

The marginal problem for sets of desirable gamble sets

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18 July 2025

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Compatibility

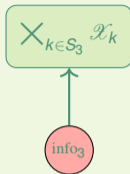
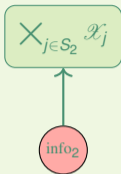
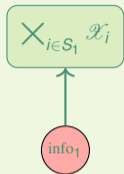
$$\prod_{i \in S_1} \mathcal{X}_i$$

$$\prod_{j \in S_2} \mathcal{X}_j$$

$$\prod_{k \in S_3} \mathcal{X}_k$$

Cartesian domains

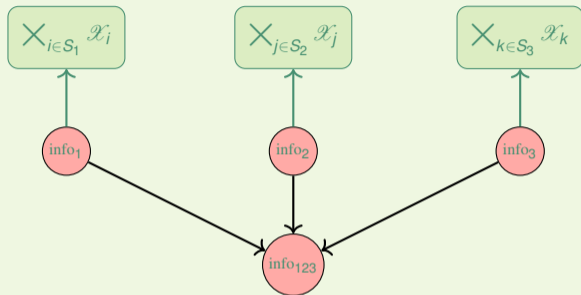
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Local information
(e.g. pmfs, desirable gambles)

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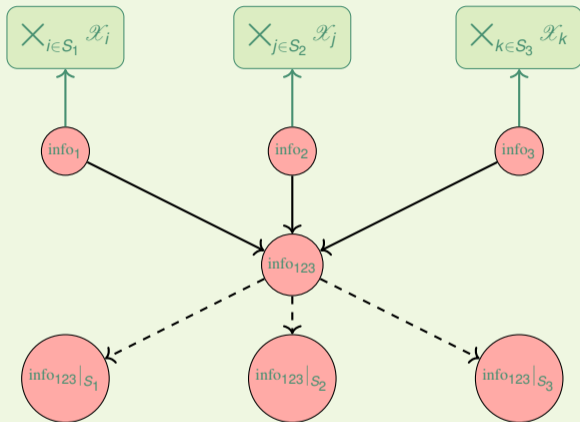


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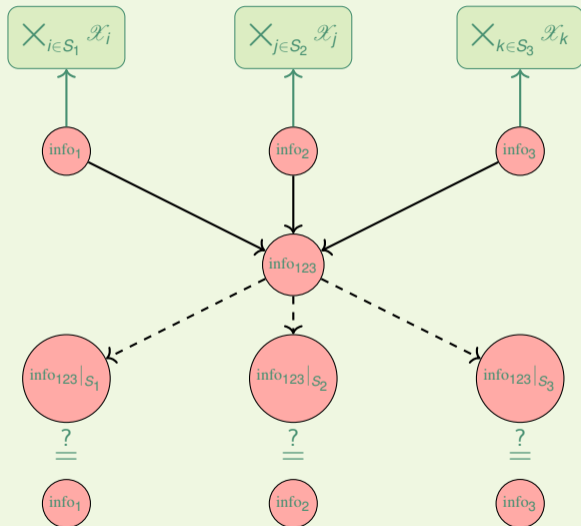
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Marginalization
(projected back to smaller scopes)

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Cartesian domains

Local information
(e.g. pmfs, desirable gambles)

Combination
(e.g. \times , natural extension)

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Question:
Do we recover the original information?

Sets of Desirable Gamble Sets

Want to express that f is desirable
or g is desirable.

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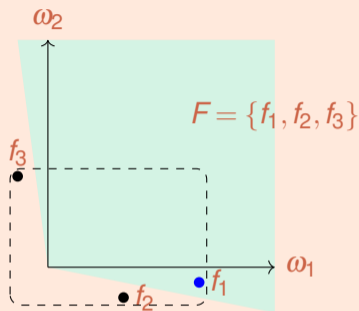
In other words, in the set $\{f, g\}$, at least
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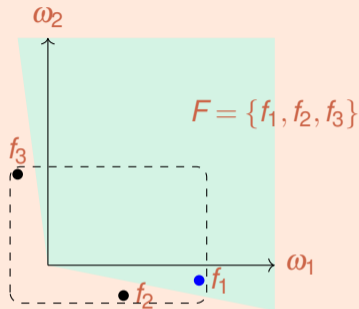


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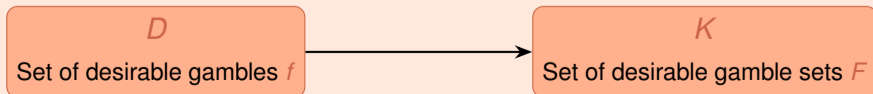
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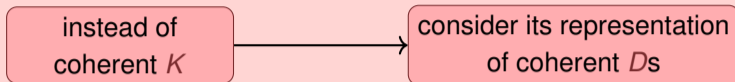
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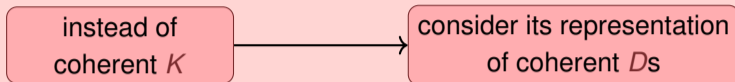
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Representations



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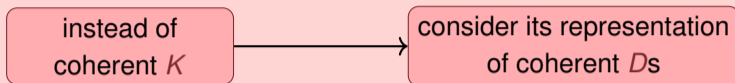


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K is *represented by* \mathcal{D}

- ⊕ *largest representation*: $\mathcal{D}_K := \{\text{coherent } Ds : K \subseteq K_D\}$
- ⊕ *finite representation*: there is a finite \mathcal{D} representing K .

Representation (continued)

Consider some K , its largest representation \mathcal{D}_K , some $D_1, D_2 \in \mathcal{D}_K$ with $D_1 \subseteq D_2$:

$$\implies K_{D_1} \subseteq K_{D_2}$$

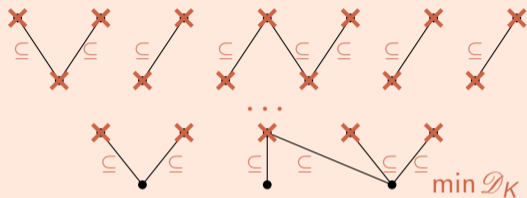
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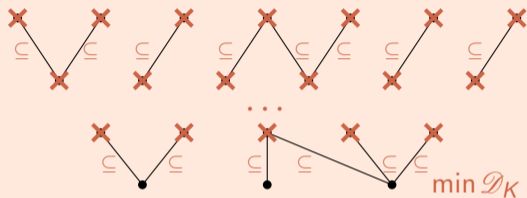


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Continuing this way, we obtain *the set of minimal elements*:

$$\min \mathcal{D}_K := \{D \in \mathcal{D}_K : (\forall D' \in \mathcal{D}_K) D' \subseteq D \Rightarrow D' = D\}$$

which satisfies

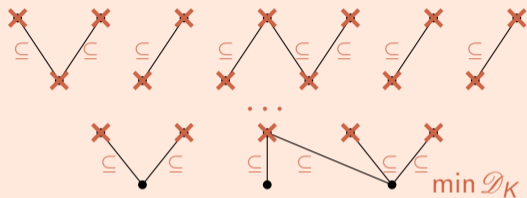
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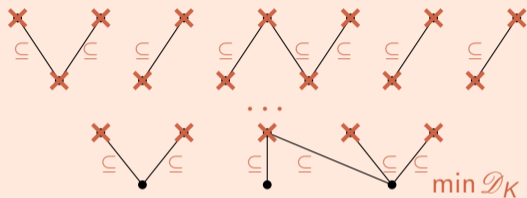
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② Is it defined only for the largest representations?

⊕ It might be empty for other representations than the largest.

Pairwise Compatibility

Consider two sets:

$$K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1}), \quad K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2}).$$

Definition

Sets of desirable gamble sets K_1 and K_2 are said to be *pairwise compatible* if:

$$\text{Marg}_{S_1 \cap S_2} K_1 = \text{Marg}_{S_1 \cap S_2} K_2.$$

In other words, their marginals agree on the common domain.

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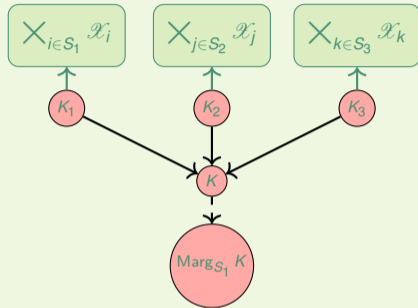
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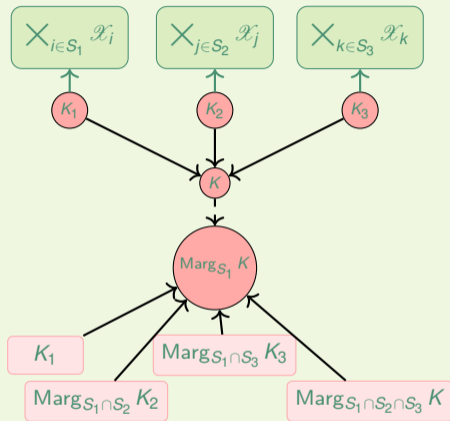
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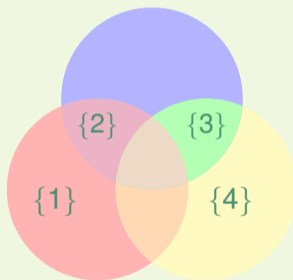
❓ What if there is some information inside the intersection of more than two domains?

RIP (Running Intersection Property)

$$(\forall \ell \in \{2, \dots, m\})(\exists i^* < \ell) S_\ell \cap S_{i^*} = S_\ell \cap \bigcup_{i < \ell} S_i$$

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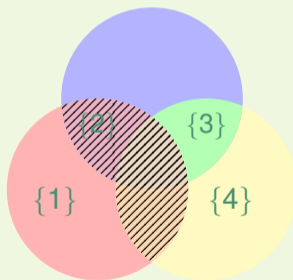
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Satisfies RIP
For some order(s)

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Compatibility for sets of desirable gamble sets

Pairwise
Compatibility

+

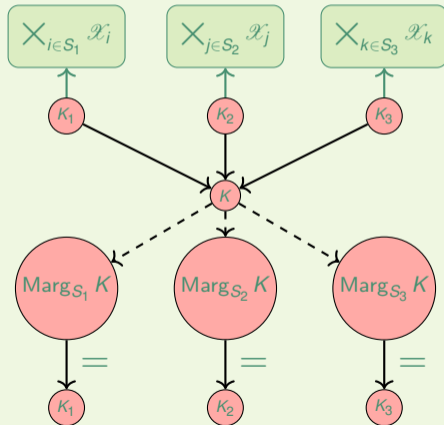
RIP

+

Finite
Representations



Compatibility



The marginal problem for sets of desirable gamble sets

1 Problem statement

Given: a finite number of belief assessments on overlapping domains, e.g. $\text{pr}_1(X_1, X_2)$ and $\text{pr}_2(X_2, X_3)$.

When can we retrieve the original information from their joint assessments?

This question is known as the marginal problem. Here, we investigate it in the context of sets of desirable gamble sets.

$\text{pr}_1(X_1, X_2)$
joint assessment X_1, X_2

$\text{pr}_2(X_2, X_3)$
joint assessment X_2, X_3

$\text{pr}(X_1, X_2, X_3)$
joint assessment X_1, X_2, X_3

$\text{pr}_1(X_1, X_2)$
marginal assessment X_1, X_2

$\text{pr}_2(X_2, X_3)$
marginal assessment X_2, X_3

Research questions:

A. How to define the marginal problem for sets of desirable gamble sets?

B. Under what conditions does a compatible joint exist?

C. What is a representation of the compatible joint?

2 Sets of desirable gambles

Possibility space Consider $n \in \mathbb{N}$ uncorrelated variables X_1, \dots, X_n taking values in finite possibility spaces $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. Let $\mathcal{X} := \{1, \dots, n\}$ be the global index set. Beliefs about X_n are expressed using gambles on $\mathcal{X} := \mathcal{X}_1 \cup \dots \cup \mathcal{X}_n$. For any subset $I \subseteq N$, the tuple of uncertain variables X_I takes values in $\mathcal{X}_I := \mathcal{X}_{1 \cup \dots \cup I}$.

Gambles A gamble f is a real-valued function on \mathcal{X} , regarded as a risky transaction: once the outcome x in \mathcal{X} is revealed, the agent receives $f(x)$ units of linear utility, which might be negative.

We collect all gambles f that the agent finds desirable in her set of desirable gambles D .

We denote the set of all gambles as $\mathcal{G}(\mathcal{X})$, and all positive gambles as $\mathcal{G}_+(\mathcal{X})$.

Coherence axioms A set of desirable gambles D is coherent if for all gambles f and g and all real $\lambda > 0$:

- D_1 : $0 \notin D$.
- D_2 : $\mathcal{G}_+(\mathcal{X}) \subseteq D$.
- D_3 : If $f \in D$ then $\lambda f \in D$.
- D_4 : If $f, g \in D$ then $f + g \in D$.

We collect all coherent sets of desirable gambles in $\mathcal{D}(\mathcal{X})$.

Marginalization For any set of desirable gambles $D \subseteq \mathcal{G}(\mathcal{X})$ and any $S \subseteq N$, its S -marginal $\text{marg}_S D \subseteq \mathcal{G}(S)$ is defined as $\text{marg}_S D := D \cap \mathcal{G}(S)$.

3 Sets of desirable gamble sets

An agent may wish to express that a gamble f is desirable or a gamble g is desirable, without committing to which one specifically is desirable. If this is the case, the agent expresses that gamble set $\{f, g\}$ contains at least one desirable gamble. To model this kind of disjunctions, we use coherent choice functions, which are equivalent to coherent sets of desirable gamble sets.

Gamble sets A gamble set F is a finite set of gambles $F \subseteq \mathcal{G}(\mathcal{X})$. If F contains at least one gamble f that the agent finds desirable, we call F a desirable gamble set.

We collect all gamble sets F that the agent finds desirable in her set of desirable gamble sets \mathcal{K} .

We denote the set of all gamble sets as $\mathcal{J}(\mathcal{X})$.

Coherence axioms A set of desirable gamble sets \mathcal{K} is coherent if for all F and G in $\mathcal{J}(\mathcal{X})$ and all $(x_{\mathcal{X} \setminus F}, y \in \mathcal{G}(F)) \subseteq \mathbb{R}$:

- K_1 : $F \neq \emptyset$.
- K_2 : $F \subseteq \mathcal{K}$ then $P_1(\{0\}) \subseteq \mathcal{K}$.
- K_3 : $\{f\} \in \mathcal{K}$ for all f in $\mathcal{G}_+(\mathcal{X})$.
- K_4 : If $F, G \in \mathcal{K}$ and if, for all $f \in F$ and $g \in G$, $(x_{\mathcal{X} \setminus F}, y) > 0$, then $(x_{\mathcal{X} \setminus F}, y) \in \mathcal{K}$.
- K_5 : If $F \subseteq \mathcal{K}$ and $F \subseteq G$ then $G \in \mathcal{K}$.

We collect all coherent sets of desirable gamble sets in $\overline{\mathcal{D}}(\mathcal{X})$.

Marginalization For any set of desirable gambles $\mathcal{K} \subseteq \overline{\mathcal{D}}(\mathcal{X})$ and any $S \subseteq N$, its S -marginal $\text{Marg}_S \mathcal{K} \subseteq \overline{\mathcal{D}}(S)$ is defined as $\text{Marg}_S \mathcal{K} := \mathcal{K} \cap \mathcal{J}(S)$.

4 Representations

Representation A set of desirable gamble sets $\mathcal{K} \subseteq \overline{\mathcal{D}}(\mathcal{X})$ is coherent if and only if there is a non-empty set of coherent sets of desirable gambles $\mathcal{D} \subseteq \mathcal{D}(\mathcal{X})$ such that $\mathcal{K} = \mathcal{K}_{\mathcal{D}} := \bigcap_{D \in \mathcal{D}} \mathcal{K}_D = \bigcap_{D \in \mathcal{D}} \{F \in \mathcal{J}(\mathcal{X}) : F \cap D \neq \emptyset\}$.

We then say \mathcal{D} is a representation of \mathcal{K} , and \mathcal{K} is represented by \mathcal{D} . Moreover, \mathcal{K} 's largest representation is $\mathcal{D}_{\mathcal{K}} := \{D : \mathcal{K} \subseteq \mathcal{K}_D\}$.

Marginalization For any representation $\mathcal{D} \subseteq \mathcal{D}(\mathcal{X})$ and any $S \subseteq N$, its S -marginal $\text{marg}_S \mathcal{D} \subseteq \mathcal{D}(S)$ is defined as $\text{marg}_S \mathcal{D} := \{D \cap S : D \in \mathcal{D}\}$.

Finite representation We say that a coherent set of desirable gamble sets $\mathcal{K} \subseteq \overline{\mathcal{D}}(\mathcal{X})$ has a finite representation if there is a finite subset $\mathcal{D} \subseteq \mathcal{D}(\mathcal{X})$ that represents \mathcal{K} .

Results:

- The marginal $\text{Marg}_S \mathcal{K}$ is coherent and represented by $\text{marg}_S \mathcal{D}$.
- For \mathcal{K} with a finite representation $\text{marg}_S \mathcal{D}_{\mathcal{K}} = \mathcal{D}_{\text{Marg}_S \mathcal{K}}$.
- For \mathcal{K} with a finite representation \mathcal{K} , \mathcal{K} has a finite representation if and only if $\text{min-}\mathcal{D}_{\mathcal{K}}$ is finite.
- For any two \mathcal{K}_1 and \mathcal{K}_2 with finite representations, $\mathcal{K}_1 \subseteq \mathcal{K}_2$ if and only if $\text{min-}\mathcal{D}_{\mathcal{K}_1} \subseteq \text{min-}\mathcal{D}_{\mathcal{K}_2}$.

5 Marginal problem

Pairwise compatibility Two coherent sets of desirable gamble sets $\mathcal{K}_1 \subseteq \overline{\mathcal{D}}(\mathcal{X}_1)$ and $\mathcal{K}_2 \subseteq \overline{\mathcal{D}}(\mathcal{X}_2)$ are pairwise compatible if $\text{Marg}_{\mathcal{X}_1 \cup \mathcal{X}_2} \mathcal{K}_1 = \text{Marg}_{\mathcal{X}_1 \cup \mathcal{X}_2} \mathcal{K}_2$.

The n -coherent $\mathcal{K}_1 \subseteq \overline{\mathcal{D}}(\mathcal{X}_1)$ and $\{1, \dots, n\}$ are pairwise compatible if any two of them are pairwise compatible.

Representation Two coherent $\mathcal{K}_1 \subseteq \overline{\mathcal{D}}(\mathcal{X}_1)$ and $\mathcal{K}_2 \subseteq \overline{\mathcal{D}}(\mathcal{X}_2)$ that have finite representations are pairwise compatible if and only if $\text{marg}_{\mathcal{X}_1 \cup \mathcal{X}_2} \mathcal{D}_{\mathcal{K}_1} = \text{marg}_{\mathcal{X}_1 \cup \mathcal{X}_2} \mathcal{D}_{\mathcal{K}_2}$.

RP The index sets $\mathcal{K}_1, \dots, \mathcal{K}_n$ satisfy the running intersection property when $\{i \in \{1, \dots, n\} : \{i\} \cap \mathcal{X}_i = \mathcal{X}_i \cap \bigcup_{j=1}^i \mathcal{X}_j\} \cap \mathcal{D}_{\mathcal{K}_i} \neq \emptyset$.

In this example, the coherent \mathcal{K}_1 of three sets can simply be composed as an intersection of any two of them with any given local RP.

A. Compatibility The n -coherent $\mathcal{K}_1 \subseteq \overline{\mathcal{D}}(\mathcal{X}_1)$, $\mathcal{K}_2 \subseteq \overline{\mathcal{D}}(\mathcal{X}_2)$, \dots , $\mathcal{K}_n \subseteq \overline{\mathcal{D}}(\mathcal{X}_n)$ are called compatible if there is a \mathcal{K} pairwise compatible with each of them, so that $\text{Marg}_{\mathcal{X}_i} \mathcal{K} = \mathcal{K}_i$.

B. Main result The n -coherent $\mathcal{K}_1 \subseteq \overline{\mathcal{D}}(\mathcal{X}_1)$, $\mathcal{K}_2 \subseteq \overline{\mathcal{D}}(\mathcal{X}_2)$, \dots , $\mathcal{K}_n \subseteq \overline{\mathcal{D}}(\mathcal{X}_n)$ are pairwise compatible if and only if $\text{min-}\mathcal{D}_{\mathcal{K}_1} \subseteq \text{min-}\mathcal{D}_{\mathcal{K}_2} \subseteq \dots \subseteq \text{min-}\mathcal{D}_{\mathcal{K}_n}$.

C. For coherent and compatible $\{\mathcal{K}_i\}_{i=1}^n$, their natural extension \mathcal{K} is represented by $\mathcal{D} := \{D_1 \cap \mathcal{X}_1 \cup \dots \cup D_n \cap \mathcal{X}_n : D_i \in \mathcal{D}_{\mathcal{K}_i}\}$.

For questions, please come to my poster!