Sets of desirable gamble sets & their representations **as Information Algebras**

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15 September 2026, Lecce

Presentation overview

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What is information algebra?

+ the goal of this paper

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sets of desirable gamble sets
+ *their information algebra*

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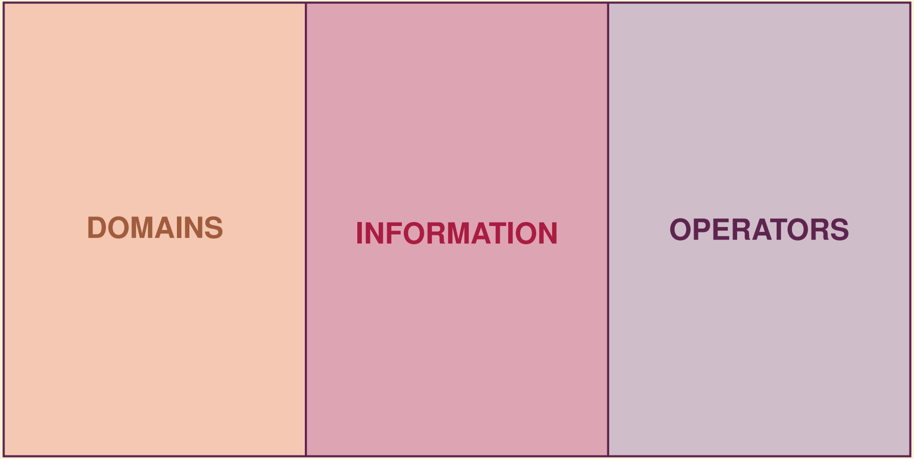
Why do we care?

Information Algebra



Information Algebra

DOMAINS

$X_N := \{X_1, \ldots, X_n\}$ finite family of uncertain variables,

X_ℓ takes values in finite Ω_ℓ;

each tuple $X_{S \subseteq N}$ takes values in $\Omega_S := \times_{s \in S} \Omega_s$.

We address the domain by its associated index set.

INFORMATION

OPERATORS

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All information in the algebra is contained in $\Phi := \bigcup_{S \subseteq N} \Phi_S$.

OPERATORS

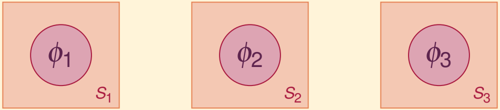
labelling operator: $d : \phi \mapsto S$,

marginalisation operator: $\pi : (\phi, S) \mapsto \pi_S(\phi)$,

combination operator: $\otimes : (\phi, \psi) \mapsto \phi \otimes \psi$.

The tuple $(\Phi, \otimes)$ ***forms a commutative semigroup.***

Information Algebra



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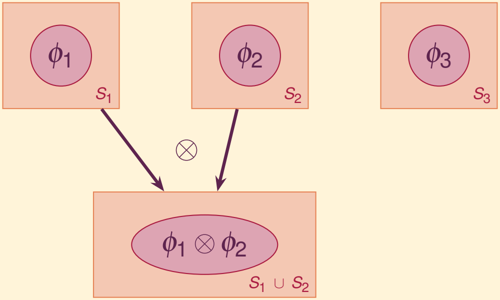

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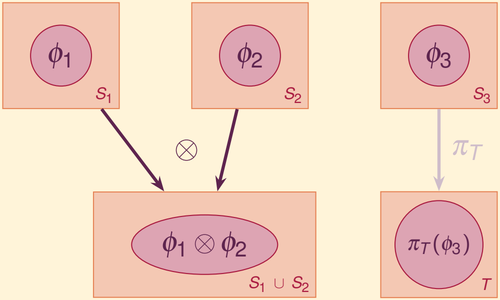

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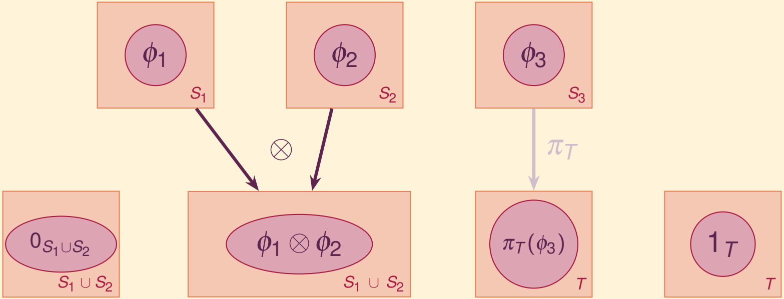

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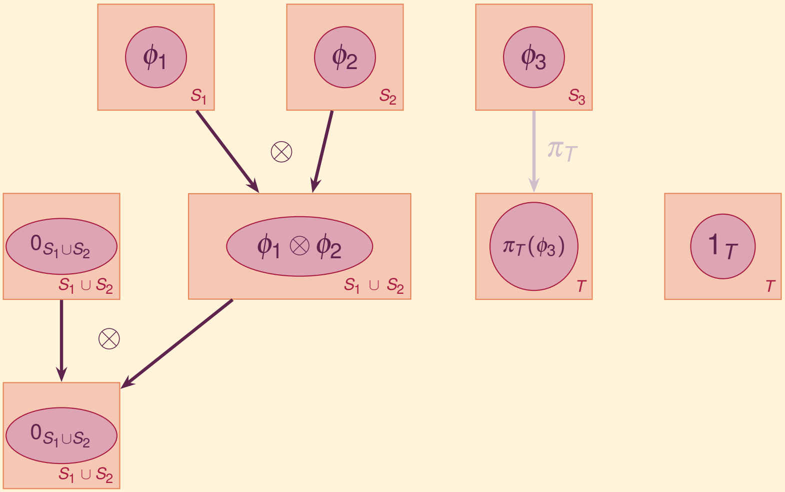

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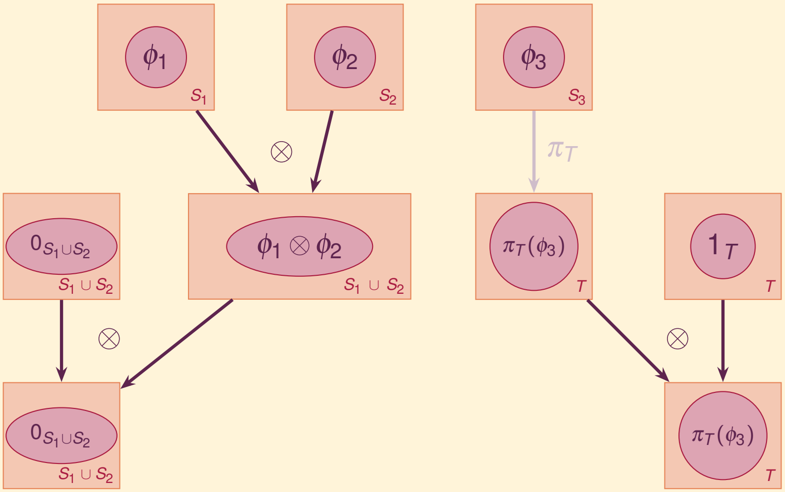

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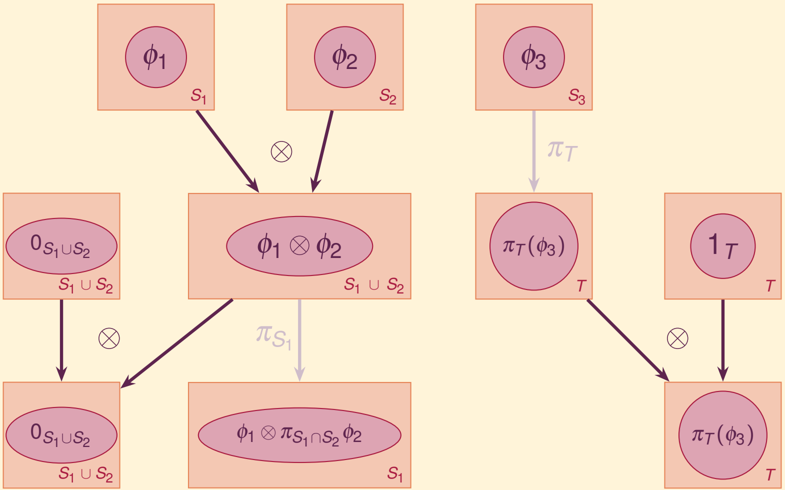

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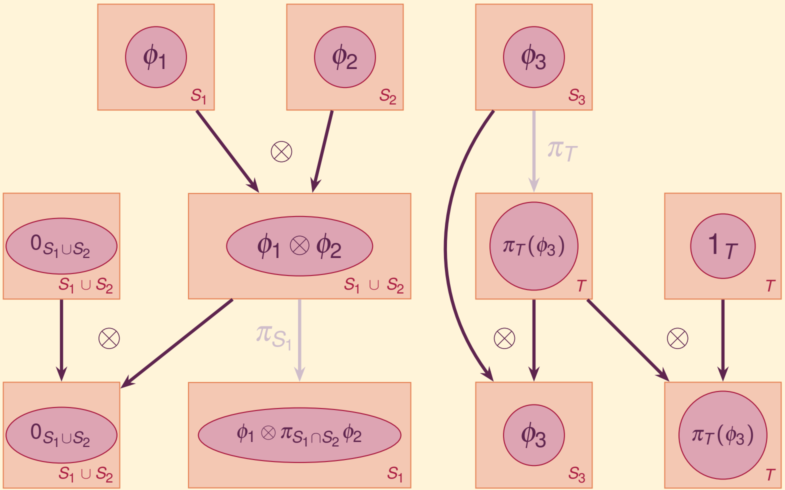

Information Algebra

♦ *a mathematically elegant framework general enough to capture many (imprecise) probabilistic models*

♦ *embeds marginalisation and closure operators within each framework*

♦ *provides structural relationships between different algebras*

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♦ *provides structural relationships between different algebras*

GOAL: ***Define information algebras of sets of desirable gamble sets and their representations, and study their relationship.***

Sets of desirable gamble sets

$\star$ = {*body temp., cough, blood pressure, ...*}
$$\Omega = \times_{*\in\star}\mathscr{X}_*$$

gambles $f : \Omega \to \mathbb{R}$

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$F = \{\alpha f + \beta f + \delta f : (\alpha, \beta, \delta) > 0\}$

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K: ***set of desirable gamble sets***

Sets of desirable gamble sets

- doing nothing is not desirable
- a miracle drug $f_{\bigstar}$ is desirable
- if $f_{💊}$, $f_{\oslash}$ are desirable, then so is $\alpha f_{💊} + \beta f_{\oslash}$

Sets of desirable gamble sets

K **is called** ***coherent*** **if:**

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Sets of desirable gamble sets

***K* is called *coherent* if:**

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K is called *coherent* if:

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K_1. if $F \in K$ then $F \setminus \{0\} \in K$;

K_2. $\{f\} \in K$ for all f in $\mathscr{L}_{>0}(\mathscr{X})$;

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K_2. $\{f\} \in K$ for all f in $\mathscr{L}_{>0}(\mathscr{X})$;

K_3. if $F, G \in \widehat{K}$ and $(\lambda^{f,g}, \mu^{f,g}) > 0$ for every pair (f,g) in $F \times G$, then $\{\lambda^{f,g} f + \mu^{f,g} g : f \in F, g \in G\} \in \widehat{K}$;

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K_3. if $\emptyset \neq \mathscr{A} \subseteq K$ and, for all $\sigma \in \Gamma_{\mathscr{A}}$, $f_\sigma \in \mathrm{cl}_{\overline{\mathscr{D}}}(\sigma(\mathscr{A}))$, then $\{f_\sigma : \sigma \in \Gamma_{\mathscr{A}}\} \in K$;

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K_4. if $F \in K$ and $F \subseteq G$, then $G \in K$.

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Coherent *K*s: $\overline{\mathscr{K}}(\Omega)$;
Closed *K*s: $\mathscr{K}(\Omega) = \overline{\mathscr{K}}(\Omega) \cup \{\mathscr{Q}(\mathscr{X})_\infty\}$;
Closure operator:

$$\mathrm{cl}_{\overline{\mathscr{K}}} : \mathscr{A} \mapsto \mathrm{cl}_{\overline{\mathscr{K}}}(\mathscr{A}) := \bigcap_{\mathscr{A} \subseteq K \in \mathscr{K}(\Omega)} K.$$

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- a miracle drug $f_{\star}$ is desirable
- if f_{pill}, f_{stop} are desirable, then so is $\alpha f_{\text{pill}} + \beta f_{\text{stop}}$

Information algebra of sets of desirable gamble sets

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$X_N := \{X_1, \ldots, X_n\}$,
$X_{S \subseteq N}$ takes values in $\Omega_S := \times_{s \in S} \Omega_s$

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OPERATORS

labelling operator: $d : K \mapsto S$,

marginalisation operator: $\mathrm{Marg} : (K, S) \mapsto \mathrm{Marg}_S K$,

combination operator: $\otimes : (K_1, K_2) \mapsto \mathrm{cl}_{\overline{\mathscr{K}}}(K_1 \cup K_2)$.

The tuple $(\mathscr{K}, \otimes)$ forms a commutative semigroup.

Collections of sets of desirable gambles: *representations*

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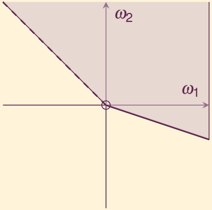
A set of desirable gambles $D \subseteq \mathscr{L}(\Omega)$ is called *coherent* if:

D_1. $0 \notin D$;

D_2. $\mathscr{L}_{>0}(\Omega) \subseteq D$;

D_3. if $f, g \in D$ and $(\lambda, \mu) > 0$, then $\lambda f + \mu g \in D$.



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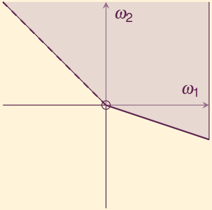

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Is there a connection between D and K?

Collections of sets of desirable gambles: *representations*

How do we construct K from $f, f \in D$?

f and f are desirable

Collections of sets of desirable gambles: *representations*

How do we construct K from $f_{\text{💊}}, f_{\oslash} \in D$?

Identify some gamble sets in K:
$\{f, g, h, \ldots, f_{\text{💊}}\}$, $\{g', f_{\oslash}\}$, $\{f', h', f_{\text{💊}}, f_{\oslash}\}$
all belong to K.

$f_{\text{💊}}$ and $f_{\oslash}$ are desirable

Collections of sets of desirable gambles: *representations*

How do we construct K from $f_{\text{💊}}, f_{\text{🚫}} \in D$?

Identify some gamble sets in K:
$\{f, g, h, \ldots, f_{\text{💊}}\}$, $\{g', f_{\text{🚫}}\}$, $\{f', h', f_{\text{💊}}, f_{\text{🚫}}\}$
all belong to K.

K defined based on this information only can be written as
$K_{\{f_{\text{💊}}, f_{\text{🚫}}\}} = \{F : F \cap \{f_{\text{💊}}, f_{\text{🚫}}\} \neq \emptyset\}$.

$f_{\text{💊}}$ and $f_{\text{🚫}}$ are desirable

Collections of sets of desirable gambles: *representations*

every gamble f in D is desirable

D is desirable

Collections of sets of desirable gambles: *representations*

every gamble f in D is desirable

every gamble set F in K contains some gamble f from D

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Collections of sets of desirable gambles: *representations*

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binary K is based on information from D: $K_D = \{F \in \mathscr{Q} : F \cap D \neq \emptyset\}$

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binary K is based on information from D: $K_D = \{F \in \mathcal{Q} : F \cap D \neq \emptyset\}$

for coherent K, collect all dominating K_Ds and corresponding Ds

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this collection of Ds is a ***representation*** *of K*

D is desirable

Information algebra of representations

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Homomorphism between the two algebras

Homomorphism between the two algebras

The mapping $h : \mathscr{A}_{\mathscr{D}} \to \mathscr{A}_K$ defined by $\mathscr{D} \mapsto K_{\mathscr{D}} = \bigcap_{D \in \mathscr{D}} K_D$ is a homomorphism if, for all $\mathscr{D}_1, \mathscr{D}_2$ representing K_1, K_2, respectively:

H$_1$. $\mathscr{D}_1 \otimes_{\mathscr{D}} \mathscr{D}_2$ represents $K_1 \otimes_K K_2$,

H$_2$. $\mathrm{marg}_S \mathscr{D} = \{\mathrm{marg}_S D : D \in \mathscr{D}\}$ represents $\mathrm{Marg}_S K$,

H$_3$. $\{\mathscr{L}_{>0}\}$ represents K_v and $\{\mathscr{L}\}$ represents $\mathscr{Q}_\infty$.

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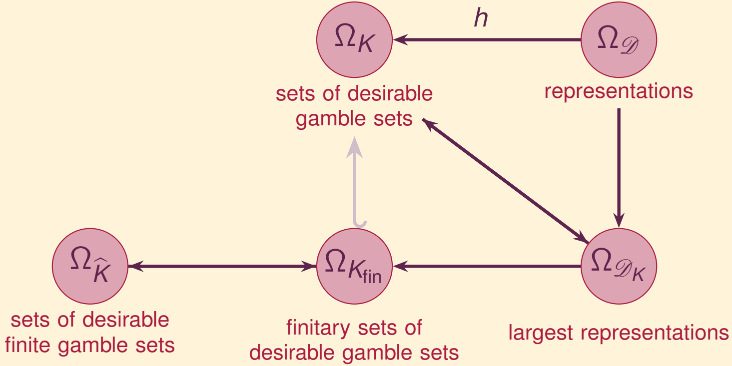
H_3. $\{\mathscr{L}_{>0}\}$ represents K_v and $\{\mathscr{L}\}$ represents $\mathscr{Q}_\infty$.

Theorem. The mapping $h : \mathscr{A}_{\mathscr{D}} \to \mathscr{A}_K$ defined by $\mathscr{D} \mapsto K_{\mathscr{D}} = \bigcap_{D \in \mathscr{D}} K_D$ is a homomorphism.

Information algebras of different types of Ks



Information algebras of different types of Ks

The mapping from a representation $\mathscr{D}$ in $\Omega_{\mathscr{D}}$ to its corresponding $\widehat{K}$ in $\Omega_{\widehat{K}}$ is a homomorphism.