Lexicographic Choice Functions Without Archimedeanicity

Arthur Van Camp, Enrique Miranda and Gert de Cooman

Abstract We investigate the connection between choice functions and lexicographic probabilities, by means of the convexity axiom considered by Seidenfeld et al. (Synthese 172:157–176, 2010 [7]) but without imposing any Archimedean condition. We show that lexicographic probabilities are related to a particular type of sets of desirable gambles, and investigate the properties of the coherent choice function this induces via maximality. Finally, we show that the convexity axiom is necessary but not sufficient for a coherent choice function to be the infimum of a class of lexicographic ones.

Keywords Choice functions · Lexicographic probabilities · Archimedeanicity · Maximality

1 Introduction

A prominent decision model under uncertainty is that of *choice functions* [5]. To be able to deal with imprecise information, Seidenfeld et al. proposed an axiomatisation of coherent choice functions in [7] that generalised Rubin's [5] to allow for incomparability. They also established a representation theorem of coherent choice functions by means of probability/utility pairs.

From an imprecise probabilities perspective, choice functions can be seen as a more general model than sets of desirable gambles, because preferences are not uniquely determined by pairwise comparisons between options. We investigated this

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idea in [10], and in particular we studied the connections between choice functions and the notions of desirability and indifference. In order to do so, we applied the above-mentioned axiomatisation [7] to gambles instead of horse lotteries, and also removed two axioms: (i) the Archimedean one, because it prevents choice functions from modelling the preferences captured by coherent sets of desirable gambles; and (ii) the convexity axiom, because that is incompatible with maximality as a decision rule, something that is closely tied in with coherent sets of desirable gambles. Although this alternative axiomatisation is more general, it also has the drawback of not leading to a Rubinesque representation theorem, or in other words, to a strong belief structure [2].

In the present paper, we add more detail to our previous findings [10] by investigating in more detail the implications of the convexity axiom, while still letting go of archimedeanicity. We show that, if a Rubinesque representation theorem were possible, it would involve lexicographic probabilities, but that unfortunately such a representation is not generally guaranteed. In establishing this, we derive some properties of coherent choice functions in terms of their so-called rejection sets.

The paper is organised as follows: in Sect. 2, we provide the basics of the theory of choice functions that we need for the rest of the paper. The connection with lexicographic probabilities and the connection with a representation theorem is addressed in Sect. 3. Some additional comments and remarks are provided in Sect. 4. Due to limitations of space, many of the proofs have been omitted.

2 Coherent Choice Functions

Consider a finite possibility space $\mathcal{X}$ in which a random variable X takes values. We denote by $\mathcal{L}$ the set of all gambles—real-valued functions—on $\mathcal{X}$. Typically, a gamble $f(X)$ is interpreted as an uncertain reward: if the actual outcome turns out to be x in $\mathcal{X}$, then the subject's capital changes by $f(x)$. For any two gambles f and g, we write $f \leq g$ when $f(x) \leq g(x)$ for all x in $\mathcal{X}$, and we write $f < g$ when $f \leq g$ and $f \neq g$. We collect all gambles f for which $f > 0$ in $\mathcal{L}_{>0}$.

For a subset O of $\mathcal{L}$, we define its *positive hull* as $\operatorname{posi}(O) := \left\{\sum_{k=1}^{n} \lambda_k f_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, f_k \in O\right\} \subseteq \mathcal{L}$, and its *convex hull* as $\operatorname{CH}(O) := \left\{\sum_{k=1}^{n} \alpha_k f_k : n \in \mathbb{N}, \alpha_k \in \mathbb{R}_{\geq 0}, \sum_{k=1}^{n} \alpha_k = 1, f_k \in O\right\} \subseteq \mathcal{L}$, where $\mathbb{R}_{>0}$ ($\mathbb{R}_{\geq 0}$) is the set of all positive (non-negative) real numbers. For any two subsets O_1 and O_2 of $\mathcal{L}$ and any λ in $\mathbb{R}$, we let $\lambda O_1 := \{\lambda f : f \in O_1\}$ and $O_1 + O_2 := \{f + g : f \in O_1, g \in O_2\}$.

We denote by $\mathcal{Q}$ the set of all non-empty *finite* subsets of $\mathcal{L}$. Elements O of $\mathcal{Q}$ are the option sets amongst which a subject can choose his preferred options.

Definition 1 A *choice function* C is a map $C \colon \mathcal{Q} \to \mathcal{Q} \cup \{\emptyset\} \colon O \mapsto C(O)$ such that $C(O) \subseteq O$.

The interpretation is that a choice function C selects the set $C(O)$ of 'best' options in the *option set* O. Our definition resembles the one commonly used in the literature [1, 7, 9], except for a (also not unusual) restriction to *finite* option sets [6, 8].

Equivalently to a choice function C, we consider its *rejection function* R, defined $R(O) \coloneqq O \setminus C(O)$ for all O in $\mathcal{Q}$. It returns the gambles that are not selected by C.

In this paper, we focus on coherent choice functions.

Definition 2 We call a choice function C on $\mathcal{Q}$ *coherent* if for all O, O_1, O_2 in $\mathcal{Q}$, f, g in $\mathcal{L}$ and λ in $\mathbb{R}_{>0}$:

C_1. $C(O) \neq \emptyset$;
C_2. if $f < g$ then $\{g\} = C(\{f, g\})$;
C_3. a. if $C(O_2) \subseteq O_2 \setminus O_1$ and $O_1 \subseteq O_2 \subseteq O$ then $C(O) \subseteq O \setminus O_1$;
b. if $C(O_2) \subseteq O_1$ and $O \subseteq O_2 \setminus O_1$ then $C(O_2 \setminus O) \subseteq O_1$;
C_4. a. if $O_1 \subseteq C(O_2)$ then $\lambda O_1 \subseteq C(\lambda O_2)$;
b. if $O_1 \subseteq C(O_2)$ then $O_1 + \{f\} \subseteq C(O_2 + \{f\})$.

These axioms are a subset of the ones studied by Seidenfeld et al. [7], translated from horse lotteries to gambles. We have not included the Archimedean axiom, which makes our definition more general. This is important in order to make the connection with the sets of desirable gambles we recall below.

In this paper, we intend to investigate in some detail the implications of an additional axiom in [7], namely

C_5. if $O \subseteq O_1 \subseteq \mathrm{CH}(O)$ then $C(O) \subseteq C(O_1)$ for all O and O_1 in $\mathcal{Q}$,

also referred to as the *convexity axiom*. One useful property we shall have occasion to use further on is the following:

Proposition 1 *Let C be a choice function on $\mathcal{L}$ satisfying* C_3*a,* C_4a *and* C_5*. Then for any $n \in \mathbb{N}, f_1, f_2, \ldots, f_n \in \mathcal{L}$ and $\lambda_1, \lambda_2, \ldots \lambda_n \in \mathbb{R}_{>0}$:*

$$0 \in C(\{0, f_1, f_2, \ldots, f_n\}) \Leftrightarrow 0 \in C(\{0, \lambda_1 f_1, \lambda_2 f_2, \ldots, \lambda_n f_n\}).$$

For two choice functions C and C', we call C *not more informative* than C'—and we write $C \sqsubseteq C'$—if $C(O) \supseteq C'(O)$ for all O in $\mathcal{Q}$. The binary relation $\sqsubseteq$ is a partial order, and for any collection $\mathcal{C}'$ of choice functions, its infimum $\inf \mathcal{C}'$ exists, and is given by $\inf \mathcal{C}'(O) = \bigcup_{C \in \mathcal{C}'} C(O)$ for all O in $\mathcal{Q}$. Coherence is preserved under arbitrary infima [10, Proposition 3], and it is easy to show that so is convexity:

Proposition 2 *For any collection $\mathcal{C}'$ of choice functions that satisfy* C_5*, its infimum* $\inf \mathcal{C}'$ *satisfies* C_5 *as well.*

One important way of defining coherent choice functions is by means of sets of desirable gambles. This connection is explored in some detail in [10]. A set of desirable gambles D is simply a subset of the vector space of gambles $\mathcal{L}$. The underlying idea is that a subject finds every gamble f in her set of desirable gambles strictly better than the status quo—she has a strict preference for the uncertain reward f over 0. As we did for choice functions, we pay special attention to *coherent* sets of desirable gambles, see for instance [3] for a detailed discussion.

Definition 3 ([3]) A set of desirable gambles D is called *coherent* when $D = \mathrm{posi}(D \cup \mathcal{L}_{>0})$ and $0 \notin D$. We collect all coherent sets of desirable gambles in the set $\bar{\mathcal{D}}$.

We may associate with any $D \in \bar{\mathcal{D}}$ a strict partial order $\prec_D$ on $\mathcal{L}$, by letting $f \prec_D g \Leftrightarrow 0 \prec_D g - f \Leftrightarrow g - f \in D$, so $D = \{f \in \mathcal{L} : 0 \prec_D f\}$; see for instance [3]. This correspondence is one-to-one.

We may also associate with a coherent set of desirable gambles D a choice function C_D based on maximality. For any O in $\mathcal{Q}$, we let $C_D(O)$ be the set of gambles that are undominated, or maximal, in O:

$$C_D(O) := \{f \in O : (\forall g \in O) g - f \notin D\} = \{f \in O : (\forall g \in O) f \nprec_D g\}.$$

Interestingly, the coherent choice function C_D associated with a coherent set of desirable gambles D need not satisfy C_5:

Proposition 3 *For any coherent set of desirable gambles D, its corresponding choice function C_D satisfies* C_5 *if and only* $\mathrm{posi}(D^c) = D^c$.

3 Lexicographic Choice Functions

Let $\bar{\mathcal{D}}_{\mathrm{L}} := \{D \in \bar{\mathcal{D}} : \mathrm{posi}(D^c) = D^c\}$. It follows from [4, Proposition 6] that a set of gambles $D \in \bar{\mathcal{D}}_{\mathrm{L}}$ induces a *linear prevision*—an expectation operator with respect to a finitely additive probability—by means of the formula $P_D(f) := \sup\{\mu \in \mathbb{R} : f - \mu \in D\}$ for all f in $\mathcal{L}$. We can make an even tighter connection with the so-called *lexicographic probabilities*.

A *lexicographic probability system* is an ℓ-tuple $p = (p_1, \ldots, p_\ell)$ of probability mass functions on $\mathcal{X}$. We associate with p its expectation operator $\mathrm{E}_p = (\mathrm{E}_{p_1}, \ldots, \mathrm{E}_{p_\ell})$, and its preference relation $\prec$ on $\mathcal{L}$:

$$f \prec g \Leftrightarrow \mathrm{E}_p(f) <_{\mathrm{L}} \mathrm{E}_p(g) \text{ for all } f \text{ and } g \text{ in } \mathcal{L},$$

where $<_{\mathrm{L}}$ denotes the usual lexicographic order between ℓ-tuples.

Proposition 4 *Given a lexicographic probability system $(p_1, \ldots, p_\ell)$, the set of desirable gambles $D := \{f \in \mathcal{L} : 0 \prec f\}$ associated with the preference relation $\prec$ is an element of $\bar{\mathcal{D}}_{\mathrm{L}}$. Conversely, given a set of desirable gambles D in $\bar{\mathcal{D}}_{\mathrm{L}}$, its associated preference relation $\prec_D$ is a preference relation based on some lexicographic probability system.*

Because of this result, we refer to the elements of $\bar{\mathcal{D}}_{\mathrm{L}}$ as *lexicographic sets of desirable gambles*, and call the elements of $\bar{\mathcal{C}}_{\mathrm{L}} := \{C_D : D \in \bar{\mathcal{D}}_{\mathrm{L}}\}$ *lexicographic choice functions*.

We gather from the discussion in Sect. 2 that the infimum of any set of lexicographic choice functions satisfies Axioms C_1–C_5. The central question that remains now, is whether any choice function that satisfies Axioms C_1–C_5 is, conversely, an infimum of lexicographic choice functions. Such a representation result would make lexicographic choice functions fulfil the role of *'dually atomic' choice functions* in our theory without the Archimedean axiom, in analogy with the theory with an Archimedean axiom [7], where the dually atomic choice functions are the ones induced by probability mass functions—see [2] for the terminology. In other words, we study the following:

> Is, in parallel with the result in [7], every choice function C that satisfies Axioms C_1–C_5 an infimum of lexicographic choice functions, or in other words, is $C(O) = \bigcup\{C'(O) : C' \in \bar{\mathcal{C}}_{\mathrm{L}}, C \sqsubseteq C'\}$ for all O in $\mathcal{Q}$?

We now show that this is not the case. In our counterexample, we focus on a binary space $\mathcal{X} = \{a, b\}$. It follows from the axioms of coherence that any coherent choice function C on a binary possibility space $\mathcal{X}$ can be determined by two sets: its associated set of desirable gambles $D_C := \{f \in \mathcal{L} : \{f\} = C\{0, f\}\}$ and a so-called *rejection set* K, which consists of the gambles g in $\mathcal{L}_{\mathrm{II}}$ and h in $\mathcal{L}_{\mathrm{IV}}$ which, taken alone, do not allow us to reject 0, but taken together, do allow us to reject 0:

$$0 \in C(\{0, g\}), 0 \in C(\{0, h\}), \text{ and } 0 \in R(\{0, g, h\}).$$

Here $\mathcal{L}_{\mathrm{II}} := \{h \in \mathcal{L} : h(a) < 0 \text{ and } h(b) > 0\}$ constitutes the second, and $\mathcal{L}_{\mathrm{IV}} := \{h \in \mathcal{L} : h(a) > 0 \text{ and } h(b) < 0\}$ the fourth quadrant, in the two-dimensional vector space $\mathcal{L}$.

In order to construct our counterexample, consider some increasing subset K of $\mathbb{R}_{>0} \times \mathbb{R}_{<0}$, and use it to define a special choice function C_K, with rejection function R_K, as follows. First of all, for any option set O, we let $0 \in R_K(\{0\} \cup O)$ if and only if

$$O \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\exists \lambda_1, \lambda_2 \in \mathbb{R}_{>0})(\exists(\rho_1, \rho_2) \in K)\{\lambda_1(-1, \rho_1), \lambda_2(1, \rho_2)\} \subseteq O. \tag{1}$$

Of course, this will define a choice function C_K *uniquely*, provided that we require that C_K should satisfy Axiom C_4b, because then, for any $O \in \mathcal{Q}$ and any $f \in O$:

$$f \in R_K(O) \Leftrightarrow 0 \in R_K(\{0\} \cup O'), \tag{2}$$

where $O' := (O - \{f\}) \setminus \{0\}$.

Proposition 5 *Any choice function C_K that is defined by Eqs. (1) and (2) satisfies Axioms* C_1, C_2, C_3a, C_4a *and* C_4b.

As far as C_5 is concerned, we have established the following:

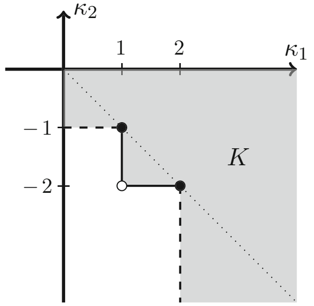

Fig. 1 The rejection set K that defines the choice function C_K in Proposition 7

Proposition 6 *Consider any increasing* $K \subseteq \mathbb{R}_{>0} \times \mathbb{R}_{<0}$*. For the choice function* C_K *on* $\mathcal{X} = \{a, b\}$ *defined by Eqs. (1) and (2), the following statements are equivalent:*

(i) C_K *satisfies* C_5.
(ii) $(\forall(\kappa_1, \kappa_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{<0})(\kappa_1 + \kappa_2 > 0 \Rightarrow (\kappa_1, \kappa_2) \in K)$.

Now, let us consider the set K as depicted in the figure above (Fig. 1).

Let C_K be the choice function associated with this set by means of Eqs. (1) and (2). It follows from the discussion above that this C_K satisfies Axioms C_1, C_2, C_3a, C_4a, C_4b and C_5. Let us show that it also satisfies Axiom C_3b.

Proposition 7 C_K *satisfies Axiom* C_3b. *As a consequence, it is a coherent choice function that satisfies* C_5.

Proof It can be checked that Axiom C_3b is equivalent to

$$(\forall O \in \mathcal{Q}, \forall g \in O)\{0, g\} \subseteq R(O) \Rightarrow 0 \in R(O \setminus \{g\}).$$

So assume that $\{0, g\} \subseteq R_K(O)$. Then $g \in R_K(O)$ and there are $(\kappa_1, \kappa_2) \in K$ such that $\{\lambda_1(-1, \kappa_1), \lambda_2(1, \kappa_2)\} \subseteq O$ for some λ_1 and λ_2 in $\mathbb{R}_{>0}$.

If $g \neq \lambda_1(-1, \kappa_1)$ and $g \neq \lambda_2(1, \kappa_2)$ then $0 \in R_K(O \setminus \{g\})$ and we are done, so assume that $g = \lambda_1(-1, \kappa_1)$ or $g = \lambda_2(1, \kappa_2)$.

If $g = \lambda_1(-1, \kappa_1)$, then $0 \in R_K(O - \{g\})$, so there are $(\kappa_1', \kappa_2') \in K$ such that $\{g + \lambda_1'(-1, \kappa_1'), g + \lambda_2'(1, \kappa_2')\} \subseteq O$ for some λ_1' and λ_2' in $\mathbb{R}_{>0}$, implying that $\{(-\lambda_1 - \lambda_1', \lambda_1\kappa_1 + \lambda_1'\kappa_1'), (-\lambda_1 + \lambda_2', \lambda_1\kappa_1 + \lambda_2'\kappa_2')\} \subseteq O$.

We now have a number of possibilities for the K defined in the figure above.

First of all, $(\frac{\lambda_1\kappa_1 + \lambda_1'\kappa_1'}{\lambda_1 + \lambda_1'}, \kappa_2) \in K$ under any of the following conditions:

(i) $\kappa_2 > -1$;
(ii) $\kappa_2 \in (-2, -1]$ (so $\kappa_1 \geq 1$) and $\kappa_1' \geq 1$;
(iii) $\kappa_2 = -2$ (so $\kappa_1 > 1$) and $\kappa_1' \geq 1$;
(iv) $\kappa_2 < -2$ (so $\kappa_1 > 2$) and $\kappa_1' \geq 2$.

So, in any of these cases, we see that $0 \in R_K(\{(-1, \frac{\lambda_1\kappa_1 + \lambda_1'\kappa_1'}{\lambda_1 + \lambda_1'})\}, 0, (1, \kappa_2))$, and therefore also $0 \in R(\{g + \lambda_1'(-1, \kappa_1'), 0, \lambda_2(1, \kappa_2)\})$, by Proposition 1. Since $\lambda_1'(-1, \kappa_1')$ $\neq 0$, we infer from Axiom C_3a that indeed $0 \in R_K(O \setminus \{g\})$.

The remaining two possibilities are:

(v) $\kappa_2 \leq -1$ (so $\kappa_1 \geq 1$) and $\kappa_1' < 1$ (so $\kappa_2' > -1$);
(vi) $\kappa_2 < -2$ (so $\kappa_1 > 2$) and $\kappa_1' \in [1, 2)$ (so $\kappa_2' \geq -2$).

There are now three possible cases.

If $\lambda_1 = \lambda_2'$, then $\lambda_1\kappa_1 + \lambda_2'\kappa_2' = \lambda_1(\kappa_1 + \kappa_2') > 0$ and therefore also $(-\lambda_1 + \lambda_2', \lambda_1\kappa_1 + \lambda_2'\kappa_2') > 0$, whence $0 \in R_K(\{0, (-\lambda_1 + \lambda_2', \lambda_1\kappa_1 + \lambda_2'\kappa_2')\})$, by Axiom C_2.

If $\lambda_1 < \lambda_2'$, then $(\kappa_1', \frac{\lambda_1\kappa_1+\lambda_2'\kappa_2'}{-\lambda_1+\lambda_2'}) \in K$, and therefore also

$$0 \in R_K\left(\left\{(-1, \kappa_1'), 0, (1, \frac{\lambda_1\kappa_1 + \lambda_2'\kappa_2'}{-\lambda_1 + \lambda_2'})\right\}\right)$$

Proposition 1 now guarantees that also

$$0 \in R_K(\{(-\lambda_1', \lambda_1'\kappa_1'), 0, (-\lambda_1 + \lambda_2', \lambda_1\kappa_1 + \lambda_2'\kappa_2')\}).$$

Since $(-\lambda_1', \lambda_1'\kappa_1') \neq g = (-\lambda_1, \lambda_1\kappa_1)$—because $\kappa_1 \geq 1$ and $\kappa_1' < 1$, or $\kappa_1 > 2$ and $\kappa_1' < 2$, we infer from Axiom C_3a that $0 \in R_K(O \setminus \{g\})$.

Finally, if $\lambda_1 > \lambda_2'$, then $(\frac{\lambda_1\kappa_1+\lambda_2'\kappa_2'}{\lambda_1-\lambda_2'}, \kappa_2') \in K$, implying that

$$0 \in R_K\left(\left\{(-1, \frac{\lambda_1\kappa_1 + \lambda_2'\kappa_2'}{\lambda_1 - \lambda_2'}), 0, (1, \kappa_2')\right\}\right).$$

Proposition 1 now guarantees that also

$$0 \in R_K(\{(-\lambda_1 + \lambda_2', \lambda_1\kappa_1 + \lambda_2'\kappa_2'), 0, (\lambda_2', \lambda_2'\kappa_2')\}).$$

Since $(-\lambda_1 + \lambda_2', \lambda_1\kappa_1 + \lambda_2'\kappa_2') \neq g = (-\lambda_1, \lambda_1\kappa_1)$, because $\lambda_2' \neq 0$, we infer from Axiom C_3a that indeed $0 \in R_K(O \setminus \{g\})$.

The proof of the case that $g = \lambda_2(1, \kappa_2)$ is similar. □

To see that our C_K is not an infimum of lexicographic choice functions, we use the following property:

Definition 4 Consider a coherent choice function C and its rejection set K. Then C is called *weakly Archimedean* if for all $f \in \mathcal{L}_{\mathrm{II}}$ and $g \in \mathcal{L}_{\mathrm{IV}}$ with $\mathrm{posi}(\{f, g\}) \cap \mathcal{L}_{\geq 0} = \emptyset$:

$$(\forall \epsilon \in \mathbb{R}_{>0})(0 \in R(\{f + \epsilon, 0, g\}) \cap R(\{f, 0, g + \epsilon\})) \Rightarrow 0 \in R(\{f, 0, g\}).$$

We use this name because the property is a strictly weaker version of the Archimedean condition in [7, Axioms 3a and 3b]; it still fulfils the role of a continuity condition, but is weak enough to be still compatible with desirability, a non-Archimedean strict preference.

Proposition 8 *An infimum of a non-empty set of lexicographic choice functions is weakly Archimedean.*

We now see that our choice function C_K from Proposition 7 is not an infimum of lexicographic choice functions, because it is not weakly Archimedean: note that $\{(1+\epsilon, -2), (1, -2+\epsilon)\} \subseteq K$ for all $\epsilon > 0$, while $(1, -2) \notin K$.

4 Discussion

We have studied to which extent it is possible to have a theory of coherent choice functions that (i) as a special case allows for choosing the maximal options in the strict binary preference expressed by the notion of desirability in imprecise probabilities—meaning that we must remove the Archimedean axiom, and that (ii) includes lexicographic probability systems as its basic building blocks. We have shown that such a theory can perfectly well incorporate the convexity axiom from [7], but that this additional axiom is not strong enough to warrant a representation theorem where every choice function is an infimum of lexicographic ones. It is still an open problem to uncover additional axioms that will guarantee such representation. We suspect that our weak archimedeanicity will play an important role in solving it.

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