,r) & \text{if } (\check{z}_1,\dots,\check{z}_{\check{n}})=\check{\pi}\check{x}\\ 0 & \text{if } (\check{z}_1,\dots,\check{z}_{\check{n}})\neq\check{\pi}\check{x}\end{cases}\\ &=\mathbb{I}_{\{\check{\pi}\check{x}\}}f(\check{z},r)=\mathbb{I}_{\{\check{y}\}}f(\check{z},r).\end{aligned}$$

So we see that $\pi^t(\mathbb{I}_{\{\check{x}\}}f)=\mathbb{I}_{\{\check{y}\}}f$ and therefore $\mathbb{I}_{\{\check{x}\}}f-\mathbb{I}_{\{\check{y}\}}f\in I_{\mathcal{P}_n}$, whence $[\mathbb{I}_{\{\check{x}\}}f]=[\mathbb{I}_{\{\check{y}\}}f]$. By Proposition 177_{251}, observe that then $\mathrm{H}_n(\mathbb{I}_{\{\check{x}\}}f)=\mathrm{H}_n(\mathbb{I}_{\{\check{y}\}}f)$ for all f in A, whence $\mathrm{H}_n(\mathbb{I}_{\{\check{x}\}}A)=\mathrm{H}_n(\mathbb{I}_{\{\check{y}\}}A)$. To show that this is also equal to $\frac{1}{|[\check{m}]|}\mathrm{H}_n(\mathbb{I}_{[\check{m}]}A)$, observe that $[\check{m}]=\{\check{\varpi}\check{x}:\check{\varpi}\in\mathcal{P}_{\check{n}}\}$, and therefore for any $\check{z}$ in $[\check{m}]$, we can select a $\check{\pi}_{\check{z}}$ in $\mathcal{P}_{\check{n}}$ such that $\check{\pi}_{\check{z}}\check{x}=\check{z}$. With this $\check{\pi}_{\check{z}}$ we construct a permutation $\pi_{\check{z}}$ in the manner described above, which satisfies $\pi^t_{\check{z}}(\mathbb{I}_{\{\check{x}\}}f)=\mathbb{I}_{\{\check{z}\}}f$ for every f in A. Use $\mathbb{I}_{[\check{m}]}f=\sum_{\check{z}\in[\check{m}]}\mathbb{I}_{\{\check{z}\}}f$ for every f in A to infer that indeed $\mathrm{H}_n(\mathbb{I}_{[\check{m}]}A)=\mathrm{H}_n(\sum_{\check{z}\in[\check{m}]}\mathbb{I}_{\{\check{z}\}}A)=\sum_{\check{z}\in[\check{m}]}\mathrm{H}_n(\mathbb{I}_{\{\check{z}\}}A)=|[\check{m}]|\mathrm{H}_n(\mathbb{I}_{\{\check{x}\}}A)$, where the last equation holds because we have already shown that $\mathrm{H}_n(\mathbb{I}_{\{\check{z}\}}A)=\mathrm{H}_n(\mathbb{I}_{\{\check{x}\}}A)$ for every $\check{z}$ in $[\check{x}]=[\check{m}]$.

Now we are ready to show that $C\rfloor\check{x}=C\rfloor\check{y}=C\rfloor\check{m}$. Since they are coherent, it suffices to show that

$$0\in C\rfloor\check{x}(A)\Leftrightarrow 0\in C\rfloor\check{y}(A)\Leftrightarrow 0\in C\rfloor\check{m}(A), \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}}\times\mathcal{R})),$$

and therefore, taking into account that $C\rfloor\check{m} = \mathrm{marg}_{\hat{n}}(C\rfloor[\check{m}])$, that

$$0 \in C(\mathbb{I}_{\{\check{x}\}}A) \Leftrightarrow 0 \in C(\mathbb{I}_{\{\check{y}\}}A) \Leftrightarrow 0 \in C(\mathbb{I}_{[\check{m}]}A), \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})).$$

Because C is exchangeable, by Theorem 180_{254} it suffices to show that

$$0 \in C(\mathrm{H}_n(\mathbb{I}_{\{\check{x}\}}A)) \Leftrightarrow 0 \in C(\mathrm{H}_n(\mathbb{I}_{\{\check{y}\}}A)) \Leftrightarrow 0 \in C(\mathrm{H}_n(\mathbb{I}_{[\check{m}]}A)),$$

for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R}))$. But we have shown above that $\mathrm{H}_n(\mathbb{I}_{\{\check{x}\}}A) = \mathrm{H}_n(\mathbb{I}_{\{\check{y}\}}A) = \frac{1}{|[\check{m}]|}\mathrm{H}_n(\mathbb{I}_{[\check{m}]}A)$, so taking coherence [and more specifically, Axiom C4a$_{20}$] into account, this is indeed the case. □

8.1.6 *Finite representation in terms of polynomials*

In Section 8.2$_{263}$, we will prove a similar representation theorem for infinite sequences. Since it no longer makes sense to *count* in such sequences, we first need to find a equivalent representation theorem in terms of something that does not depend on counts. More specifically, we need, for every n in $\mathbb{N}$ another order-isomorphic linear space to $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$, that allows for embedding: the linear space for any $n_1 \leq n_2$ (both in $\mathbb{N}$) must be a subspace of the one for n_2.

All the maps in this section have been introduced by De Cooman et al. [31, 32]. We use their ideas and work with polynomials on the $\mathcal{X}$-simplex $\Sigma_{\mathcal{X}} \coloneqq \{\theta \in \mathbb{R}^{\mathcal{X}} : \theta \geq 0, \sum_{x\in\mathcal{X}} \theta_x = 1\}$. We consider the special subset $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ of $\mathcal{L}(\Sigma_{\mathcal{X}} \times \mathcal{R})$: $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ are the *polynomial vector-valued gambles* on $\Sigma_{\mathcal{X}} \times \mathcal{R}$, which are those gambles h such that for every r in $\mathcal{R}$, $h(\bullet, r)$ is the restriction to $\Sigma_{\mathcal{X}}$ of a multivariate polynomial p_r on $\mathbb{R}^{\mathcal{X}}$, in the sense that $h(\theta, r) = p_r(\theta)$ for all θ in $\Sigma_{\mathcal{X}}$. We call p_r then a representation of $h(\bullet, r)$. It will be useful to introduce a notation for polynomial vector-valued gambles with fixed degree n in $\mathbb{N}$: $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$ is the collection of all polynomial vector-valued gambles h such that for every r in $\mathcal{R}$, $h(\bullet, r)$ has at least one representation whose degree is not higher than n. As shown in References [20, 31], both $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ and $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$ are linear subspaces of $\mathcal{L}(\Sigma_{\mathcal{X}} \times \mathcal{R})$, and, as wanted, for $n_1 \leq n_2$, $\mathcal{V}^{n_1}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ is a subspace of $\mathcal{V}^{n_2}(\Sigma_{\mathcal{X}} \times \mathcal{R})$.

Special polynomial gambles are the *Bernstein gambles*:

Definition 50 (Bernstein gambles)**.** *Consider any n in $\mathbb{N}$ and any m in $\mathcal{N}^n$. Define the* Bernstein basis polynomial *B_m on $\mathbb{R}^{\mathcal{X}}$ as $B_m(\theta) \coloneqq \binom{n}{m}\prod_{x\in\mathcal{X}}\theta_x^{m_x}$ for all θ in $\mathbb{R}^{\mathcal{X}}$. The restriction of B_m to $\Sigma_{\mathcal{X}}$ is called a* Bernstein gamble*, which we also denote as B_m.*

As mentioned by De Cooman and Quaeghebeur [31] and proved explicitly by De Bock et al. [20], the set of all Bernstein gambles constitutes a basis for the linear space $\mathcal{V}^n(\Sigma_{\mathcal{X}})$:

Proposition 185 ([31, Appendix B], [20, Proposition 14]). *Consider any n in $\mathbb{N}$. The set of Bernstein gambles $\{B_m : m \in \mathcal{N}^n\}$ constitutes a basis for the linear space $\mathcal{V}^n(\Sigma_{\mathcal{X}})$. Therefore, each element p of $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$ can be uniquely written as $p(\theta, r) = \sum_{m \in \mathcal{N}^n} \alpha(m, r) B_m(\theta)$ for every θ in $\Sigma_{\mathcal{X}}$ and r in $\mathcal{R}$.*

As we have seen, to preserve coherence between two ordered linear spaces, we need a linear order isomorphism. So we wonder whether there is one between $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ and $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$. In Section 8.1.2$_{251}$ we have seen that there is one between $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ and $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$, namely $\tilde{\mathrm{H}}_n$. Therefore, it suffices to find one between $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$ and $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$. Consider the map

$$\mathrm{CoM}_n : \mathcal{L}(\mathcal{N}^n \times \mathcal{R}) \to \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}) : h \mapsto \sum_{m \in \mathcal{N}^n} h(m, \bullet) B_m.$$

Before we can establish that CoM_n is a linear order isomorphism, we need to provide the linear space $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$ with an order $\leq_B^n$. We use the proper cone $\{0\} \cup \mathrm{posi}(\{B_m : m \in \mathcal{N}^n\})$ to define the order $\leq_B^n$:

$$h_1 \leq_B^n h_2 \Leftrightarrow (\forall r \in \mathcal{R}) h_2(\bullet, r) - h_1(\bullet, r) \in \{0\} \cup \mathrm{posi}(\{B_m : m \in \mathcal{N}^n\}),$$

for all h_1 and h_2 in $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$.

The following proposition is essentially due to De Cooman and Quaeghebeur [31]: it suffices to apply their result point-wise, for every r in $\mathcal{R}$.

Proposition 186 ([31, Section 4.9], [20, Section 4.5]). *Consider any n in $\mathbb{N}$. Then the map CoM_n is a linear order isomorphism between the ordered linear spaces $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$ and $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$.*

The linear order isomorphism CoM_n helps us to define a linear order isomorphism between the linear spaces $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ and $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$, a final tool needed for a representation theorem in terms of polynomial gambles. Indeed, consider the map $\mathrm{M}_n := \mathrm{CoM}_n \circ \mathrm{H}_n$:

$$\mathrm{M}_n : \mathcal{L}(\mathcal{X}^n \times \mathcal{R}) \to \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}) : f \mapsto \mathrm{M}_n(f)$$

where $\mathrm{M}_n(f)(\theta, r) := \mathrm{M}_n(f(\bullet, r)|\theta)$ for all θ in $\Sigma_{\mathcal{X}}$ and r in $\mathcal{R}$. $\mathrm{M}_n(\bullet|\theta)$ is the linear expectation operator associated with the multinomial distribution whose parameters are n and θ, and is for every g in $\mathcal{L}(\mathcal{X}^n)$ given by $\mathrm{M}_n(g|\theta) := \sum_{m \in \mathcal{N}^n} \sum_{y \in [m]} g(y) \prod_{x \in \mathcal{X}} \theta_x^{m_x}$. We introduce its version

$$\tilde{\mathrm{M}}_n := \mathrm{CoM}_n \circ \tilde{\mathrm{H}}_n, \tag{8.10}$$

mapping $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$ to $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$. $\tilde{\mathrm{M}}_n$ is a composition of two linear order isomorphisms, and is therefore a linear order isomorphism itself. Due to Proposition 177$_{251}$, considering any $\tilde{f}$ in $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$, M_n is constant on $\tilde{f}$, and the value it takes on any element of $\tilde{f}$ is exactly $\tilde{\mathrm{M}}_n(\tilde{f})$.

$$\begin{tikzcd}
\mathcal{L}(\mathcal{N}^n \times \mathcal{R}) \arrow[rr, leftrightarrow, "\mathrm{CoM}_n"] \arrow[ddr, leftrightarrow, "\tilde{\mathrm{H}}_n"'] & & \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}) \arrow[ddl, leftrightarrow, "\tilde{\mathrm{M}}_n"] \\
& \mathcal{L}(\mathcal{X}^n \times \mathcal{R}) \arrow[ul, "\mathrm{H}_n"'] \arrow[ur, "\mathrm{M}_n"] \arrow[d, "{[\bullet]}"] & \\
& \mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n} &
\end{tikzcd}$$

Figure 8.2: Commuting diagram for CoM_n, $\tilde{\mathrm{M}}_n$ and $\tilde{\mathrm{H}}_n$

The commuting diagram in Figure 8.2 illustrates the surjections $[\bullet]$, H_n and M_n, and the bijections $\tilde{\mathrm{H}}_n$, $\tilde{\mathrm{M}}_n$ and CoM_n. It shows that both $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$ and $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$ are order-isomorphic to $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n}$, so they are both suitable to define a representing choice function on: in Theorem 180_{254} we used the space $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$, and here, in Theorem 187, we will use the other equivalent space $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$.

Theorem 187 (Finite polynomial representation)**.** *Consider any choice function C_n on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$. Then C_n is exchangeable if and only if there is a unique representing choice function $\tilde{C}$ on $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$ such that*

$$C_n(A) = \{f \in A : \mathrm{M}_n(f) \in \tilde{C}(\mathrm{M}_n(A))\} \textit{ for all } A \textit{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})).$$

Furthermore, in that case, $\tilde{C}$ is given by $\tilde{C}(\mathrm{M}_n(A)) = \mathrm{M}_n(C_n(A))$ for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$. Finally, C_n is coherent if and only if $\tilde{C}$ is. We call $\tilde{C}$ the frequency representation *of C_n.*

Similarly, consider any set of desirable gambles $D_n \subseteq \mathcal{L}(\mathcal{X}^n \times \mathcal{R})$. Then D_n is exchangeable if and only if there is a unique representing set of desirable gambles $\tilde{D} \subseteq \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$ such that $D_n = \bigcup \tilde{\mathrm{M}}_n^{-1}(\tilde{D})$. Furthermore, in that case, $\tilde{D}$ is given by $\tilde{D} = \mathrm{M}_n(D_n)$. Finally, D_n is coherent if and only if $\tilde{D}$ is. We call $\tilde{D}$ the frequency representation *of D_n.*

Proof. The part for desirability has essentially already been proved in Reference [31, Theorem 21]. Here, we give a shorter alternative proof that also works for choice functions.

Let C'' on $\mathcal{L}(\mathcal{N}^n \times \mathcal{R})$ and $D'' \subseteq \mathcal{L}(\mathcal{N}^n \times \mathcal{R})$ be the representing choice function and set of desirable gambles from Theorem 180_{254}, and let $\tilde{C}$ be defined by

$$\mathrm{CoM}_n(f) \in \tilde{C}(\mathrm{CoM}_n(A)) \Leftrightarrow f \in C''(A)$$

for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{N}^n \times \mathcal{R}))$ and f in A, and $\tilde{D} := \mathrm{CoM}_n(D'')$. Since CoM_n is a linear order isomorphism, $\tilde{C}$ and $\tilde{D}$ are unique, and $\mathrm{M}_n(f) \in \tilde{C}(\mathrm{M}_n(A)) \Leftrightarrow \mathrm{H}_n(f) \in C''(\mathrm{H}_n(A))$ for all A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n) \times \mathcal{R})$ and f in A, and $\tilde{D} = \mathrm{CoM}_n(\mathrm{H}_n(D_n)) = \mathrm{M}_n(D_n)$, and all the coherence properties are preserved, from which the statements follow. $\square$

8.1.7 *Conditioning in terms of polynomials*

It turns out that conditioning an exchangeable choice function can be done very easily using the frequency representation. Assume that we observe a count vector $\check{m}$ in $\mathcal{N}^{\check{n}}$, and we condition the choice function C on $[\check{m}]$, to obtain the conditional choice function $C\rfloor\check{m} := \operatorname{marg}_{\hat{n}}(C\rfloor[\check{m}])$ on $\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})$, that is exchangeable by Propositions 183_{257} and 184_{258}. What is its frequency representation?

Proposition 188. *Consider any coherent and exchangeable choice function C on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$, and any $\check{m}$ in $\mathcal{N}^n$. If $\tilde{C}$ on $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$ is the frequency representation of C, then the frequency representation of $C\rfloor\check{m}$ is the choice function $\tilde{C}\rfloor\check{m}$ on $\mathcal{V}^{\hat{n}}(\Sigma_{\mathcal{X}} \times \mathcal{R})$, defined by*

$$\hat{h} \in \tilde{C}\rfloor\check{m}(\hat{A}) \Leftrightarrow B_{\check{m}}\hat{h} \in \tilde{C}(B_{\check{m}}\hat{A}), \textit{ for all } \hat{A} \textit{ in } \mathcal{Q}(\mathcal{V}^{\hat{n}}(\Sigma_{\mathcal{X}} \times \mathcal{R})) \textit{ and } \hat{h} \textit{ in } \hat{A}.$$

Proof. Since C is exchangeable with frequency representation $\tilde{C}$, by Theorem 182_{256} we have that

$$C(A) = \{f \in A : \mathrm{M}_n(f) \in \tilde{C}(\mathrm{M}_n(A))\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})).$$

Consider any $\check{x}$ in $[\check{m}]$. Then $C\rfloor\check{m} = C\rfloor\check{x}$ by Proposition 184_{258}, so

$$C\rfloor\check{m}(\hat{A}) = C\rfloor\check{x}(\hat{A}) = \{\hat{f} \in \hat{A} : \mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{f}) \in \tilde{C}(\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{A}))\} \text{ for all } \hat{A} \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})).$$

It suffices to show that $\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{f}) = \frac{1}{|[\check{m}]|}B_{\check{m}}\mathrm{M}_{\hat{n}}(\hat{f})$ and $\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{A}) = \frac{1}{|[\check{m}]|}B_{\check{m}}\mathrm{M}_{\hat{n}}(\hat{A})$, since then indeed

$$\begin{aligned}
C\rfloor\check{m}(\hat{A}) &= \{\hat{f} \in \hat{A} : \mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{f}) \in \tilde{C}(\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{A}))\} \\
&= \{\hat{f} \in \hat{A} : |[\check{m}]|\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{f}) \in \tilde{C}(|[\check{m}]|\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{A}))\} \\
&= \{\hat{f} \in \hat{A} : B_{\check{m}}\mathrm{M}_{\hat{n}}(\hat{f}) \in \tilde{C}(B_{\check{m}}\mathrm{M}_{\hat{n}}(\hat{A}))\} \\
&= \{\hat{f} \in \hat{A} : \mathrm{M}_{\hat{n}}(\hat{f}) \in \tilde{C}\rfloor\check{m}(\mathrm{M}_{\hat{n}}(\hat{A}))\} \text{ for all } \hat{A} \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})),
\end{aligned}$$

taking coherence of $\tilde{C}$ [more specifically, Axiom $\mathrm{C4a}_{20}$] into account, so $\tilde{C}\rfloor\check{m}$ is the frequency representation of $C\rfloor\check{m}$. Since M_n works element-wise on $\mathbb{I}_{\{\check{x}\}}\hat{A}$, it even suffices to show that $\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{f}) = \frac{1}{|[\check{m}]|}B_{\check{m}}\mathrm{M}_{\hat{n}}(\hat{f})$ for every $\hat{f}$ in $\hat{A}$. Lemma 189 establishes this. □

Lemma 189. *Consider any $\check{n} < n$, $\check{m}$ in $\mathcal{N}^{\check{n}}$, $\check{x}$ in $\check{m}$ and $\hat{f}$ in $\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})$. Then $\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{f}) = \frac{1}{|[\check{m}]|}B_{\check{m}}\mathrm{M}_{\hat{n}}(\hat{f})$*

Proof. Consider any $\hat{f}$ in $\hat{A}$, any θ in $\Sigma_{\mathcal{X}}$ and any r in $\mathcal{R}$. Then

$$\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{f})(\theta, r) = \sum_{m \in \mathcal{N}^n} \sum_{y \in [m]} \mathbb{I}_{\{\check{x}\}}\hat{f}(y, r) \prod_{z \in \mathcal{X}} \theta_z^{m_z}.$$

Consider some m in $\mathcal{N}$. Since

$$\mathbb{I}_{\{\check{x}\}}\hat{f}(y,r) = \begin{cases} \hat{f}(y_{\check{n}+1},\dots,y_n,r) & \text{if } (y_1,\dots,x_{\check{n}}) = \check{x} \\ 0 & \text{if } (y_1,\dots,x_{\check{n}}) \neq \check{x} \end{cases} \text{ for all } y = (y_1,\dots,y_n) \text{ in } [m] \tag{8.11}$$

we have that $m \geq \check{m} \Leftrightarrow \sum_{y\in[m]} \mathbb{I}_{\{\check{x}\}}\hat{f}(y,r) \neq 0$, so

$$\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{f})(\theta,r) = \sum_{\substack{m\in\mathcal{N}^n \\ m\geq\check{m}}} \sum_{y\in[m]} \mathbb{I}_{\{\check{x}\}}\hat{f}(y,r) \prod_{z\in\mathcal{X}} \theta_z^{m_z}.$$

Consider some m in $\mathcal{N}^n$. Then $m \geq \check{m}$ if and only if $\hat{m} := m - \check{m} \geq 0$, so $m \geq \check{m}$ if and only if $\hat{m} \in \mathcal{N}^{\hat{n}}$. In that case, $\hat{m}$ is a count vector itself. Assume that $m \geq \check{m}$. Then

$$\begin{aligned} \sum_{y\in[m]} \mathbb{I}_{\{\check{x}\}}\hat{f}(y,r) &= \sum_{(y_1,\dots,y_n)\in[m]} \mathbb{I}_{\{\check{x}\}}\hat{f}(y_1,\dots,y_n,r) \\ &= \sum_{(y_1,\dots,y_{\check{n}})=\check{x}} \sum_{(y_{\check{n}+1},\dots,y_n)\in[\hat{m}]} \mathbb{I}_{\{\check{x}\}}\hat{f}(y_1,\dots,y_n,r) \\ &= \sum_{(y_{\check{n}+1},\dots,y_n)\in[\hat{m}]} \hat{f}(y_{\check{n}+1},\dots,y_n,r) = \sum_{y\in[\hat{m}]} \hat{f}(y,r), \end{aligned}$$

where the third equality follows from Equation (8.11). Furthermore,

$$\prod_{z\in\mathcal{X}} \theta_z^{m_z} = \prod_{z\in\mathcal{X}} \theta_z^{\check{m}_z+\hat{m}_z} = \prod_{z\in\mathcal{X}} \theta_z^{\check{m}_z} \prod_{z\in\mathcal{X}} \theta_z^{\hat{m}_z} = \frac{1}{|[\check{m}]|} B_{\check{m}}(\theta) \prod_{z\in\mathcal{X}} \theta_z^{\hat{m}_z}.$$

Taking these observations into account, we find that

$$\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{f})(\theta,r) = \frac{1}{|[\check{m}]|} B_{\check{m}}(\theta) \sum_{\hat{m}\in\mathcal{N}^{\hat{n}}} \sum_{y\in[\hat{m}]} \hat{f}(y,r) \prod_{z\in\mathcal{X}} \theta_z^{\hat{m}_z} = \frac{1}{|[\check{m}]|} B_{\check{m}}(\theta)\mathrm{M}_{\hat{n}}(\hat{f})(\theta,r).$$

Since our choice of θ in $\Sigma_{\mathcal{X}}$ and r in $\mathcal{R}$ was arbitrary, therefore indeed $\mathrm{M}_n(\mathbb{I}_{\{\check{x}\}}\hat{f}) = \frac{1}{|[\check{m}]|} B_{\check{m}} \mathrm{M}_{\hat{n}}(\hat{f})$. □

Interestingly, as we have seen, for conditioning frequency representations, the Bernstein gambles play the role of indicators.

8.2 COUNTABLE EXCHANGEABILITY

In the previous section, we assumed a finite sequence $X_1,\dots, X_n$ to be exchangeable, and inferred representation theorems. Here, we will consider the countable sequence $X_1, \dots, X_n, \dots$ to be exchangeable, and derive representation theorems for such assessments. We will call $\mathcal{X}_{\mathbb{N}} := \times_{j\in\mathbb{N}} \mathcal{X}$, the set of all possible countable sequences where each variable takes values in $\mathcal{X}$.

First, we need a way to relate gambles on different domains. As in Chapter 7_{221}, this will be done using *cylindrical extension*. We extend Definition 46_{222} to countable sequences:

$$f^*(x,r) := f(x_1,\dots,x_n,r) \text{ for all } x := (x_1,\dots,x_n,\dots) \text{ in } \mathcal{X}_{\mathbb{N}} \text{ and } r \text{ in } \mathcal{R}.$$

for any gamble f on $\mathcal{X}^n \times \mathcal{R}$.

Formally, f^* belongs to $\mathcal{L}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$ while f belongs to $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$. However, they contain the same information, and therefore, are indistinguishable from a behavioural point of view. We will resort to our simplifying device from Remark 7.1223 of identifying f with its cylindrical extension f^*.

Next, we need a way to relate choice functions and sets of desirable gambles on different domains: We will base ourselves again on Chapter 7221 and do this using marginalisation. We extend Definition 47223 to infinite sequences: Given any choice function C on $\mathcal{L}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$ and any n in $\mathbb{N}$, its $\mathcal{X}^n$-marginal C_n is determined by

$$C_n(A) := C(A) \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n) \times \mathcal{R}). \tag{8.12}$$

Similarly, given any set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$ and any n in $\mathbb{N}$, its $\mathcal{X}^n$-marginal D_n is $D_n := D \cap \mathcal{L}(\mathcal{X}^n \times \mathcal{R})$.

Coherence is preserved under marginalisation [it is an immediate consequence of the definition; see, amongst others, Reference [29, Proposition 6] for sets of desirable gambles].

Proposition 190. *Consider any coherent choice function C on $\mathcal{L}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$ and any coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$. Then for every n in $\mathbb{N}$, their $\mathcal{X}^n$-marginals C_n and D_n are coherent.*

8.2.1 *Vector-valued gambles of finite structure*

Before we can explain what it means to assess a countable sequence to be exchangeable, we need to realise that now there are infinitely many variables. From an operational point of view, it will be impossible to describe choice between gambles that depend upon an infinite number of variables. Indeed, since we can never observe the actual outcome in a finite time, gambles will never be actually paid off, and hence every assessment is essentially without any risk. But, it does make operational and behavioural sense to consider choices between gambles of *finite structure*: gambles that each depend on a finite number of variables only. See Reference [20, Section 3.2] for a discussion.

Definition 51 (Gambles of finite structure)**.** *We will call any vector-valued gamble that depends only on a finite number of variables a* vector-valued gamble of finite structure*. We collect all such gambles in the set $\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$:*

$$\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R}) := \{f \in \mathcal{L}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R}) : (\exists n \in \mathbb{N}) f \in \mathcal{L}(\mathcal{X}^n \times \mathcal{R})\} = \bigcup_{n \in \mathbb{N}} \mathcal{L}(\mathcal{X}^n \times \mathcal{R}).$$

$\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$ is a linear space, with the usual ordering $\leq$: for any f and g in $\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$, $f \leq g \Leftrightarrow f(x,r) \leq g(x,r)$ for all x in $\mathcal{X}_{\mathbb{N}}$ and r in $\mathcal{R}$.

Due to our finitary approach, we can even establish a converse result to Proposition 190, whose proof is a straightforward verification of all the axioms.

Proposition 191. *Consider any choice function C on $\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$, and any set of desirable gambles $D \subseteq \overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$. If for every n in $\mathbb{N}$, its $\mathcal{X}^n$-marginal C_n on $\mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ is coherent, then C is coherent. Similarly, if for every n in $\mathbb{N}$, its $\mathcal{X}^n$-marginal $D_n \subseteq \mathcal{L}(\mathcal{X}^n \times \mathcal{R})$ is coherent, then D is coherent.*

Proof. We restrict ourselves to proving this for choice functions; the proof for desirability can be found in Reference [20, Proposition 4]. Consider any A in $\mathcal{Q}(\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R}))$. Then any f in A depends—besides on the value of $\mathcal{R}$—on a finite number of variables $n_f \in \mathbb{N}$. Let $n \coloneqq \max\{n_f : f \in A\}$, which is a well-defined natural number since A is finite. Every (cylindrical extension of) f in A is then a gamble on $\mathcal{X}^n$. It follows then from Equation (8.12) that $C(A) = C_n(A)$.

The proof follows readily once we realise that, following the procedure above, for every option set A there is some n in $\mathbb{N}$ such that $C(A) = C_n(A)$; for Axiom C3$_{20}$, we need to consider $A \cup A_1 \cup A_2$ rather than A. □

8.2.2 *Set of indifferent gambles*

If a subject assesses the sequence of variables $X_1, \ldots, X_n, \ldots$ to be exchangeable, this means that he is indifferent between any gamble f in $\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$ and its permuted variant $\pi^t f$, for any π in $\mathcal{P}_n$, where n now is the (finite) number of variables that f depends upon: his set of indifferent gambles is

$$I_{\mathcal{P}} \coloneqq \{f \in \overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R}) : (\exists n \in \mathbb{N}) f \in I_{\mathcal{P}_n}\} = \bigcup_{n \in \mathbb{N}} I_{\mathcal{P}_n}.$$

If we want to use $I_{\mathcal{P}}$ to define countable exchangeability, it must be a coherent set of indifferent gambles.

Proposition 192. *The set $I_{\mathcal{P}}$ is a coherent set of indifferent gambles.*

Proof. For Axiom I1$_{176}$, since, by Proposition 176$_{249}$, $0 \in I_{\mathcal{P}_n}$ for every n in $\mathbb{N}$, also $0 \in I_{\mathcal{P}}$. For Axiom I2$_{176}$, consider any f in $I_{\mathcal{P}}$, then there is some n in $\mathbb{N}$ for which $f \in I_{\mathcal{P}_n}$. By Proposition 176$_{249}$, we infer that indeed $f \nless 0$ and $f \ngtr 0$. For Axioms I3$_{176}$ and I4$_{176}$, consider any f_1, f_2 and f_3 in $I_{\mathcal{P}}$ and any λ in $\mathbb{R}$. Then there are n_i in $\mathbb{N}$ such that $f_i \in I_{\mathcal{P}_{n_i}}$, for every i in $\{1,2,3\}$. Let $n \coloneqq \max\{n_1, n_2, n_3\}$. Then f_1, f_2 and f_3 are elements of $I_{\mathcal{P}_n}$, so $\lambda f_1 \in I_{\mathcal{P}_n}$ and $f_2 + f_3 \in I_{\mathcal{P}_n}$ by Proposition 176$_{249}$. Then indeed $\lambda f_1 \in I_{\mathcal{P}}$ and $f_2 + f_3 \in I_{\mathcal{P}}$, so $I_{\mathcal{P}}$ is indeed a linear hull. □

Countable exchangeability is now easily defined, similar to the definition for the finite case.

Definition 52. *A choice function C on $\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$ is called* (countably) exchangeable *if C is compatible with $I_{\mathcal{P}}$. Similarly, a set of desirable gambles $D \subseteq \overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$ is called* (countably) exchangeable *if it is compatible with $I_{\mathcal{P}}$.*

This definition is closely related to its finite counterpart:

Proposition 193. *Consider any coherent choice function C on $\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}}\times\mathcal{R})$. Then C is exchangeable if and only if for every choice of n in $\mathbb{N}$, the $\mathcal{X}^n$-marginal C_n of C is exchangeable. Similarly, consider any coherent set of desirable gambles $D\subseteq\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}}\times\mathcal{R})$. Then D is exchangeable if and only if for every choice of n in $\mathbb{N}$, the $\mathcal{X}^n$-marginal D_n of D is exchangeable.*

Proof. The proof for sets of desirable real-valued gambles, in the more general context of *partial* exchangeability, can be found in Reference [20, Proposition 18], and can be trivially extended to vector-valued gambles.

We give the proof for choice functions. For necessity, assume that C is exchangeable, or equivalently, that C is compatible with $I_{\mathcal{P}}$. Use Proposition 134_{187} to infer that then, equivalently,

$$(\forall\tilde{h}\in I_{\mathcal{P}})(\forall\tilde{A}\in\mathcal{Q}(\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}}\times\mathcal{R})))\big(\{0,\tilde{h}\}\subseteq\tilde{A}\Rightarrow(0\in C(\tilde{A})\Leftrightarrow\tilde{h}\in C(\tilde{A}))\big).\quad(8.13)$$

Consider any n in $\mathbb{N}$. We need to prove that then C_n is compatible with $I_{\mathcal{P}_n}$, or equivalently, that

$$(\forall h\in I_{\mathcal{P}_n})(\forall A\in\mathcal{Q}(\mathcal{L}(\mathcal{X}^n\times\mathcal{R})))\big(\{0,h\}\subseteq A\Rightarrow(0\in C_n(A)\Leftrightarrow h\in C_n(A))\big).\quad(8.14)$$

So consider any A in $\mathcal{Q}_0(\mathcal{L}(\mathcal{X}^n\times\mathcal{R}))$ and h in A. Then A is an element of $\mathcal{Q}(\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}}\times\mathcal{R}))$ and h an element of $\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}}\times\mathcal{R})$, so $0\in C(A)\Leftrightarrow h\in C(A)$ by Equation (8.13). Therefore, after marginalising, $0\in C_n(A)\Leftrightarrow h\in C_n(A)$, so C_n is compatible with $I_{\mathcal{P}_n}$, and by Definition $52_{\curvearrowleft}$ therefore indeed exchangeable.

For sufficiency, assume that C_n is exchangeable for every n in $\mathbb{N}$—so it satisfies Equation (8.14) for every n in $\mathbb{N}$. We need to prove that then C is exchangeable. Using Equation (8.13), it suffices to consider any $\tilde{A}$ in $\mathcal{Q}(\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}}\times\mathcal{R}))$ such that $0\in\tilde{A}$, and any $\tilde{h}$ in $\tilde{A}$, and prove that $0\in C(\tilde{A})\Leftrightarrow\tilde{h}\in C(\tilde{A})$. Since $\tilde{A}\cup\{\tilde{h}\}$ consists of gambles of finite structure, there is some (sufficiently large) n in $\mathbb{N}$ for which $\tilde{A}\in\mathcal{Q}(\mathcal{L}(\mathcal{X}^n\times\mathcal{R}))$, and therefore also $\tilde{h}\in\mathcal{L}(\mathcal{X}^n\times\mathcal{R})$. Then by Equation (8.14), $0\in C_n(\tilde{A})\Leftrightarrow\tilde{h}\in C_n(\tilde{A})$, so $0\in C(\tilde{A})\Leftrightarrow\tilde{h}\in C(\tilde{A})$, whence C is compatible with $I_{\mathcal{P}}$, and therefore indeed exchangeable. □

8.2.3 *A representation theorem for countable sequences*

We will look for a representation result that is similar to the one in Section $8.1.6_{259}$. However, since we no longer deal with finite sequences of length n, now the representing choice function will no longer be defined on $\mathcal{V}^n(\Sigma_{\mathcal{X}}\times\mathcal{R})$, but instead on $\mathcal{V}(\Sigma_{\mathcal{X}}\times\mathcal{R})$.

Consider the commuting diagram of Figure 8.3, where a dashed line represents an embedding: for every n in $\mathbb{N}$, $\mathcal{V}^n(\Sigma_{\mathcal{X}}\times\mathcal{R})$ is a subspace of $\mathcal{V}(\Sigma_{\mathcal{X}}\times\mathcal{R})$. This shows the importance of the polynomial representation.

As we have seen, in order to define coherent choice functions on some linear space, we need to provide it with a vector ordering. Similar to what we did before, we use the proper cone $\{0\}\cup\operatorname{posi}(\{B_m:m\in\mathcal{N}^n,n\in\mathbb{N}\})$ to define the order $\leq_B$ on $\mathcal{V}(\Sigma_{\mathcal{X}}\times\mathcal{R})$:

$$h_1\leq_B h_2\Leftrightarrow(\forall r\in\mathcal{R})h_2(\bullet,r)-h_1(\bullet,r)\in\{0\}\cup\operatorname{posi}(\{B_m:m\in\mathcal{N}^n,n\in\mathbb{N}\})$$

$$\begin{tikzcd}
\mathcal{L}(\mathcal{N}^n \times \mathcal{R}) \arrow[rr, leftrightarrow, "\mathrm{CoM}_n"] \arrow[dr, leftrightarrow, bend right=20, "\tilde{\mathrm{H}}_n"'] & & \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}) \arrow[dl, leftrightarrow, bend left=20, "\tilde{\mathrm{M}}_n"] \arrow[dd, dashed] \\
& \mathcal{L}(\mathcal{X}^n \times \mathcal{R}) \arrow[ul, "\mathrm{H}_n"'] \arrow[ur, "\mathrm{M}_n"] \arrow[d, "{[\bullet]}"] & \\
& \mathcal{L}(\mathcal{X}^n \times \mathcal{R})/I_{\mathcal{P}_n} & \mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})
\end{tikzcd}$$

Figure 8.3: Commuting diagram for countable exchangeability

for all h_1 and h_2 in $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$.

Keeping Propositions 190_{264} and 191_{265} in mind, the following results are not surprising.

Proposition 194. *Consider any choice function C' on $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$. Then C' is coherent if and only if for every n in $\mathbb{N}$ the choice function C'_n, defined by $C'_n(A) \coloneqq C'(A)$ for all A in $\mathcal{Q}(\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$, is coherent.*

Proof. We only prove sufficiency, since necessity is trivial. So consider any C' on $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ such that for every n in $\mathbb{N}$, C'_n is coherent. We prove that then C' is coherent.

For Axiom $\mathrm{C1}_{20}$, consider any A in $\mathcal{Q}(\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. Then every polynomial in the finite set A has a certain degree; let n be the maximum of those degrees. Then $A \in \mathcal{Q}(\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$, whence indeed $C'(A) = C'_n(A) \neq \emptyset$, since C'_n is coherent.

For Axiom $\mathrm{C2}_{20}$, consider any h_1 and h_2 in $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ such that $h_1 \leq_B h_2$. Then for every r in $\mathcal{R}$, $h_2(\bullet, r) - h_1(\bullet, r) \in \operatorname{posi}(\{B_m : m \in \mathcal{N}^n, n \in \mathbb{N}\})$. Consider, for every r in $\mathcal{R}$, a representing polynomial p_r of $h_1(\bullet, r)$, and let n_1 be the degree of the representing polynomial in the finite set $\{p_r : r \in \mathcal{R}\}$ with highest degree. Then, for every r in $\mathcal{R}$, $h_1(\bullet, r)$ is represented by a polynomial in $\mathcal{V}^{n_1}(\Sigma_{\mathcal{X}} \times \mathcal{R})$. Similarly, we find that, for every r in $\mathcal{R}$, $h_2(\bullet, r)$ is represented by a polynomial in $\mathcal{V}^{n_2}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ for some n_2 in $\mathbb{N}$. Let $n \coloneqq \max\{n_1, n_2\}$. Then, for every r in $\mathcal{R}$, there is a representing polynomial of $h_2(\bullet, r) - h_1(\bullet, r)$ whose degree is not higher than n, so $h_2 - h_1 \in \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$ and therefore $h_2(\bullet, r) - h_1(\bullet, r) \in \operatorname{posi}(\{B_m : m \in \mathcal{N}^n\})$. This guarantees that $h_1 \leq_B^n h_2$, whence indeed $\{h_2\} = C'_n(\{h_1, h_2\}) = C'(\{h_1, h_2\})$, since C'_n is coherent.

For Axiom $\mathrm{C3}_{20}$, consider any A, A_1 and A_2 in $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$. Using the similar construction as for Axiom $\mathrm{C2}_{20}$, we find that then $A \cup A_1 \cup A_2 \subseteq \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$, for some n in $\mathbb{N}$. Then A, A_1 and A_2 all are elements of $\mathcal{Q}(\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. For Axiom $\mathrm{C3a}_{20}$, assume that $C'(A_2) \subseteq A_2 \setminus A_1$ and $A_1 \subseteq A_2 \subseteq A$. Then $C'_n(A_2) \subseteq A_2 \setminus A_1$ and therefore, since C'_n is coherent, indeed $C'(A) = C'_n(A) \subseteq A \setminus A_1$. For Axiom $\mathrm{C3b}_{20}$, assume that $C'(A_2) \subseteq A_2 \setminus A_1$ and $A \subseteq A_1$. Then $C'_n(A_2) \subseteq A_2 \setminus A_1$ and therefore, since C'_n is coherent, indeed $C'(A_2 \setminus A) = C'_n(A_2 \setminus A) \subseteq A_2 \setminus A_1$.

For Axiom $\mathrm{C4}_{20}$, consider any h in $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$, any λ in $\mathbb{R}_{>0}$ and any A_1 and A_2 in $\mathcal{Q}(\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. Using the similar construction as for Axiom $\mathrm{C2}_{20}$, we find that then $\{h\} \cup A_1 \cup A_2 \subseteq \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$, for some n in $\mathbb{N}$. Then $h \in \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$ and A_1 and A_2 both are elements of $\mathcal{Q}(\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. Assume that $A_1 \subseteq C'(A_2) = C'_n(A_2)$, then, since C'_n is coherent, indeed $\lambda A_1 \subseteq C'_n(\lambda A_2) = C'(\lambda A_2)$ and $A_1 + \{h\} \subseteq C'_n(A_2 + \{h\}) = C'(A_2 + \{h\})$. $\square$

Theorem 195 (Countable representation)**.** *Consider any choice function C on $\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$. Then C is coherent and exchangeable if and only if there is a coherent representing choice function $\tilde{C}$ on $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ such that, for every n in $\mathbb{N}$, the $\mathcal{X}^n$-marginal C_n of C is determined by*

$$C_n(A) = \{f \in A : \mathrm{M}_n(f) \in \tilde{C}(\mathrm{M}_n(A))\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})). \quad (8.15)$$

In that case, $\tilde{C}$ is uniquely determined via its corresponding rejection function $\tilde{R}$ by $\tilde{R}(\tilde{A}) := \bigcup_{n\in\mathbb{N}} \tilde{R}_n(\tilde{A} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$ for all $\tilde{A}$ in $\mathcal{Q}(\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$, with $\tilde{R}_n(\mathrm{M}_n(A)) := \mathrm{M}_n(R_n(A))$ for every A in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R}))$, and where we let $\tilde{R}_n(\emptyset) := \emptyset$ for notational convenience. We call $\tilde{C}$ the frequency representation *of C.*

Similarly, consider any set of desirable gambles $D \subseteq \overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$. Then D is coherent and exchangeable if and only if there is a unique representing $\tilde{D} \subseteq \mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ such that, for every n in $\mathbb{N}$, the $\mathcal{X}^n$-marginal D_n is given by $D_n = \bigcup \tilde{\mathrm{M}}_n^{-1}(\tilde{D} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. In that case, $\tilde{D}$ is given by $\tilde{D} = \bigcup_{n\in\mathbb{N}} \mathrm{M}_n(D_n)$.

Proof. We begin with the representation of choice functions. That C is exchangeable is, by Proposition 193_{266}, equivalent to C_n is exchangeable, for every n in $\mathbb{N}$. Therefore, by Theorem 187_{261} this is equivalent to:

$$(\forall n \in \mathbb{N}) C_n(A) = \{f \in A : \mathrm{M}_n(f) \in \tilde{C}_n(\mathrm{M}_n(A))\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})),$$

where, as a consequence, for every n in $\mathbb{N}$, $\tilde{C}_n$ is uniquely given by

$$\tilde{C}_n(\mathrm{M}_n(A)) = \mathrm{M}_n(C_n(A)) \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})). \quad (8.16)$$

That C is coherent, is by Propositions 190_{264} and 191_{265} equivalent to C_n is coherent, for every n in $\mathbb{N}$, and using Theorem 187_{261}, to $\tilde{C}_n$ is coherent, for every n in $\mathbb{N}$. We prove that this is necessary and sufficient for the existence of some coherent choice function $\tilde{C}$ on $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ that, for every n in $\mathbb{N}$, satisfies Equation (8.15).

We start by showing the converse implication (sufficiency). Assume that there is some coherent choice function $\tilde{C}$ on $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ that for every n in $\mathbb{N}$ satisfies Equation (8.15). Consider any n in $\mathbb{N}$. By Proposition $194_{\curvearrowleft}$, then the choice function $\tilde{C}_n$ on $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$, determined by $\tilde{C}_n(\tilde{A}_n) = \tilde{C}(\tilde{A}_n)$ for every $\tilde{A}_n$ in $\mathcal{Q}(\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$, is coherent. Furthermore, by Equation (8.15), it satisfies indeed

$$C_n(A) = \{f \in A : \mathrm{M}_n(f) \in \tilde{C}_n(\mathrm{M}_n(A))\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})).$$

Since $\tilde{C}_n(\tilde{A}_n) = \tilde{C}(\tilde{A}_n)$ for every $\tilde{A}_n$ in $\mathcal{Q}(\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$, we find that also

$$\tilde{R}(\tilde{A}_n) = \tilde{R}_n(\tilde{A}_n) \text{ for all } \tilde{A}_n \text{ in } \mathcal{Q}(\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})). \quad (8.17)$$

Next, we show that then $\tilde{R}(\tilde{A}) = \bigcup_{n\in\mathbb{N}} \tilde{R}_n(\tilde{A} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$ for all $\tilde{A}$ in $\mathcal{Q}(\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$, thus making it unique. Consider any $\tilde{A}$ in $\mathcal{Q}(\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. Then, by the definition of a polynomial gamble, there is some n^* in $\mathbb{N}$ such that $\tilde{A} \in \mathcal{Q}(\mathcal{V}^{n^*}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$, whence $\tilde{A} = \tilde{A} \cap \mathcal{V}^{n^*}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ and therefore $\tilde{R}(\tilde{A}) = \tilde{R}(\tilde{A} \cap \mathcal{V}^{n^*}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. By Equation (8.17) therefore $\tilde{R}(\tilde{A}) = \tilde{R}_{n^*}(\tilde{A} \cap \mathcal{V}^{n^*}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$, whence indeed $\tilde{R}(\tilde{A}) \subseteq \bigcup_{n\in\mathbb{N}} \tilde{R}_n(\tilde{A} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. Conversely, consider any h in $\bigcup_{n\in\mathbb{N}} \tilde{R}_n(\tilde{A} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. Then $h \in \tilde{R}_{n^*}(\tilde{A} \cap \mathcal{V}^{n^*}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$ for some n^* in $\mathbb{N}$, and therefore, by Equation (8.17), $h \in \tilde{R}(\tilde{A} \cap \mathcal{V}^{n^*}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. By Axiom R3a$_{20}$, therefore indeed $h \in \tilde{R}(\tilde{A})$.

We complete the proof by showing the direct implication (necessity). Assume that

$$(\forall n \in \mathbb{N}) C_n(A) = \{f \in A : \mathrm{M}_n(f) \in \tilde{C}_n(\mathrm{M}_n(A))\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})),$$

and that $\tilde{C}_n$ is coherent for every n in $\mathbb{N}$. Let the rejection function $\tilde{R}$ on $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ by determined by $\tilde{R}(\tilde{A}) := \bigcup_{n\in\mathbb{N}} \tilde{R}_n(\tilde{A} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$ for all $\tilde{A}$ in $\mathcal{Q}(\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. The proof is finished if we can show that it satisfies Equation (8.17) for every n in $\mathbb{N}$, because, if it does, Equation (8.15) is then satisfied, and furthermore, by Proposition 194$_{267}$ it is then coherent. Consider any n^* in $\mathbb{N}$. We need to show that $\bigcup_{n\in\mathbb{N}} \tilde{R}_n(\tilde{A}_{n^*} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})) = \tilde{R}_{n^*}(\tilde{A}_{n^*})$ for all $\tilde{A}_{n^*}$ in $\mathcal{Q}(\mathcal{V}^{n^*}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. Observe that $\tilde{A}_{n^*} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}) = \tilde{A}_{n^*}$ for every $n \geq n^*$, whence by Lemma 196, $\bigcup_{n\in\mathbb{N}} \tilde{R}_n(\tilde{A}_{n^*} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})) = \bigcup_{n\leq n^*} \tilde{R}_n(\tilde{A}_{n^*} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R}))$. Furthermore, since $\tilde{A}_{n^*} \cap \mathcal{V}^{n_1}(\Sigma_{\mathcal{X}} \times \mathcal{R}) \subseteq \tilde{A}_{n^*} \cap \mathcal{V}^{n_2}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ whenever $n_1 \leq n_2$, also $\tilde{R}_{n_1}(\tilde{A}_{n^*} \cap \mathcal{V}^{n_1}(\Sigma_{\mathcal{X}} \times \mathcal{R})) = \tilde{R}_{n_2}(\tilde{A}_{n^*} \cap \mathcal{V}^{n_1}(\Sigma_{\mathcal{X}} \times \mathcal{R})) \subseteq \tilde{R}_{n_2}(\tilde{A}_{n^*} \cap \mathcal{V}^{n_2}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$, where the first equality follows from Lemma 196, and the second one from Axiom R3a$_{20}$. This implies that indeed $\bigcup_{n\leq n^*} \tilde{R}_n(\tilde{A}_{n^*} \cap \mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})) = \tilde{R}_{n^*}(\tilde{A}_{n^*} \cap \mathcal{V}^{n^*}(\Sigma_{\mathcal{X}} \times \mathcal{R})) = \tilde{R}_{n^*}(\tilde{A}_{n^*})$.

The representation of sets of desirable gambles is a trivial extension to *vector-valued* gambles of the proof given in Reference [20, Theorem 22]. □

Lemma 196. *Consider any coherent choice function C on $\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$ and assume that, for every n in $\mathbb{N}$, its $\mathcal{X}^n$-marginal C_n is given by*

$$C_n(A) = \{f \in A : \mathrm{M}_n(f) \in \tilde{C}_n(\mathrm{M}_n(A))\} \textit{ for all } A \textit{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})), \quad (8.18)$$

where $\tilde{C}_n$ is a coherent choice function on $\mathcal{V}^n(\Sigma_{\mathcal{X}} \times \mathcal{R})$. Then, for every $n_1 \leq n_2$ in $\mathbb{N}$ and $\tilde{A}_{n_1}$ in $\mathcal{V}^{n_1}(\Sigma_{\mathcal{X}} \times \mathcal{R})$:

$$\tilde{C}_{n_1}(\tilde{A}_{n_1}) = \tilde{C}_{n_2}(\tilde{A}_{n_1}).$$

Proof. Since M_{n_1} is surjective, we can find A_{n_1} in $\mathcal{Q}(\mathcal{L}(\mathcal{X}^{n_1} \times \mathcal{R}))$ such that $\mathrm{M}_{n_1}(A_{n_1}) = \tilde{A}_{n_1}$. We will show that $\tilde{R}_{n_1}(\tilde{A}_{n_1}) = \tilde{R}_{n_2}(\tilde{A}_{n_1})$.

To show that $\tilde{R}_{n_1}(\tilde{A}_{n_1}) \subseteq \tilde{R}_{n_2}(\tilde{A}_{n_1})$, consider any h in $\tilde{R}_{n_1}(\tilde{A}_{n_1})$, and let f_{n_1} be an element of A_{n_1} such that $\mathrm{M}_{n_1}(f_{n_1}) = h$. Then $f_{n_1} \in R_{n_1}(A_{n_1})$ by Equation (8.18). If we denote the cylindrical extension of A_{n_1} to $\mathcal{X}^{n_2}$ by A_{n_2}—and the cylindrical extension of f_{n_1} by f_{n_2}—, then $f_{n_2} \in R_{n_2}(A_{n_2})$ because R_{n_1} and R_{n_2} are related through marginalisation. But $R_{n_2}(A_{n_2}) = \{f \in A_{n_2} : \mathrm{M}_{n_2}(f) \in \tilde{R}_{n_2}(\mathrm{M}_{n_2}(A_{n_2}))\}$ by Equation (8.18), and by Lemma 197$_{\curvearrowright}$, $R_{n_2}(A_{n_2}) = \{f \in A_{n_2} : \mathrm{M}_{n_2}(f) \in \tilde{R}_{n_2}(\mathrm{M}_{n_1}(A_{n_1}))\}$. Since $f_{n_2} \in R_{n_2}(A_{n_2})$, indeed $h = \mathrm{M}_{n_1}(f_{n_1}) = \mathrm{M}_{n_2}(f_{n_2}) \in \tilde{R}_{n_2}(\mathrm{M}_{n_1}(A_{n_1})) = \tilde{R}_{n_2}(\tilde{A}_{n_1})$.

The proof that also $\tilde{R}_{n_2}(\tilde{A}_{n_1}) \subseteq \tilde{R}_{n_1}(\tilde{A}_{n_1})$ is completely similar [with the same notation, $h \in \tilde{R}_{n_2}(\tilde{A}_{n_1})$ implies that $f_{n_1} \in R_{n_2}(A_{n_1})$ by Equation (8.18), which implies that $f_{n_1} \in R_{n_1}(A_{n_1})$ because R_{n_1} and R_{n_2} are related through marginalisation, which in turn implies that $h \in \tilde{R}_{n_1}(\tilde{A}_{n_1})$]. □

Lemma 197 ([20, Lemma 27]). *Consider any $n_1 \leq n_2$ in $\mathbb{N}$, and any gamble f_{n_1} in $\mathcal{L}(\mathcal{X}^{n_1} \times \mathcal{R})$. Denote its cylindrical extension to $\mathcal{X}^{n_2}$ by f_{n_2}. Then* $\mathrm{M}_{n_1}(f_{n_1}) = \mathrm{M}_{n_2}(f_{n_2})$.

8.2.4 *Conditioning and countable representation*

We use the same notation and ideas as in Reference [31, Section 5.2] for desirability, and generalise it to choice models.

Suppose we have a coherent and exchangeable choice function C on $\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$ with associated frequency representation $\tilde{C}$ on $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$. Suppose that we observe the values $\check{x}$ of the first $\check{n}$ variables, with associated count vector $\check{m} := T(\check{x})$. We have seen in Proposition 193_{266} that, for every n in $\mathbb{N}$, the $\mathcal{X}^n$-marginal C_n is exchangeable, and in Proposition 183_{257} that $C_n \rfloor \check{m}$ is exchangeable (if $n > \check{m}$). But what about $C \rfloor \check{m}$?

Theorem 198. *Consider any coherent and exchangeable choice function C on $\overline{\mathcal{L}}(\mathcal{X}_{\mathbb{N}} \times \mathcal{R})$ with associated frequency representation $\tilde{C}$ on $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$. After conditioning on a sample with count vector $\check{m}$ in $\mathcal{N}^{\check{n}}$, $C \rfloor \check{m}$ is still exchangeable and coherent, and has frequency representation $\tilde{C} \rfloor \check{m}$, defined by*

$$h \in \tilde{C}\rfloor\check{m}(\tilde{A}) \Leftrightarrow B_{\check{m}} h \in \tilde{C}(B_{\check{m}}\tilde{A}), \text{ for every } \tilde{A} \text{ in } \mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R}) \text{ and } h \text{ in } \tilde{A}. \quad (8.19)$$

Proof. Use Theorem 195_{268} to infer that $\tilde{C}$ is coherent. We first show that $\tilde{C}\rfloor\check{m}$ is coherent. For Axiom $\mathrm{C1}_{20}$, consider any $\tilde{A}$ in $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$. Since $\tilde{C}(B_{\check{m}}\tilde{A}) \neq \emptyset$, indeed also $\tilde{C}\rfloor\check{m}(\tilde{A}) \neq \emptyset$.

For Axiom $\mathrm{C2}_{20}$, consider any h_1 and h_2 in $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ such that $h_1 \prec_B h_2$—meaning $h_1 \preceq_B h_2$ and $h_1 \neq h_2$. Then $(\forall r \in \mathcal{R}) h_1(\bullet, r) - h_2(\bullet, r) \in \mathrm{posi}(\{B_m : m \in \mathcal{N}^n, n \in \mathbb{N}\})$. Consider any r in $\mathcal{R}$. Then $h_1(\bullet, r) - h_2(\bullet, r) = \sum_{i=1}^{\ell} \lambda_i B_{m^i}$ for some ℓ in $\mathbb{N}$, $\lambda_1, \ldots, \lambda_\ell$ in $\mathbb{R}_{>0}$, $n_1, \ldots, n_\ell$ in $\mathbb{N}$ and $m^1 \in \mathcal{N}^{n_1}, \ldots, m^\ell \in \mathcal{N}^{n_\ell}$, and therefore $(h_1(\bullet, r) - h_2(\bullet, r)) B_{\check{m}} = \sum_{i=1}^{\ell} \lambda_i B_{m^i} B_{\check{m}}$. Use the result that, for any θ in $\Sigma_{\mathcal{X}}$,

$$\begin{aligned} B_{m^i+\check{m}}(\theta) = |[m^i + \check{m}]| \prod_{z \in \mathcal{X}} \theta_z^{m_z^i + \check{m}_z} &= |[m^i + \check{m}]| \Big(\prod_{z \in \mathcal{X}} \theta_z^{m_z^i} \Big) \Big(\prod_{z \in \mathcal{X}} \theta_z^{\check{m}_z} \Big) \\ &= \frac{|[m^i + \check{m}]|}{|[m^i]||[\check{m}]|} B_{m^i}(\theta) B_{\check{m}}(\theta), \end{aligned}$$

so $B_{m^i+\check{m}} = \frac{|[m^i+\check{m}]|}{|[m^i]||[\check{m}]|} B_{m^i} B_{\check{m}}$, to infer that

$$(h_1(\bullet, r) - h_2(\bullet, r)) B_{\check{m}} = \sum_{i=1}^{\ell} \lambda_i \frac{|[m^i]||[\check{m}]|}{|[m^i + \check{m}]|} B_{m^i + \check{m}},$$

so $B_{\check{m}} h_1(\bullet, r) - B_{\check{m}} h_2(\bullet, r) \in \mathrm{posi}(\{B_m : m \in \mathcal{N}^n, n \in \mathbb{N}\})$. Since the choice of r in $\mathcal{R}$ was arbitrary, $B_{\check{m}} h_1 \prec_B B_{\check{m}} h_2$, so $B_{\check{m}} h_1 \notin \tilde{C}(\{B_{\check{m}} h_1, B_{\check{m}} h_2\})$ and therefore indeed $h_1 \notin \tilde{C}\rfloor\check{m}(\{h_1, h_2\})$.

For Axiom $\mathrm{C3a}_{20}$, consider any $\tilde{A}$, $\tilde{A}_1$ and $\tilde{A}_2$ in $\mathcal{Q}(\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$ such that $\tilde{C}\rfloor\check{m}(\tilde{A}_2) \subseteq \tilde{A}_2 \setminus \tilde{A}_1$ and $\tilde{A}_1 \subseteq \tilde{A}_2 \subseteq \tilde{A}$. Then $\tilde{C}(B_{\check{m}}\tilde{A}_2) \subseteq B_{\check{m}}(\tilde{A}_2 \setminus \tilde{A}_1) = B_{\check{m}}\tilde{A}_2 \setminus B_{\check{m}}\tilde{A}_1$

and $B_{\check{m}}\tilde{A}_1 \subseteq B_{\check{m}}\tilde{A}_2 \subseteq B_{\check{m}}\tilde{A}$, and therefore, by coherence, $\tilde{C}(B_{\check{m}}\tilde{A}) \subseteq B_{\check{m}}\tilde{A} \setminus B_{\check{m}}\tilde{A}_1$. But then indeed $\tilde{C}\rfloor\check{m}(\tilde{A}) \subseteq \tilde{A} \setminus \tilde{A}_1$.

For Axiom C3b$_{20}$, consider any $\tilde{A}$, $\tilde{A}_1$ and $\tilde{A}_2$ in $\mathcal{Q}(\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$ such that $C\rfloor\check{m}(\tilde{A}_2) \subseteq \tilde{A}_2 \setminus \tilde{A}_1$ and $\tilde{A} \subseteq \tilde{A}_1$. Then $C(B_{\check{m}}\tilde{A}_2) \subseteq B_{\check{m}}(\tilde{A}_2 \setminus \tilde{A}_1) = B_{\check{m}}\tilde{A}_2 \setminus B_{\check{m}}\tilde{A}_1$ and $B_{\check{m}}\tilde{A} \subseteq B_{\check{m}}\tilde{A}_1$, and therefore, $C(B_{\check{m}}(\tilde{A}_2 \setminus \tilde{A})) = C(B_{\check{m}}\tilde{A}_2 \setminus B_{\check{m}}\tilde{A}) \subseteq B_{\check{m}}\tilde{A}_2 \setminus B_{\check{m}}\tilde{A}_1 = B_{\check{m}}(\tilde{A}_2 \setminus \tilde{A}_1)$. But then indeed $C\rfloor\check{m}(\tilde{A}_2 \setminus \tilde{A}) \subseteq \tilde{A}_2 \setminus \tilde{A}_1$.

For Axiom C4a$_{20}$, consider any $\tilde{A}_1$ and $\tilde{A}_2$ in $\mathcal{Q}(\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$ and any λ in $\mathbb{R}_{>0}$ for which $\tilde{A}_1 \subseteq C\rfloor\check{m}(\tilde{A}_2)$. Then $B_{\check{m}}\tilde{A}_1 \subseteq C(B_{\check{m}}\tilde{A}_2)$, and therefore, $B_{\check{m}}\lambda\tilde{A}_1 \subseteq C(B_{\check{m}}\lambda\tilde{A}_2)$. But then indeed $\lambda\tilde{A}_1 \subseteq C\rfloor\check{m}(\lambda\tilde{A}_2)$.

For Axiom C4b$_{20}$, consider any $\tilde{A}_1$ and $\tilde{A}_2$ in $\mathcal{Q}(\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R}))$ and any h in $\mathcal{V}(\Sigma_{\mathcal{X}} \times \mathcal{R})$ for which $\tilde{A}_1 \subseteq C\rfloor\check{m}(\tilde{A}_2)$. Then $B_{\check{m}}\tilde{A}_1 \subseteq C(B_{\check{m}}\tilde{A}_2)$, and therefore, $B_{\check{m}}(\tilde{A}_1 + \{h\}) = B_{\check{m}}\tilde{A}_1 + \{B_{\check{m}}h\} \subseteq C(B_{\check{m}}\tilde{A}_2 + \{B_{\check{m}}h\}) = C(B_{\check{m}}(\tilde{A}_2 + \{h\}))$. But then indeed $\lambda\tilde{A}_1 + \{h\} \subseteq C\rfloor\check{m}(\lambda\tilde{A}_2 + \{h\})$.

To finish the proof, it suffices to show that $\tilde{C}\rfloor\check{m}$ is the frequency representation of $C\rfloor\check{m}$. To establish this, we will show that, for every n in $\mathbb{N}$ such that $n > \check{n}$, the $\mathcal{X}^n$-marginal $C_n\rfloor\check{m}$ of C conditional on $\check{m}$, is given by

$$C_n\rfloor\check{m}(\hat{A}) = \{f \in \hat{A} : \mathrm{M}_{\hat{n}}(f) \in \tilde{C}\rfloor\check{m}(\mathrm{M}_{\hat{n}}(\hat{A}))\} \text{ for all } \hat{A} \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})),$$

where $\hat{n} := n - \check{n} > 1$.

Consider any n in $\mathbb{N}$ such that $n > \check{n}$. Since C is exchangeable, by Theorem 195$_{268}$, we get

$$C_n(A) = \{f \in A : \mathrm{M}_n(f) \in \tilde{C}(\mathrm{M}_n(A))\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^n \times \mathcal{R})).$$

Therefore

$$C_n\rfloor\check{m}(\hat{A}) = \{f \in \hat{A} : \mathrm{M}_n(\mathbb{I}_{[\check{m}]}f) \in \tilde{C}(\mathrm{M}_n(\mathbb{I}_{[\check{m}]}\hat{A}))\} \text{ for all } \hat{A} \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})).$$

By Lemma 189$_{262}$, we have that then

$$C_n\rfloor\check{m}(\hat{A}) = \{f \in \hat{A} : B_{\check{m}}\mathrm{M}_{\hat{n}}(f) \in \tilde{C}(B_{\check{m}}\mathrm{M}_{\hat{n}}(\hat{A}))\} \text{ for all } \hat{A} \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})),$$

and by Equation (8.19), therefore indeed

$$C_n\rfloor\check{m}(\hat{A}) = \{f \in \hat{A} : \mathrm{M}_{\hat{n}}(f) \in \tilde{C}\rfloor\check{m}(\mathrm{M}_{\hat{n}}(\hat{A}))\} \text{ for all } \hat{A} \text{ in } \mathcal{Q}(\mathcal{L}(\mathcal{X}^{\hat{n}} \times \mathcal{R})). \qquad \square$$

8.3 CONCLUSION

We have studied exchangeability and we have found counterparts to de Finetti's finite and countable representation results, in the general setting of choice functions. We have shown that an exchangeability assessment is a particular indifference assessment, where we identified the set of indifferent options. The main idea that made (finite) representation possible is the linear order isomorphism $\tilde{\mathrm{H}}_n^{-1}$ between the quotient space and the set of gambles on count vectors, indicating that (finitely) exchangeable choice functions can be represented by a choice function that essentially represents preferences between gambles on the unknown composition m of an urn with n balls of types $\mathcal{X}$. Alternatively, for the countable case, we have shown that there is a polynomial representation.

9

CONCLUSIONS

The main conclusion of this dissertation is that with coherent choice functions, we are able to generalise surprisingly many existing concepts in imprecise probabilities, and more specifically, for sets of desirable gambles. We have introduced an axiomatisation of coherent choice that is compatible with desirability. Reduced to pairwise comparison, the rationality criteria for choice functions imply the rationality criteria for desirability. Below, I will highlight some of the most important results, and will look ahead at a number of interesting problems that still remain.

An important property of coherent choice functions, is that coherence is preserved under arbitrary infima. Its proof is basic, but the result is nevertheless crucial: it allows for conservative reasoning with coherent choice functions. We apply this essential property widely in this dissertation. For instance, we use it to find an explicit expression for the natural extension of a partially specified choice function. As it turns out, the natural extension for choice functions reduces to its well-known counterpart for sets of desirable gambles. Furthermore, if we start out from a purely binary assessment, the result of our natural extension (for choice functions) is a purely binary choice function itself. Even though this seems fairly natural, this result is not at all obvious, and it seems to suggest that our axiomatisation is well suited for the connection between choice functions and desirability.

The additional Property $C5_{25}$ (which Seidenfeld et al. [67] use as rationality axiom) that we consider, is also preserved under arbitrary infima. It furthermore is also a productive axiom—just as Axioms $C2_{20}$–$C4_{20}$—as can be clearly seen from its version $(2.30)_{78}$ for rejection sets. However, finding a manageable, constructive, expression for the natural extension under convexity—the least informative coherent 'convex' extension of a given assessment—remains an open problem. Obvious modifications to the natural extension $R_{\mathcal{B}}$ of an assessment $\mathcal{B}$ that avoids complete rejection turn out not to be up for the job: it seems that another idea will be necessary. Similar remarks hold for the weaker Property C6_{25}.

Another example that uses the idea of conservative reasoning, is our treatment of indifference: given a coherent set of indifferent options, we find the least informative coherent choice function that is compatible with it. It suffices to consider as a representing choice function the vacuous choice function on the (representing) quotient space. We combine this with a direct assessment as well: this leads to the natural extension under indifference.

Our ability to treat indifference in the light of conservative reasoning should be credited to the fact that we can define choice functions on arbitrary options that form a linear space. Furthermore, in this way, under some mild conditions, we can also embed Seidenfeld et al.'s [67] account of choice functions into our framework. However, we are less general in one respect: they allow for possibly infinite but closed option sets, while we only consider finite option sets. Many of our proofs depend on the option sets being finite; consider for instance the proofs of some of the direct consequences of coherence: Propositions 24_{38}, 25_{39}, 31_{42} and 34_{44}, as well as the result in Lemma 43_{52} that is crucial for proving that coherence is preserved for suprema of *chains* of coherent choice functions, and on which the proof that there are maximal choice functions depends.

Next to choice functions and rejection functions, we consider the equivalent model of choice relations. They are sometimes more elegant to work with, and help in clarifying the connection with desirability. But in Section 2.9_{76} we see yet another equivalent model: rejection sets. They seem to have an even tighter connection with desirability: consider for instance Proposition 68_{78} that establishes that the rejection set of a purely binary choice function is determined by the rejection set that consists of option sets of cardinality two. It is an interesting open question whether some concepts or discussions in this dissertation can be simplified by adopting a different mathematical language. An example where the use of rejection sets—or rather its variant of *coordinate* rejection sets—turns out to be very useful, is the characterisation in Section 4.4_{146} of coherent (and convex) choice functions on binary possibility spaces, and the counterexample in the same section, showing that Property C5_{25} does not guarantee representation in terms of lexicographic choice functions. One of the crucial observations that makes this possible, is the result in Proposition 117_{162} that choice functions on binary possibility spaces that satisfy Property C6_{25}, are completely determined by their (coordinate) rejection sets, whose elements consist of option sets of three elements. It is an open question whether such easy characterisation of coherent choice functions generalises to more than binary possibility spaces. However, Example 8_{80} shows that, even for ternary possibility spaces, there is no limit on the size of the elements of the rejection sets, which therefore will tend to complicate matters considerably.

Another interesting open problem is that of representation of coherent choice functions, in terms of maximal (or other) choice functions. Ideally, we would like to establish the following two properties. We would like to have representation in terms of maximal choice functions, in the sense that every coherent choice function is an infimum of its dominating maximal choice functions. Moreover, we would like the maximal choice functions to be purely binary choice functions corresponding to maximal sets of desirable options. Even though none of these properties is established, we have showed that not both can hold: the coherent rejection function in Example 16_{109} is no infimum of purely binary rejection functions. It is an interesting open problem to find out whether one of these two properties holds, and, if so, which one.

With respect to finding the maximal choice functions, we feel that rejection sets might help, too. For desirability, an important property that helps finding the maximal sets of desirable options, is $u \notin D \Rightarrow 0 \notin \operatorname{posi}(D \cup \{-u\})$ for any coherent set of desirable gambles D and non-zero option u. We have a strong intuition that this may be generalised in some way to choice models by using rejection sets, and we suspect that Lemma 80_{96} will play a role.

But there is an end to all human endeavour, and my work on this dissertation, however much I've enjoyed it, has run its course. It is my firm hope that my findings here may help me (at some later time) or others solve these and other important open questions.

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