Sets of Desirable Gamble Sets and their Representations as Information Algebras

Justyna Dąbrowska, Arthur Van Camp, and Erik Quaeghebeur

Eindhoven University of Technology
Groene Loper 3 5612AE Eindhoven, The Netherlands,
{j.s.dabrowska,a.van.camp,e.quaeghebeur}@tue.nl

Abstract. We study sets of desirable gamble sets, which are closely related to choice functions, from the perspective of information algebras, which is a framework for reasoning with partial information in a multivariate setting. We show that closed sets of desirable gamble sets, as well as their representations consisting of sets of closed sets of desirable gambles, form information algebras. For a natural map between these algebras, the marginalisation and combination operators commute. Finally, we prove that this mapping is indeed a homomorphism, guaranteeing that the two algebras contain the same information and induce the same marginalisation and combination operators.

Keywords: information algebra, sets of desirable gamble sets

1 Introduction

In this paper, we study sets of desirable (arbitrary) gamble sets from the perspective of information algebras. Information algebras provide a general mathematical framework for representing, combining and marginalising pieces of information on different domains. Coherent lower previsions and coherent sets of desirable gambles have already been established to form information algebras by Casanova et al. [1]. We establish the more general sets of desirable gamble sets as an information algebra as well, and show that their representations, consisting of collections of sets of desirable gambles, form an information algebra homomorphic to it.

Our main inspiration for this connection is the *marginal problem*, which we solved for sets of desirable *finite* gamble sets which are required to have a finite representation [2]. However, both requirements can be dropped by looking at coherent sets of desirable gamble sets from the information algebra perspective. Indeed, Casanova et al. [1, Section 6] point out that the marginal problem can be solved for *all* information algebras.

Next, in Section 2, we introduce sets of desirable gambles, sets of desirable gambles sets and their representations in the multivariate setting [5,10]. Our goal, in Section 3, is to show that both coherent sets of desirable gamble sets and their representations form such algebras and that there exists a homomorphism between them.

2 Sets of desirable gamble sets and their representations

Consider an uncertain variable X taking values in a finite possibility space Ω. A *gamble* f is a real-valued function that maps each possibility $\omega \in \Omega$ to a value $f(\omega) \in \mathbb{R}$ in a linear real utility scale. Having a gamble f yields the reward $f(\omega)$ after the real value $\omega \in \Omega$ is determined, so it may increase as well as decrease the agent's wealth. The set of all gambles on Ω, denoted $\mathcal{L}(\Omega)$, forms an $|\Omega|$-dimensional linear space under pointwise addition and scalar multiplication. The set of all positive gambles is denoted $\mathcal{L}_{>0}(\Omega) \coloneqq \{f \in \mathcal{L}(\Omega) : f > 0\}$.[1]

2.1 Sets of desirable gambles

An agent's preferences over gambles reveal information about her beliefs. They can be described by the gambles she prefers over 0, or in other words, finds desirable. Such preferences are modeled by a *set of desirable gambles.*

Definition 1 (Coherent set of desirable gambles [6]). *A set of desirable gambles $D \subseteq \mathcal{L}(\Omega)$ is called* coherent *if:*
D_1. $0 \notin D$;
D_2. $\mathcal{L}_{>0}(\Omega) \subseteq D$;
D_3. *if* $f, g \in D$ *and* $(\lambda, \mu) > 0$, *then* $\lambda f + \mu g \in D$.
The set of all coherent sets of desirable gambles is denoted by $\overline{\mathcal{D}}(\Omega) \subseteq \mathscr{P}(\mathcal{L}(\Omega))$, where $\mathscr{P}(\bullet)$ denotes the power set of its argument.

We say that D_1 is at least as informative as D_2 when $D_1 \supseteq D_2$; then D_2 is at least as conservative as D_1. The partially ordered set $(\overline{\mathcal{D}}(\Omega), \subseteq)$ is a complete meet-semilattice [6]. As a consequence, for every non-empty family of coherent sets $\{D_\ell : \ell \in L\}$, its infimum is given by $\bigcap_{\ell \in L} D_\ell$ and is itself coherent. This structure induces a closure operator

$$\mathrm{cl}_{\overline{\mathcal{D}}} : \mathscr{P}(\mathcal{L}(\Omega)) \to \overline{\mathcal{D}}(\Omega) \cup \{\mathcal{L}(\Omega)\} : A \mapsto \mathrm{cl}_{\overline{\mathcal{D}}}(A) \coloneqq \bigcap_{A \subseteq D \in \overline{\mathcal{D}}(\Omega) \cup \{\mathcal{L}(\Omega)\}} D$$

which maps each set of gambles A to its smallest coherent superset when such a superset exists, and then such an A is called *consistent*; otherwise, it maps A to the set of all gambles $\mathcal{L}(\Omega)$. We call $\mathrm{cl}_{\overline{\mathcal{D}}}(A)$ the *natural extension* of A and collect all sets of desirable gambles that are *closed* (under $\mathrm{cl}_{\overline{\mathcal{D}}}$) in $\mathcal{D}(\Omega) \coloneqq \overline{\mathcal{D}}(\Omega) \cup \{\mathcal{L}(\Omega)\}$. The smallest element of $\overline{\mathcal{D}}(\Omega)$ is $D_v \coloneqq \mathrm{cl}_{\overline{\mathcal{D}}}(\emptyset) = \mathcal{L}_{>0}(\Omega)$.

2.2 Sets of desirable gamble sets

When comparing two gambles f and g, the agent may be unable to judge whether f or g is desirable, knowing only that at least one of them is. In this case, the set $\{f, g\}$ contains a desirable gamble and is therefore called a *desirable*

[1] For every vector v, we take $v > 0$ to mean that all v's entries are non-negative and it has at least one (strictly) positive entry.

gamble set. The agent's assessments are collected in a *set of desirable gamble sets* $K \subseteq \mathscr{P}(\mathcal{L}(\Omega)) =: \mathcal{Q}(\Omega)_\infty$, and each element of $\mathcal{Q}(\Omega)_\infty$ is called a *gamble set.*

Under mild conditions, coherent sets of desirable gamble sets provide a representation that is equivalent to coherent choice functions [9]. Sets of desirable finite gamble sets were first introduced by De Bock & De Cooman in [3], and later generalised to sets of desirable gamble sets with arbitrary cardinality [4,7][2].

To introduce coherence, we need some additional machinery. For every $\emptyset \neq \mathcal{A} \subseteq \mathcal{Q}(\Omega)_\infty$, we call a map $\sigma : \mathcal{A} \to \mathcal{L}(\Omega) : F \mapsto \sigma(F)$ such that $\sigma(F) \in F$ a *selection map on* $\mathcal{A}$, and collect all such selection maps in $\Gamma_{\mathcal{A}}$. The set of all gambles selected by the selection map σ is then $\sigma(\mathcal{A}) \coloneqq \{\sigma(F) : F \in \mathcal{A}\} \subseteq \mathcal{L}(\Omega)$.

Definition 2 (Coherent set of desirable gamble sets [7,4]). *A set of desirable gamble sets* $K \subseteq \mathcal{Q}(\Omega)_\infty$ *is called* coherent *if:*
K_0. $\emptyset \notin K$;
K_1. *if* $F \in K$ *then* $F \setminus \{0\} \in K$;[3]
K_2. $\{f\} \in K$ *for all* f *in* $\mathcal{L}_{>0}(\Omega)$;
K_3. *if* $\emptyset \neq \mathcal{A} \subseteq K$ *and, for all* $\sigma \in \Gamma_{\mathcal{A}}$, $f_\sigma \in \mathrm{cl}_{\overline{\mathcal{D}}}(\sigma(\mathcal{A}))$, *then* $\{f_\sigma : \sigma \in \Gamma_{\mathcal{A}}\} \in K$;
K_4. *if* $F \in K$ *and* $F \subseteq G$, *then* $G \in K$.
The set of all coherent sets of desirable gamble sets is denoted by $\overline{\mathcal{K}}(\Omega)$.

We say that K_1 is at least as informative as K_2 when $K_1 \supseteq K_2$; then K_2 is at least as conservative as K_1. The partially ordered set $(\overline{\mathcal{K}}(\Omega), \subseteq)$ is also a complete meet-semilattice [7]. As a consequence, here too, for every non-empty family of coherent sets $\{K_\ell : \ell \in L\}$, its infimum is given by $\bigcap_{\ell \in L} K_\ell$ and is itself coherent. This structure induces a closure operator

$$\mathrm{cl}_{\overline{\mathcal{K}}} : \mathscr{P}(\mathcal{Q}(\Omega)_\infty) \to \overline{\mathcal{K}}(\Omega) \cup \{\mathcal{Q}(\Omega)_\infty\} : \mathcal{A} \mapsto \mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A}) \coloneqq \bigcap\nolimits_{\mathcal{A} \subseteq K \in \overline{\mathcal{K}}(\Omega) \cup \{\mathcal{Q}(\Omega)_\infty\}} K$$

which maps each $\mathcal{A} \subseteq \mathcal{Q}(\Omega)_\infty$ to its smallest coherent superset, when such a superset exists and then such an $\mathcal{A}$ is also called *consistent*; otherwise, it maps $\mathcal{A}$ to $\mathcal{Q}(\Omega)_\infty$. We call $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A})$ the *natural extension* of $\mathcal{A}$ and collect all sets of desirable gamble sets that are *closed* (under $\mathrm{cl}_{\overline{\mathcal{K}}}$) in $\mathcal{K}(\Omega) \coloneqq \overline{\mathcal{K}}(\Omega) \cup \{\mathcal{Q}(\Omega)_\infty\}$. The smallest element of $\mathcal{K}(\Omega)$ is $K_\nu(\Omega) \coloneqq \mathrm{cl}_{\overline{\mathcal{K}}}(\emptyset) = \{F \in \mathcal{Q}(\Omega)_\infty : (\exists f \in \mathcal{L}_{>0}(\Omega)) f \in F\}$ [7, Proposition 3].

2.3 Representation

Sets of desirable gamble sets are generally difficult to work with, so we often opt to work with alternative representations, consisting of sets of desirable gambles.

If the agent has a set of desirable gambles D, then the information in D can also be expressed using a set of desirable gamble sets: every singleton $\{f\}$ for $f \in D$ is itself a desirable gamble set. The corresponding set of desirable gamble

[2] There, the authors worked with *sets of things*, a generalisation of gamble sets. Since we are interested in a multivariate setting, we stick to *gamble sets*.

[3] The version of this axiom generalised to things [7,4] corresponds to 'if $F \in K$ then $F \setminus \{f : 0 \geq f\} \in K$', which is equivalent to Axiom K_1 under the other axioms.

sets is $K_D \coloneqq \mathrm{cl}_{\overline{\mathcal{K}}}(\{\{f\} : f \in D\}) = \{F \in \mathcal{Q}(\Omega)_\infty : F \cap D \neq \emptyset\}$, which we call *binary* [3,7, contain more information]. For example, the smallest coherent set of desirable gamble sets is binary, as $K_v(\Omega) = \{F \in \mathcal{Q}(\Omega)_\infty : F \cap D_v \neq \emptyset\} = K_{D_v}$.

Theorem 3 (Representation, based on [7, Theorem 26]). *For any set of desirable gamble sets K, the following two expressions are equivalent:*
(i) $K \in \mathcal{K}$*;*
(ii) there is a non-empty set $\mathfrak{D} \subseteq \mathcal{D}$ of closed sets of desirable gambles such that $K = K_{\mathfrak{D}} \coloneqq \bigcap\{K_D : D \in \mathfrak{D}\}$, in which case $\mathfrak{D}$ represents *K.*
Moreover, K's (unique) largest representing set is $\mathfrak{D}_K \coloneqq \{D \in \mathcal{D} : K \subseteq K_D\}$.

This important theorem shows that sets of desirable gamble sets can be characterised – represented – by a set of binary preferences.

2.4 Marginalisation

To talk about marginalisation, we move to a multivariate setting. Let $X_N = \{X_1, \dots, X_n\}$ be a finite family of uncertain variables indexed by the global index set $N = \{1, \dots, n\}$, where each uncertain variable X_ℓ takes values in the finite possibility space Ω_ℓ. We denote by Ω_S the Cartesian product of the possibility spaces associated with the variables in X_S for every $S \subseteq N$, that is, $\Omega_S \coloneqq \bigtimes_{s \in S} \Omega_s$. For the global index set N, we let Ω_N be Ω.

For every index set $S \subseteq N$, let X_S be a tuple of uncertain variables taking values in Ω_S. Then for every gamble $f \in \mathcal{L}(\Omega)$ and every $\omega_S \in \Omega_S$, the partial map $f(\omega_S, \bullet)$ is a gamble on $\Omega_{N \setminus S}$. Moreover, for every gamble $f \in \mathcal{L}(\Omega_S)$ for some $S \subseteq N$, turning it into a gamble on the global domain Ω is done using *cylindrical extension*.

Definition 4 (Cylindrical extension). *Given two disjoint subsets I and J of N and any gamble f on Ω_I, we let its* cylindrical extension *f^J to $\Omega_{I \cup J}$ be defined by $f^J(\omega_I, \omega_J) \coloneqq f(\omega_I)$ for all ω_I in Ω_I and ω_J in Ω_J. Similarly, given any set of gambles $F \subseteq \mathcal{L}(\Omega_I)$, we let its* cylindrical extension *$F^J \subseteq \mathcal{L}(\Omega_{I \cup J})$ be defined as $F^J \coloneqq \{f^J : f \in F\}$.*

Both gambles f on Ω_I and f^J on $\Omega_{I \cup J}$ represent the same information, so we do not distinguish them notationally and write $f = f^J$. This allows us to relate gambles and gamble sets on different domains: $\mathcal{L}(\Omega_I) \subseteq \mathcal{L}(\Omega_{I \cup J})$, and $\mathcal{Q}(\Omega_I)_\infty \subseteq \mathcal{Q}(\Omega_{I \cup J})_\infty$. Conversely, suppose that we have beliefs contained in a model on Ω_T, and wish to model information on a subset Ω_S for some $S \subseteq T$. For both sets of desirable gambles and sets of desirable gamble sets, this operation is called *marginalisation*.

Definition 5 (Marginalisation). *Consider any $S \subseteq T$. For any set of desirable gambles $D \subseteq \mathcal{L}(\Omega_T)$, its S-*marginal *$\mathrm{marg}_S D \subseteq \mathcal{L}(\Omega_S)$ is defined as* $\mathrm{marg}_S D \coloneqq D \cap \mathcal{L}(\Omega_S)$. *Similarly, for any set of desirable gamble sets $K \subseteq \mathcal{Q}(\Omega_T)_\infty$, its S-*marginal *$\mathrm{Marg}_S K \subseteq \mathcal{Q}(\Omega_S)_\infty$ is defined as* $\mathrm{Marg}_S K \coloneqq K \cap \mathcal{Q}(\Omega_S)_\infty$.

For sets of desirable gambles, marginalisation preserves the order $\subseteq$ and coherence, and since $\mathrm{marg}_S \mathcal{L}(\Omega) = \mathcal{L}(\Omega_S)$, also closedness.

Proposition 6 (based on [10, Proposition 11]). *Marginalisation preserves the order* $\subseteq$ *and closedness for sets of desirable gamble sets. Additionally, for any closed* $K \in \mathcal{K}(\Omega)$ *with representation* $\mathfrak{D}$*, its* S*-marginal* $\mathrm{Marg}_S K$ *is represented by* $\mathrm{marg}_S \mathfrak{D} \coloneqq \{\mathrm{marg}_S D : D \in \mathfrak{D}\}$.

3 Information algebras

In this section we introduce information algebras [8] and establish the main result of our paper, namely that sets of desirable gamble sets and their representations both form information algebras that moreover are homomorphic. Information algebras can be either labelled or unlabelled (sometimes called *domain-free*), and it has been shown that the resulting structures are equivalent [1]. In this paper, we choose to work with a labelled variant.

The primitive elements of an information algebra are called *pieces of information* or *information pieces*, which we take to be belief models. Given a (possibly empty) tuple of variables $X_S \coloneqq (X_s)_{s \in S}$, the set Φ_S collects the information pieces on X_S. For a finite global index set N, we let $\Phi \coloneqq \bigcup_{S \subseteq N} \Phi_S$ be the set of all pieces of information, whose generic elements are denoted by ϕ or ψ.

Definition 7. *[1, Definition 12] A* labelled information algebra *is a two-sorted structure* $(\Phi, N, d, \otimes, \{0_S\}_{S \subseteq N}, \{1_S\}_{S \subseteq N}, \pi)$ *satisfying the following properties, for all* ϕ *and* ψ *in* Φ*:*

I$_1$. $d(\phi \otimes \psi) = d(\phi) \cup d(\psi)$, [Labelling]
I$_2$. $d(\pi_S(\phi)) = S$ *for all* $S \subseteq d(\phi)$, [Marginalisation]
I$_3$. $\pi_T(\pi_S(\phi)) = \pi_T(\phi)$ *for all* $T \subseteq S \subseteq d(\phi)$, [Transitivity]
I$_4$. $\pi_S(\phi \otimes \psi) = \phi \otimes \pi_{S \cap T}(\psi)$*, if* $S = d(\phi)$ *and* $T = d(\psi)$, [Combination]
I$_5$. $1_S \otimes 1_T = 1_{S \cup T}$ *and* $0_S \otimes 0_T = 0_{S \cup T}$*, for all* $S, T \subseteq N$, [Nullity and neutrality]
I$_6$. $\phi \otimes \pi_S(\phi) = \phi$ *for all* $S \subseteq d(\phi)$, [Idempotency]
I$_7$. $\pi_T(1_S) = 1_T$ *for all* $T \subseteq S \subseteq N$, [Stability]

where
- $d : \Phi \to \mathscr{P}(N)$*:* $\phi \mapsto d(\phi)$ *is a labelling operator,*
- $(\Phi, \otimes)$ *is a* commutative semigroup *with combination operator* $\otimes : \Phi \times \Phi \to \Phi$*:* $(\phi, \psi) \mapsto \phi \otimes \psi$*, with the additional requirement that for all* $S \subseteq N$*, there exist elements* 0_S *and* 1_S *with* $d(0_S) = d(1_S) = S$ *such that for all* $\phi \in \Phi$ *with* $d(\phi) = S$, $0_S \otimes \phi = \phi \otimes 0_S = 0_S$ *and* $1_S \otimes \phi = \phi \otimes 1_S = \phi$,
- $(\mathscr{P}(N), \cap, \cup)$ *is the lattice constructed from the power set of the non-empty index set* N *ordered by inclusion,*
- $\pi : \Phi \times \mathscr{P}(N) \to \Phi$*:* $(\phi, S) \mapsto \pi_S(\phi)$ *is a* marginalisation operator *extracting the information about the variables indexed by* S *from the piece of information* ϕ.

3.1 Algebra of sets of desirable gamble sets

The set of all pieces of information for a given index set $S \subseteq N$ is $(\Phi_{\mathcal{K}})_S \coloneqq \mathcal{K}(\Omega_S)$, and the global set of all information pieces in the algebra is $\Phi_{\mathcal{K}} \coloneqq \bigcup_{S \subseteq N} (\Phi_{\mathcal{K}})_S =$

$\bigcup_{S\subseteq N}\mathcal{K}(\Omega_S)$. Let $d_{\mathcal{K}}$ denote the labelling operator that maps each closed set of desirable gamble sets $K \subseteq \mathcal{Q}(\Omega_S)_\infty$ to the corresponding index set $d_{\mathcal{K}}(K) = S$ of its domain. We identify the marginalisation operator π with the operator

$$\mathrm{Marg} : \Phi_{\mathcal{K}} \times \mathscr{P}(N) \to \Phi_{\mathcal{K}} : (K, S) \mapsto \mathrm{Marg}_{S\cap d(K)}(K).$$

For every ϕ in the $\Phi_{\mathcal{K}}$ defined above and $S \subseteq N$, the fact that marginalisation preserves closedness implies that $\pi_S(\phi)$ belongs to $\Phi_{\mathcal{K}}$, too. We define the combination operator $\otimes_{\mathcal{K}}$ as

$$\otimes_{\mathcal{K}} : \Phi_{\mathcal{K}} \times \Phi_{\mathcal{K}} \to \Phi_{\mathcal{K}} : (K_1, K_2) \mapsto K_1 \otimes_{\mathcal{K}} K_2 \coloneqq \mathrm{cl}_{\overline{\mathcal{K}}}(K_1 \cup K_2).$$

The only non-coherent element of $(\Phi_{\mathcal{K}})_S$ – the set of all gamble sets $\mathcal{Q}(\Omega_S)_\infty$ – is the result of the combination of two elements whose union does not have a coherent superset.

Theorem 8 (Algebra of sets of desirable gamble sets). *Closed sets of desirable gamble sets form an information algebra:*

$$\mathcal{A}_{\mathcal{K}} \coloneqq \left(\bigcup_{S\subseteq N}\mathcal{K}(\Omega_S), N, d_{\mathcal{K}}, \otimes_{\mathcal{K}}, \{\mathcal{Q}(\Omega_S)_\infty\}_{S\subseteq N}, \{K_v(\Omega_S)\}_{S\subseteq N}, \mathrm{Marg}\right).$$

3.2 Algebra of representations

In this section, we construct an information algebra consisting of non-empty collections of closed sets of desirable gambles as primitive elements. Taking into account Theorem 3, the primitive elements of this algebra are representations of primitive elements of the algebra $\Phi_{\mathcal{K}}$. This algebra of representations will generally be much larger, since a closed set of desirable gamble sets may admit multiple (often infinitely many) representations.

Consider the collection of all non-empty sets of closed sets of desirable gambles $(\Phi_{\mathfrak{D}})_S \coloneqq \mathscr{P}(\mathcal{D}(\Omega_S)) \setminus \{\emptyset\}$ on a given domain Ω_S indexed by $S \subseteq N$, which contains all representations of elements of $\mathcal{K}(\Omega_S)$, and let $\Phi_{\mathfrak{D}} \coloneqq \bigcup_{S\subseteq N}(\Phi_{\mathfrak{D}})_S$. On the other hand, every element of the power set $(\Phi_{\mathfrak{D}})_S$ represents *some* $K \in \mathcal{K}(\Omega_S)$: indeed, consider any $\mathfrak{D} \in (\Phi_{\mathfrak{D}})_S$, then Theorem 3 guarantees that $K_{\mathfrak{D}}$ is closed. This tells us that every element in $\Phi_{\mathfrak{D}}$ represents some element in $\Phi_{\mathcal{K}}$, and vice versa, that every element in $\Phi_{\mathcal{K}}$ is represented by some element in $\Phi_{\mathfrak{D}}$.

For any set of closed sets of desirable gambles $\phi_S \in (\Phi_{\mathfrak{D}})_S$ on the domain Ω_S indexed by $S \subseteq N$, we let $d_{\mathfrak{D}}$ be the operator that returns the corresponding index set $d_{\mathfrak{D}}(\phi_S) = S$. We define the combination operator $\otimes_{\mathfrak{D}}$ as

$$\begin{aligned}&\otimes_{\mathfrak{D}} : \Phi_{\mathfrak{D}} \times \Phi_{\mathfrak{D}} \to \Phi_{\mathfrak{D}} :\\ &\qquad (\mathfrak{D}_1, \mathfrak{D}_2) \mapsto \mathfrak{D}_1 \otimes_{\mathfrak{D}} \mathfrak{D}_2 \coloneqq \{\mathrm{cl}_{\overline{\mathcal{D}}}(D_1 \cup D_2) : D_1 \in \mathfrak{D}_1, D_2 \in \mathfrak{D}_2\}.\end{aligned}$$

This pointwise definition of the combination operator $\otimes_{\mathfrak{D}}$ assures that every result of a combination is closed, so the combination operator is well-defined for the algebra of representations. The marginalisation operator is a pointwise applied version of the marginalisation operator for sets of desirable gambles: $\mathrm{marg} : \Phi_{\mathfrak{D}} \times \mathscr{P}(N) \to \Phi_{\mathfrak{D}} : (\phi, S) \mapsto \mathrm{marg}_{S\cap d_{\mathfrak{D}}(\phi)}(\phi) \coloneqq \{\mathrm{marg}_{S\cap d_{\mathfrak{D}}(\phi)} D : D \in \phi\}$. Since marginalisation preserves closedness, $\mathrm{marg}_S(\phi)$ belongs to $\Phi_{\mathfrak{D}}$, as desired.

Theorem 9 (Algebra of representations). *Non-empty sets of closed sets of desirable gambles form an information algebra:*

$$\mathcal{A}_{\mathfrak{D}} := \big(\bigcup_{S\subseteq N}\mathscr{P}(\mathcal{D}(\Omega_S)), N, d_{\mathfrak{D}}, \otimes_{\mathfrak{D}}, \{\{\mathcal{L}(\Omega_S)\}\}_{S\subseteq N}, \{\{\mathcal{L}_{>0}(\Omega_S)\}\}_{S\subseteq N}, \mathrm{marg}\big).$$

3.3 Homomorphism between the two information algebras

Based on Theorem 3, it is natural to consider a mapping between the information algebra of closed sets of desirable gamble sets $\mathcal{A}_{\mathcal{K}}$ and the algebra of their representations $\mathcal{A}_{\mathfrak{D}}$. We begin by defining a homomorphism of two algebras.

Definition 10. *Let* $\mathcal{A}_a = (\Phi_a, N, d_a, \otimes_a, \{(0_a)_S\}_{S\subseteq N}, \{(1_a)_S\}_{S\subseteq N}, \pi_a)$ *and* $\mathcal{A}_b = (\Phi_b, N, d_b, \otimes_b, \{(0_b)_S\}_{S\subseteq N}, \{(1_b)_S\}_{S\subseteq N}, \pi_b)$ *be two information algebras. A mapping* $h : \Phi_a \to \Phi_b$ *is called a* homomorphism from $\mathcal{A}_a$ to $\mathcal{A}_b$ *if, for all* $\phi, \psi \in \Phi_a$*:*
H_1. $h(\phi \otimes_a \psi) = h(\phi) \otimes_b h(\psi)$*,*
H_2. $h\big((\pi_a)_S(\phi)\big) = (\pi_b)_S(h(\phi))$ *for all* $S \subseteq N$*,*
H_3. $h\big((1_a)_S\big) = (1_b)_S$ *and* $h\big((0_a)_S\big) = (0_b)_S$ *for all* $S \subseteq N$*.*

Consider the mapping $\gamma : \Phi_{\mathfrak{D}} \to \Phi_{\mathcal{K}} : \mathfrak{D} \mapsto K_{\mathfrak{D}} = \bigcap_{D\in\mathfrak{D}} K_D$ that maps a representation to its corresponding closed set of desirable gamble sets. Note that γ has no inverse, since a closed K may have multiple representations. We show that γ is a homomorphism.

That γ satisfies property H_2 is established in Proposition 6. For every $S \subseteq N$, the null element $\{\mathcal{L}(\Omega_S)\}$ of $\mathcal{A}_{\mathfrak{D}}$ gets mapped to the corresponding null element $\gamma(\{\mathcal{L}(\Omega_S)\}) = K_{\mathcal{L}(\Omega_S)} = \mathcal{Q}(\Omega)_\infty$ of $\mathcal{A}_K$, and the unit element $\{\mathcal{L}_{>0}(\Omega_S)\}$ gets mapped to the corresponding unit element $\gamma(\{\mathcal{L}_{>0}(\Omega_S)\}) = K_{\mathcal{L}_{>0}(\Omega_S)} = K_{\mathrm{v}}(\Omega)$ of $\mathcal{A}_K$, so γ satisfies property H_3. We establish that γ also satisfies the remaining property H_1.

Proposition 11. *Consider any two non-empty sets of closed sets of desirable gambles* $\mathfrak{D}_1, \mathfrak{D}_2 \in \Phi_{\mathfrak{D}}$*. Then* $K_{\mathfrak{D}_1\otimes\mathfrak{D}_2} = K_{\mathfrak{D}_1} \otimes K_{\mathfrak{D}_2}$*.*

We summarise this in the main result of this paper.

Theorem 12. *The mapping* γ *is a homomorphism from* $\mathcal{A}_{\mathfrak{D}}$ *to* $\mathcal{A}_{\mathcal{K}}$*.*

4 Conclusions

We have established that closed sets of desirable gamble sets form an information algebra, and therefore serve as a general framework for reasoning in a multivariate context. This implies, for instance, that they solve the marginal problem. We have also shown that their representations – non-empty sets of closed sets of desirable gambles – also form an information algebra, and have identified a homomorphism between these two algebras. This is important because it allows us to transfer results from the representations, which are typically easier to work with, to sets of desirable gamble sets.

Related work often considers sets of desirable *finite* gamble sets, where each K consists of finite gamble sets. Similar arguments show that closed sets of desirable finite gamble sets form an information algebra, but whether there is a homomorphism from the algebra of their representations remains open: specifically, our proof for Proposition 11 relies crucially on the fact that our sets of desirable gamble sets contain infinite gamble sets.

Our homomorphism γ is not injective, since a closed set of desirable gamble sets may have multiple representations. It remains open whether there is a *canonical representation* that forms an information algebra isomorphic to the algebra of sets of desirable gamble sets. A natural candidate is the unique *largest representation*, but it is unclear whether combination and marginalisation preserve it. Another avenue for further research is to investigate whether existing results for information algebras – such as the construction of junction trees – extend to the algebras we have introduced. Finally, the properties given by the homomorphism are crucial to describe Bayesian networks for choice functions.

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