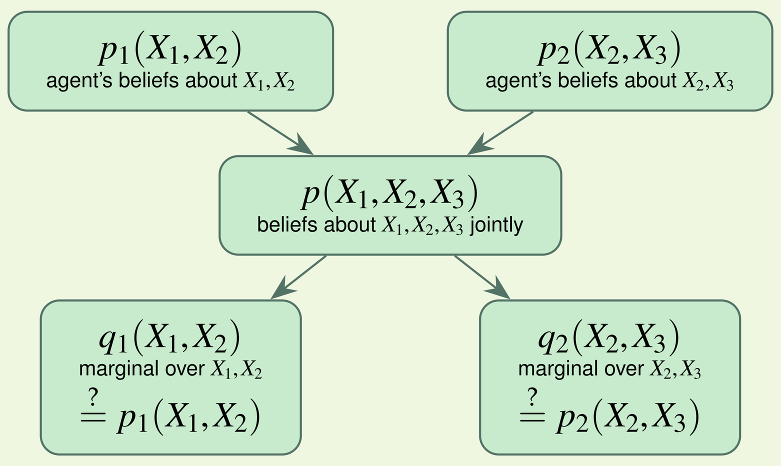
The marginal problem for sets of desirable gamble sets

1 Problem statement

Given a finite number of belief assessments on overlapping domains, e.g. pmfs $p_1(X_1,X_2)$ and $p_2(X_2,X_3)$.

When can we retrieve the original information from their joint assessment?

This question is known as the **marginal problem**. Here, we investigate it in the context of sets of desirable gamble sets.



Research questions:

A. How to define the marginal problem for sets of desirable gamble sets ?

B. Under what conditions does a compatible joint exist ?

C. What is a representation of the compatible joint ?

2 Sets of desirable gambles

Possibility space Consider $n \in \mathbb{N}$ uncertain variables $X_1, \ldots, X_n$ taking values in finite possibility spaces $\mathscr{X}_1, \ldots, \mathscr{X}_n$, respectively. Let $N := \{1,\ldots,n\}$ be the global index set. Beliefs about X_N are expressed using gambles on $\mathscr{X} := \times_{k=1}^{n} \mathscr{X}_k$. For any subset $I \subseteq N$, the tuple of uncertain variables X_I takes values in $\mathscr{X}_I := \times_{k\in I} \mathscr{X}_k$.

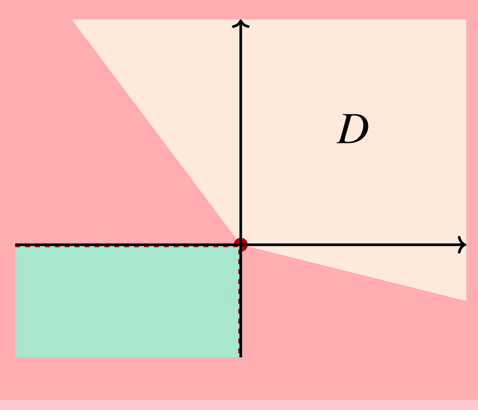
Gambles A **gamble** f is a real-valued function on $\mathscr{X}$, regarded as a risky transaction: once the outcome x in $\mathscr{X}$ is revealed, the agent receives $f(x)$ units of linear utility, which might be negative.

We collect all gambles f that the agent finds desirable in her **set of desirable gambles** D.

We denote the set of all gambles as $\mathscr{L}(\mathscr{X})$, and all positive gambles as $\mathscr{L}_{>0}(\mathscr{X})$.

Coherence axioms A set of desirable gambles D is **coherent** if for all gambles f and g and all real $\lambda > 0$:

- D_1. $0 \notin D$;
- D_2. $\mathscr{L}_{>0}(\mathscr{X}) \subseteq D$;
- D_3. if $f \in D$ then $\lambda f \in D$;
- D_4. if $f, g \in D$ then $f + g \in D$.



We collect all coherent sets of desirable gambles in $\overline{\mathscr{D}}(\mathscr{X})$.

Marginalization For any set of desirable gambles $D \subseteq \mathscr{L}(\mathscr{X})$ and any $S \subseteq N$, its **S-marginal** $\mathrm{marg}_S D \subseteq \mathscr{L}(\mathscr{X}_S)$ is defined as

$$\mathrm{marg}_S D := D \cap \mathscr{L}(\mathscr{X}_S).$$

3 Sets of desirable gamble sets

An agent may wish to express that a gamble f is desirable or a gamble g is desirable, without committing to which one specifically is desirable. If this is the case, the agent expresses that gamble set $\{f,g\}$ contains at least one desirable gamble. To model this kind of disjunctions, we use coherent **choice functions**, which are equivalent to coherent **sets of desirable gamble sets**.

Gamble sets A **gamble set** F is a finite set of gambles $F \subseteq \mathscr{L}(\mathscr{X})$. If F contains at least one gamble f that the agent finds desirable, we call F a **desirable gamble set**.

We collect all gamble sets F that the agent finds desirable in her **set of desirable gamble sets** K.

We denote the set of all gamble sets as $\mathscr{Q}(\mathscr{X})$.

Coherence axioms A set of desirable gambles sets K is **coherent** if for all F and G in $\mathscr{Q}(\mathscr{X})$ and all $\{\lambda_{f,g}, \mu_{f,g} : f \in F, g \in G\} \subseteq \mathbb{R}$:

- K_0. $\emptyset \notin K$;
- K_1. if $F \in K$ then $F \setminus \{0\} \in K$;
- K_2. $\{f\} \in K$, for all f in $\mathscr{L}_{>0}$;
- K_3. if $F, G \in K$ and if, for all f in F and g in G, $(\lambda_{f,g}, \mu_{f,g}) > 0$, then
$$\{\lambda_{f,g} f + \mu_{f,g} g : f \in F, g \in G\} \in K;$$
- K_4. if $F \in K$ and $F \subseteq G$ then $G \in K$.

We collect all coherent sets of desirable gamble sets in $\overline{\mathscr{K}}(\mathscr{X})$.

Marginalization For any set of desirable gambles $K \subseteq \mathscr{Q}(\mathscr{X})$ and any $S \subseteq N$, its **S-marginal** $\mathrm{Marg}_S K \subseteq \mathscr{Q}(\mathscr{X}_S)$ is defined as

$$\mathrm{Marg}_S K := K \cap \mathscr{Q}(\mathscr{X}_S).$$

4 Representations

Representation A set of desirable gamble sets $K \subseteq \mathscr{Q}(\mathscr{X})$ is coherent if and only if there is a non-empty set of coherent sets of desirable gambles $\mathscr{D} \subseteq \overline{\mathscr{D}}(\mathscr{X})$ such that

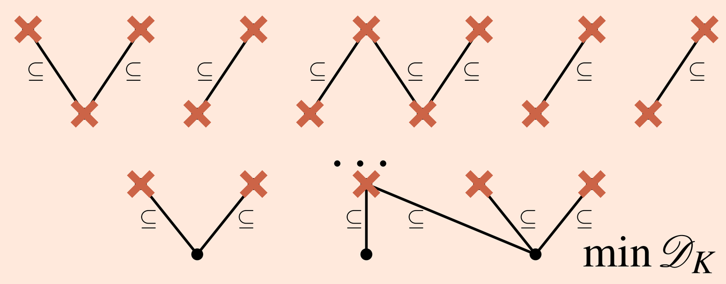
$$K = K_{\mathscr{D}} := \bigcap_{D\in\mathscr{D}} K_D = \bigcap_{D\in\mathscr{D}} \{F \in \mathscr{Q}(\mathscr{X}) : F \cap D \neq \emptyset\}.$$

We then say $\mathscr{D}$ is a **representation** of K, and K is represented by $\mathscr{D}$. Moreover, K's largest representation is $\mathscr{D}_K := \{D : K \subseteq K_D\}$.

Marginalization For any representation $\mathscr{D} \subseteq \overline{\mathscr{D}}(\mathscr{X})$ and any $S \subseteq N$, its **S-marginal** $\mathrm{marg}_S \mathscr{D} \subseteq \overline{\mathscr{D}}(\mathscr{X}_S)$ is defined as

$$\mathrm{marg}_S \mathscr{D} := \{\mathrm{marg}_S D : D \in \mathscr{D}\}.$$

Finite representation We say that a coherent set of desirable gamble sets $K \subseteq \mathscr{Q}(\mathscr{X})$ has a **finite representation** if there is a finite subset $\mathscr{D} \subseteq \overline{\mathscr{D}}(\mathscr{X})$ that represents K.



Consider D_1 and D_2 from some representation $\mathscr{D}$ such that $D_1 \subseteq D_2$. Then $K_{D_1} \subseteq K_{D_2}$, and excluding D_2 from $\mathscr{D}$ will not affect the intersection:

$$K = \bigcap_{D\in\mathscr{D}} K_D = \bigcap_{D\in\mathscr{D}\setminus\{D_2\}} K_D.$$

By excluding such K_D's, we obtain a potentially smaller representation of K.

Minimal elements For every representation $\mathscr{D} \subseteq \overline{\mathscr{D}}$, the set

$$\min \mathscr{D} := \{D \in \mathscr{D} : (\forall D' \in \mathscr{D}) D' \subseteq D \Rightarrow D' = D\}$$

contains **$\mathscr{D}$'s minimal elements**.

Minimal elements representation For any coherent set of desirable gamble sets K, we have that $\min \mathscr{D}_K \neq \emptyset$, so the poset $(\mathscr{D}_K, \subseteq)$ has minimal elements. Moreover, $\mathscr{D}_K = \uparrow \min \mathscr{D}_K$. As a consequence $K = K_{\min \mathscr{D}_K}$ so $\min \mathscr{D}_K$ represents K.

Results:

- ✓ The marginal $\mathrm{Marg}_S K$ is coherent and represented by $\mathrm{marg}_S \mathscr{D}$.
- ✓ For K with a finite representation $\mathrm{marg}_S \mathscr{D}_K = \mathscr{D}_{\mathrm{Marg}_S K}$.
- ✓ For K with a finite representation $\mathscr{D}$, $\min \mathscr{D}$ is also a representation of K.
- ✓ K has a finite representation if and only if $\min \mathscr{D}_K$ is finite.
- ✓ For any two K_1 and K_2 with finite representations, $K_1 = K_2$ if and only if $\min \mathscr{D}_1 = \min \mathscr{D}_2$.

5 Marginal problem

Pairwise compatibility Two coherent sets of desirable gamble sets $K_1 \subseteq \mathscr{Q}(\mathscr{X}_{S_1})$ and $K_2 \subseteq \mathscr{Q}(\mathscr{X}_{S_2})$ are **pairwise compatible** if

$$\mathrm{Marg}_{S_1\cap S_2} K_1 = \mathrm{Marg}_{S_1\cap S_2} K_2.$$

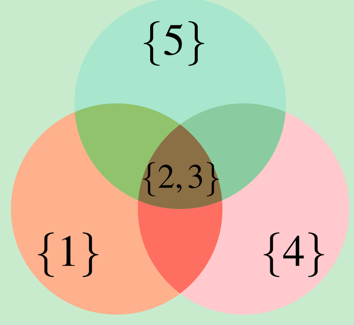
The m coherent $K_\ell \subseteq \mathscr{Q}(\mathscr{X}_{S_\ell})$, $\ell \in \{1,\ldots,m\}$ are pairwise compatible if any two of them are pairwise compatible.

Representation Two coherent $K_1 \subseteq \mathscr{Q}(\mathscr{X}_{S_1})$ and $K_2 \subseteq \mathscr{Q}(\mathscr{X}_{S_2})$ that have finite representations are pairwise compatible if and only if

$$\mathrm{marg}_{S_1\cap S_2} \mathscr{D}_{K_1} = \mathrm{marg}_{S_1\cap S_2} \mathscr{D}_{K_2}.$$

RIP The index sets $S_1, \ldots, S_m$ satisfy the **running intersection property** when

$$(\forall \ell \in \{2,\ldots,m\})(\exists i^\star < \ell) S_\ell \cap S_{i^\star} = S_\ell \cap \bigcup_{i<\ell} S_i. \quad \text{(RIP)}$$



In this example, the intersection of three sets can always be expressed as an intersection of any two of them (with any given labeling).

A. Compatibility The m coherent $K_\ell \subseteq \mathscr{Q}(\mathscr{X}_{S_\ell})$, $\ell \in \{1,\ldots,m\}$ are called **compatible** if there is a K pairwise compatible with each of them, so that $\mathrm{Marg}_{S_\ell} K = K_\ell$.

✓ Compatible K is the natural extension $\mathrm{cl}_{\overline{\mathscr{K}}}(\bigcup_{\ell\leq m} K_\ell)$.

B. Main result The m coherent $K_\ell \subseteq \mathscr{Q}(\mathscr{X}_{S_\ell})$, $\ell \in \{1,\ldots,m\}$ with finite representations are compatible if $S_1, \ldots, S_m$ satisfy (RIP) and $K_1, \ldots, K_m$ are pairwise compatible.

C. For coherent and compatible $\{K_\ell\}_{\ell=1}^m$, their natural extension K is represented by

$$\mathscr{D} := \{\mathrm{cl}_{\overline{\mathscr{D}}}(D_1 \cup \cdots \cup D_m) : D_\ell \in \mathscr{D}_{K_\ell} \text{ compatible}\}.$$

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