Independent natural extension for choice functions

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A journey through our results by an example

Consider two random variables X and Y.

$X \in \mathscr{X}$ finite

$Y \in \mathscr{Y}$ finite

An agent's beliefs about them is expressed using credal sets $\mathscr{M}_X$ and $\mathscr{M}_Y$.

$\mathscr{M}_X \subseteq \operatorname{int}\Sigma_{\mathscr{X}}$ arbitrary

$\mathscr{M}_Y \subseteq \operatorname{int}\Sigma_{\mathscr{Y}}$ finite

The agent uses **maximality** for X.

$$f \in C(A) \Leftrightarrow (\forall g \in A)(\exists p \in \mathscr{M}_X)E_p(g) \leq E_p(f)$$

"*f is choiceworthy in A if there is no gamble g that has higher p-expectation for every p in $\mathscr{M}_X$.*"

The agent uses **E-admissibility** for Y.

$$f \in C(A) \Leftrightarrow (\exists p \in \mathscr{M}_Y)(\forall g \in A)E_p(g) \leq E_p(f)$$

"*f is choiceworthy in A if there is a p in $\mathscr{M}_Y$ for which f wins against every gamble g in A.*"

? How can we model these two different decision rules in one framework?

i We use **sets of desirable gamble sets**.

They are based on **choice functions** [introduced by *Teddy Seidenfeld et al, Coherent Choice Functions under Uncertainty, Synthese 2010*], first formulated by [*Jasper De Bock and Gert de Cooman, A desirability-based axiomatisation for coherent choice functions, SMPS 2018*].
A **set of desirable gamble sets** K collects all the gamble sets that contain desirable gambles:

$$A \in K \Leftrightarrow A \text{ contains a gamble that is preferred over } 0 \Leftrightarrow 0 \notin C(A \cup \{0\}).$$

K_X based on maximality:

$$A \in K_X \Leftrightarrow (\exists g \in A)(\forall p \in \mathscr{M}_X)E_p(g) > 0$$

"*Gamble set A is desirable if it contains a gamble g that has a positive p-expectation for every p in $\mathscr{M}_X$.*"

K_Y based on E-admissibility:

$$A \in K_Y \Leftrightarrow (\forall p \in \mathscr{M}_Y)(\exists g \in A)E_p(g) > 0.$$

"*Gamble set A is desirable if for every p in $\mathscr{M}_Y$ it contains a gamble g with positive p-expectation.*"

Sets of desirable gamble sets can model even more general decision rules.

? How can we express an assessment of independence?

i We use **epistemic independence**.

K on $\mathscr{X} \times \mathscr{Y}$ expresses **epistemic independence** when

$$\operatorname{marg}_X(K \rfloor E_Y) = \operatorname{marg}_X K \quad \text{and} \quad \operatorname{marg}_Y(K \rfloor E_X) = \operatorname{marg}_Y K \quad \text{for all } \emptyset \neq E_X \subseteq \mathscr{X} \text{ and } \emptyset \neq E_Y \subseteq \mathscr{Y}.$$

"*Two variables, X and Y, are epistemically independent when learning about one of the variables doesn't influence our beliefs about the other.*"

? How can we combine K_X and K_Y using epistemic independence?

i We use the **independent natural extension**.

K is the **independent natural extension** of K_X and K_Y if K is the smallest coherent and epistemically independent set of desirable gamble sets such that $\operatorname{marg}_X K = K_X$ and $\operatorname{marg}_Y K = K_Y$.

$$A \in K \Leftrightarrow (\forall p_Y \in \mathscr{M}_Y)(\exists g \in A)(\forall p_X \in \mathscr{M}_X)E_{p_X \times p_Y}(g) > 0$$

This is, to the best of our knowledge, the first time that E-admissibility and maximality are combined using independence to obtain a joint decision rule. We used a result in [*Gert de Cooman et al, Independent natural extension, Artificial Intelligence, 2011*].

Our paper shows how to combine any finite number of sets of desirable gamble sets using epistemic independence.

Contrasting with S-independence

[*Teddy Seidenfeld et al, Coherent Choice Functions under Uncertainty, Synthese 2010*] introduced the following interesting but different independence notion, studied in great detail by [*Jasper De Bock and Gert de Cooman, On a notion of independence proposed by Teddy Seidenfeld, 2021*]:

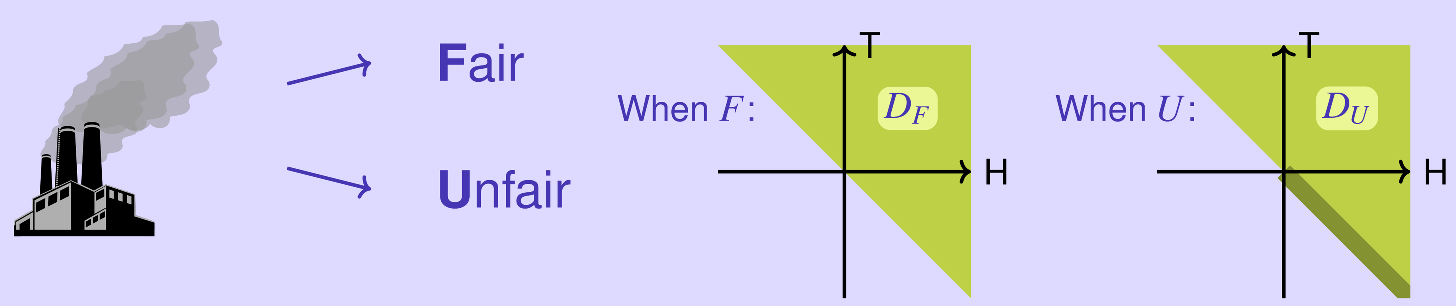
"*When X and Y are [**S-independent**] then it is not reasonable to spend resources in order to use the observed value of one of them, say X, to choose between options that depend solely on the value of the other variable, Y.*"

? How does S-independence relate to epistemic independence?

i Neither implies the other.

It is clear from the analysis in [*Jasper De Bock and Gert de Cooman, On a notion of independence proposed by Teddy Seidenfeld, 2021*] that epistemic independence does not imply S-independence. We show that it is not implied by it:

Coin factory:



With $D = \operatorname{posi}(\mathbb{I}_{\{F\}}D_F + \mathbb{I}_{\{U\}}D_U + \mathscr{L}_{>0})$ we find that $A \in K_D \Leftrightarrow (\exists g \in A)g \in D$ does not satisfy epistemic independence, but does satisfy S-independence.