The marginal problem for sets of desirable gamble sets

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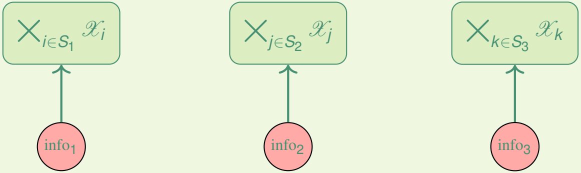
Compatibility

$\times_{i \in S_1} \mathscr{X}_i$ $\times_{j \in S_2} \mathscr{X}_j$ $\times_{k \in S_3} \mathscr{X}_k$ **Cartesian domains**

Compatibility



Cartesian domains

Local information
(e.g. pmfs, desirable gambles)

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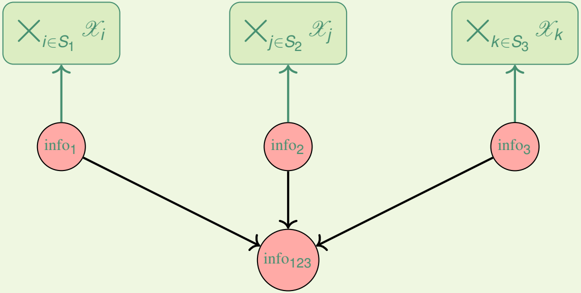

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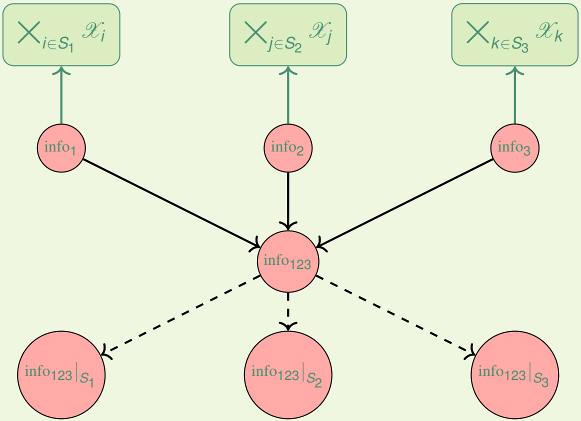

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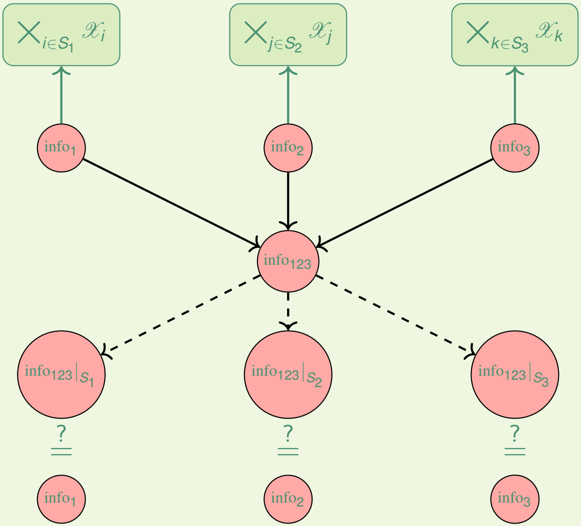

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Question:
Do we recover the original information?

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Want to express that *f* is desirable
or *g* is desirable.

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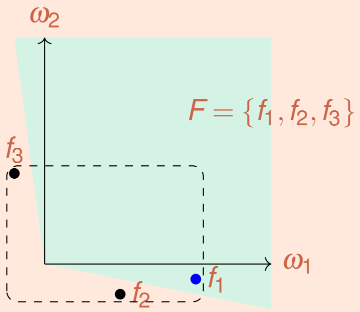

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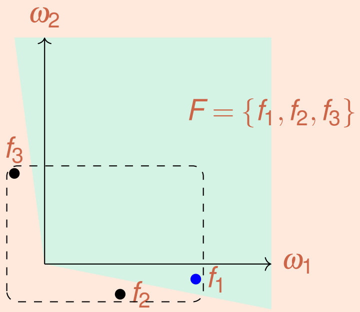

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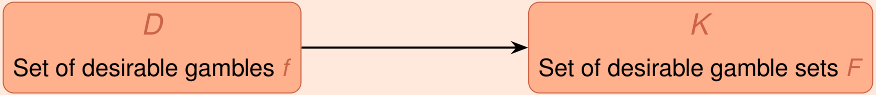

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$\mathscr{D}$ is a *representation* of K
K is *represented by* $\mathscr{D}$

⊕ *largest representation*: $\mathscr{D}_K := \{\text{coherent } D\text{s} : K \subseteq K_D\}$
⊕ *finite representation*: there is a finite $\mathscr{D}$ representing K.

Representation (continued)

Consider some K, its largest representation $\mathscr{D}_K$, some $D_1, D_2 \in \mathscr{D}_K$ with $D_1 \subseteq D_2$:

$$\implies K_{D_1} \subseteq K_{D_2}$$

$$\implies \bigcap_{D \in \mathscr{D}_K} K_D = \bigcap_{D \in \mathscr{D}_K \setminus \{D_2\}} K_D$$

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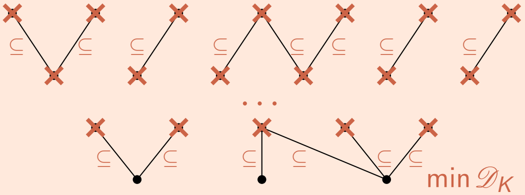

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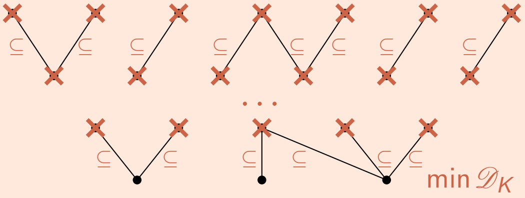

Continuing this way, we obtain *the set of minimal elements*:

$$\min \mathscr{D}_K := \{D \in \mathscr{D}_K : (\forall D' \in \mathscr{D}_K) D' \subseteq D \implies D' = D\}$$

which satisfies

- $\min \mathscr{D}_K \neq \emptyset$;
- $\mathscr{D}_K = \uparrow \min \mathscr{D}_K$;
- $\min \mathscr{D}_K$ represents K.

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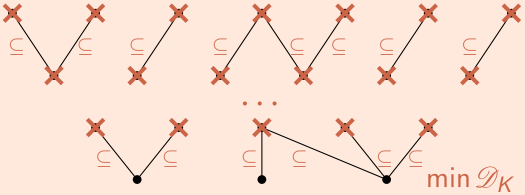

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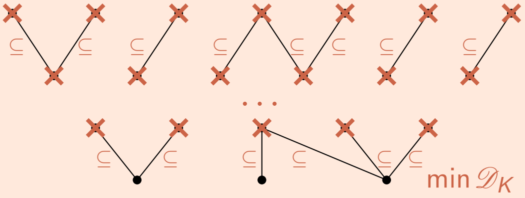

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+ It might be empty for other representations than the largest.

Pairwise Compatibility

Consider two sets:

$$K_1 \subseteq \mathscr{Q}(\mathscr{X}_{S_1}), \quad K_2 \subseteq \mathscr{Q}(\mathscr{X}_{S_2}).$$

Definition

Sets of desirable gamble sets K_1 and K_2 are said to be *pairwise compatible* if:

$$\mathrm{Marg}_{S_1 \cap S_2} K_1 = \mathrm{Marg}_{S_1 \cap S_2} K_2.$$

In other words, their marginals agree on the common domain.

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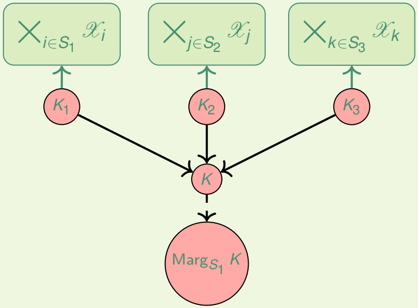

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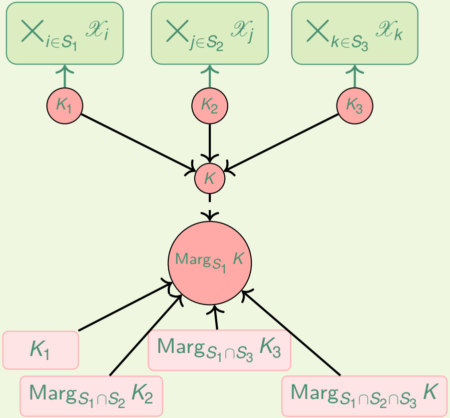

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? What if there is some information inside the intersection of more than two domains?

RIP (Running Intersection Property)

$$(\forall \ell \in \{2, \ldots, m\})(\exists i^\star < \ell) S_\ell \cap S_{i^\star} = S_\ell \cap \bigcup_{i<\ell} S_i$$

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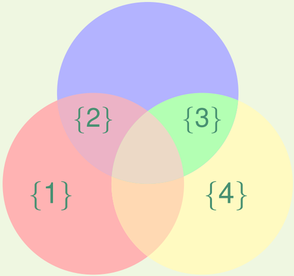

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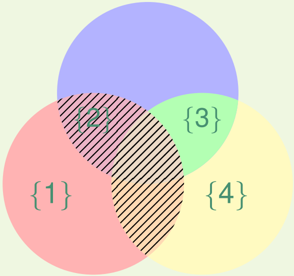

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Compatibility for sets of desirable gamble sets

Pairwise Compatibility

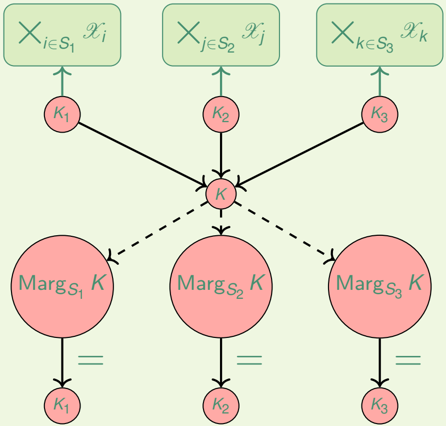
+

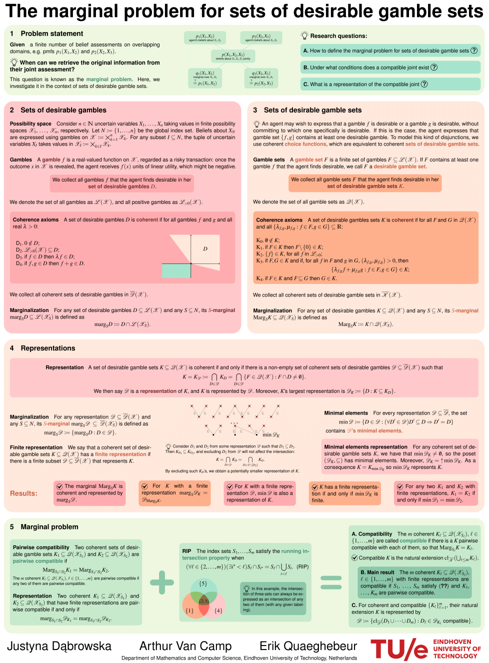
RIP

+

Finite Representations

Compatibility





For questions, please come to my poster!