Proceedings of Machine Learning Research 147:320–330, 2021 ISIPTA 2021

Independent Natural Extension for Choice Functions

Arthur Van Camp ARTHUR.VANCAMP@BRISTOL.AC.UK
Kevin Blackwell KEVIN.BLACKWELL@BRISTOL.AC.UK
Jason Konek JASON.KONEK@BRISTOL.AC.UK
Department of Philosophy, University of Bristol, United Kingdom

Abstract

We investigate epistemic independence for choice functions in a multivariate setting. This work is a continuation of earlier work of one of the authors [23], and our results build on the characterization of choice functions in terms of sets of binary preferences recently established by De Bock and De Cooman [7]. We obtain the independent natural extension in this framework. Given the generality of choice functions, our expression for the independent natural extension is the most general one we are aware of, and we show how it implies the independent natural extension for sets of desirable gambles, and therefore also for less informative imprecise-probabilistic models. Once this is in place, we compare this concept of epistemic independence to another independence concept for choice functions proposed by Seidenfeld [22], which De Bock and De Cooman [1] have called S-independence. We show that neither is more general than the other.

Keywords: choice function, set of desirable gamble sets, epistemic independence, S-independence, natural extension

1. Introduction

The framework of sets of desirable gamble sets has incredible expressive power for representing imprecise probabilistic beliefs. A strict generalization of the more familiar framework of sets of desirable gambles, it also allows us to model preferences that are not expressible merely by making binary comparisons between the available options. The dark side of this great power is that sets of desirable gamble sets are very difficult to work with; it will generally be conceptually and computationally intractable to begin modelling some credal state representing many variables with a set of desirable gamble sets that expresses the agent's beliefs about the entire possibility space. However, it will frequently be a much more manageable starting point to begin with some simpler local assessments about various particular ways of partitioning up the possibility space; the utility of this approach is greatly dependent on our ability to usefully combine these assessments into models of the entire space.

The first step is to show how to combine local assessments on uncertain variables that the agent regards as independent, and this is the project of our paper. We investigate the least informative way of combining these assessments into a joint set of desirable gamble sets, called the *independent natural extension*. The first few sections of the paper are devoted to developing the formal and notational details that we will need to represent the various concepts at work, recapitulating work by other authors; our novel contributions begin in Section 5.

2. Sets of Desirable Gamble Sets

Consider a finite possibility space $\mathscr{X}$ in which an uncertain variable X takes values. We denote by $\mathscr{L}(\mathscr{X})$ the set of all gambles—real-valued functions—on $\mathscr{X}$, often denoted by $\mathscr{L}$ when it is clear from the context what the possibility space is. We interpret a gamble f as an uncertain reward: if the actual outcome turns out to be x in $\mathscr{X}$, then the agent's capital changes by $f(x)$. For any two gambles f and g, we write $f \leq g$ when $f(x) \leq g(x)$ for all x in $\mathscr{X}$, and we write $f < g$ when $f \leq g$ and $f \neq g$. We identify a real constant α with the (constant) gamble that maps every element of $\mathscr{X}$ to α. We collect all the non-negative gambles—the gambles f for which $f \geq 0$—in the set $\mathscr{L}(\mathscr{X})_{\geq 0}$ (often denoted by $\mathscr{L}_{\geq 0}$), the positive ones—for which $f > 0$—in $\mathscr{L}(\mathscr{X})_{>0}$ (often denoted by $\mathscr{L}_{>0}$), and the non-positive ones—for which $f \leq 0$—in $\mathscr{L}(\mathscr{X})_{\leq 0}$ (often denoted by $\mathscr{L}_{\leq 0}$).

We denote by $\mathscr{Q}(\mathscr{X})$ the set of all finite but non-empty subsets of $\mathscr{L}(\mathscr{X})$, also denoted by $\mathscr{Q}$ when it is clear from the context what $\mathscr{X}$ is. $\mathscr{Q}$ is a subset of the power set $\mathscr{P}(\mathscr{L})$ of $\mathscr{L}$. Elements of $\mathscr{Q}$ are the *gamble sets*.

De Bock and de Cooman [7] established a useful equivalent representation to choice functions, introduced in an imprecise-probabilistic context by Seidenfeld et al. [22]:

Definition 1 (Set of desirable gamble sets) *A set of desirable gamble sets K on $\mathscr{X}$ is a subset of $\mathscr{Q}(\mathscr{X})$. We collect all the sets of desirable gamble sets in $\mathscr{K} := \mathscr{P}(\mathscr{Q})$.*

The idea is that the set of desirable gamble sets K collects all the gamble sets that contain at least one gamble that our agent strictly prefers over the status quo represented by 0, the gamble that will leave your capital unchanged whatever the outcome. A set of desirable gamble sets K is an equivalent representation to a choice function C: they are linked by $K = \{A \in \mathscr{Q} \colon 0 \notin C(\{0\} \cup A)\}$. Therefore all our results will apply for choice functions as well. We will use sets of desirable gamble sets mainly for practical reasons: they are easier to work with.

De Bock and De Cooman [7] gave an axiomatization of *coherent* sets of desirable gamble sets—sets of desirable gamble sets of rational agents. We refer to their article for a justification of the axioms.

Definition 2 (Coherent set of desirable gamble sets) *A set of desirable gamble sets $K \subseteq \mathscr{Q}$ is called* coherent *if for all A, A_1 and A_2 in $\mathscr{Q}$, all $\{\lambda_{f,g}, \mu_{f,g} \colon f \in A_1, g \in A_2\} \subseteq \mathbb{R}$, and all f in $\mathscr{L}$:*

K_0. $\emptyset \notin K$;

K_1. $A \in K \Rightarrow A \setminus \{0\} \in K$;

K_2. $\{f\} \in K$, *for all f in $\mathscr{L}_{>0}$;*

K_3. *if $A_1, A_2 \in K$ and if, for all f in A_1 and g in A_2, $(\lambda_{f,g}, \mu_{f,g}) > 0$, then $\{\lambda_{f,g} f + \mu_{f,g} g \colon f \in A_1, g \in A_2\} \in K$;*

K_4. *if $A_1 \in K$ and $A_1 \subseteq A_2$ then $A_2 \in K$.*

We collect all the coherent sets of desirable gamble sets in the collection $\overline{\mathscr{K}}(\mathscr{X})$, often simply denoted by $\overline{\mathscr{K}}$.

In this definition, we let $A_1 + A_2 := \{f + g \colon f \in A_1, g \in A_2\}$ be the Minkowski addition of two gamble sets A_1 and A_2. For any m and n in $\mathbb{N} \cup \{0\}$,[1] we define $m{:}n$ as the set $\{m, \ldots, n\}$, which we take to be the empty set when $n < m$. We will use both notations throughout. We denote any sequence $(\lambda_1, \ldots, \lambda_n)$ by $\lambda_{1:n}$, and define $\lambda_{1:n} > 0 \Leftrightarrow ((\forall j \in \{1, \ldots, n\})\lambda_j \geq 0$ and $(\exists j \in \{1, \ldots, n\})\lambda_j > 0)$. In other words, this means that $\lambda_{1:n} > 0 \Leftrightarrow (\lambda_{1:n} \geq 0$ and $\neg(\lambda_{1:n} = 0))$, where we let '$\geq$' and '$=$' work point-wisely on $(\lambda_1, \ldots, \lambda_n)$. This short-hand notation is used in item K_3 of this definition where $(\lambda_{f,g}, \mu_{f,g}) > 0$ means '$\lambda_{f,g} \geq 0$ and $\mu_{f,g} \geq 0$, with at least one of the real numbers $\lambda_{f,g}$ and $\mu_{f,g}$ strictly positive'.

Given two sets of desirable gamble sets K_1 and K_2, we follow De Bock and De Cooman [7] in calling K_1 *at most as informative as* K_2 if $K_1 \subseteq K_2$. The resulting partially ordered set $(\mathscr{K}, \subseteq)$ is a complete lattice where intersection serves the role of infimum, and union that of supremum. De Bock and De Cooman [7, Theorem 8] furthermore show that the partially ordered set $(\overline{\mathscr{K}}, \subseteq)$ of coherent sets of desirable gamble sets is a complete meet-semilattice: given an arbitrary family $\{K_i \colon i \in I\} \subseteq \overline{\mathscr{K}}$, its infimum $\inf\{K_i \colon i \in I\} = \bigcap_{i \in I} K_i$ is a coherent set of desirable gamble sets. This allows for conservative reasoning: it makes it possible to extend a partially specified set of desirable gamble sets to the most conservative—least informative—coherent one that includes it. This procedure is called *natural extension*:

Definition 3 ([7, Definition 9]) *For any assessment $\mathscr{A} \subseteq \mathscr{Q}$, we let $\mathbf{K}(\mathscr{A}) := \{K \in \overline{\mathscr{K}} \colon \mathscr{A} \subseteq K\}$. We call the assessment $\mathscr{A}$* consistent *if $\mathbf{K}(\mathscr{A}) \neq \emptyset$, and we then call $\mathrm{cl}_{\overline{\mathscr{K}}}(\mathscr{A}) := \bigcap \mathbf{K}(\mathscr{A})$ the* natural extension *of $\mathscr{A}$.*

1. We let $\mathbb{N}$ be the positive natural numbers. We let $\mathbb{R}_{>0} := \{x \in \mathbb{R} \colon x > 0\}$ be the positive real numbers.

One of the main results of De Bock and De Cooman [7] is their expression for the natural extension:

Theorem 4 ([7, Theorem 10]) *Consider any assessment $\mathscr{A} \subseteq \mathscr{Q}$. Then $\mathscr{A}$ is consistent if and only if $\emptyset \notin \mathscr{A}$ and $\{0\} \notin \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$. If this is the case, then $\mathrm{cl}_{\overline{\mathscr{K}}}(\mathscr{A}) = \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}))$.*

Here we used the set $\mathscr{L}^{\mathrm{s}}(\mathscr{X})_{>0} := \{\{f\} \colon f \in \mathscr{L}(\mathscr{X})_{>0}\}$—often denoted simply by $\mathscr{L}^{\mathrm{s}}_{>0}$—and the following two operations on $\mathscr{K}$ defined by $\mathrm{Rs}(K) := \{A \in \mathscr{Q} \colon (\exists B \in K) B \setminus \mathscr{L}_{\leq 0} \subseteq A\}$ and

$$\mathrm{Posi}(K) := \left\{ \left\{ \sum_{k=1}^{m} \lambda_k^{f_{1:m}} f_k \colon f_{1:m} \in \bigtimes_{k=1}^{m} A_k \right\} \colon m \in \mathbb{N}, \right.$$
$$\left. A_1, \ldots, A_m \in K, \left(\forall f_{1:m} \in \bigtimes_{k=1}^{m} A_k \right) \lambda_{1:m}^{f_{1:m}} > 0 \right\}$$

for all K in $\mathscr{K}$. Both Rs and Posi are *closure operators*: they are extensive, monotone and idempotent.

Binary Choice and Representation A set of desirable gamble sets K collects all the gamble sets A that contain at least one gamble that the agent strictly prefers over 0. For instance, the agent may know that one of $\{f_1, f_2\}$ is preferred over 0, but she may not know which one it is. So K can represent more than binary choice: indeed, she may have no preference in the binary choices $\{0, f_1\}$ and $\{0, f_2\}$, but in the ternary choice $\{0, f_1, f_2\}$ reject 0. In this section we will quickly summarize relevant known results about the binary choices captured by a set of desirable gamble sets.

Binary choices can be modelled by a *set of desirable gambles* which collects all the gambles that the agent prefers to zero. They were introduced by Seidenfeld et al. [21], and have been studied extensively by Walley [24, 25], De Cooman and Quaeghebeur [12], De Cooman and Miranda [11] and Quaeghebeur [18], amongst others. Formally, a set of desirable gambles is a subset $D \subseteq \mathscr{L}$ of gambles that are preferred over 0. We collect all the sets of desirable gambles in $\mathscr{D} := \mathscr{P}(\mathscr{L})$.

Definition 5 (Coherent set of desirable gambles) *A set of desirable gambles D is called* coherent *if for all f and g in $\mathscr{L}$, and λ and μ in $\mathbb{R}$:*

D_1. $0 \notin D$;

D_2. $\mathscr{L}_{>0} \subseteq D$;

D_3. *if $f, g \in D$ and $(\lambda, \mu) > 0$, then $\lambda f + \mu g \in D$.*

We collect all the coherent sets of desirable gambles in $\overline{\mathscr{D}}$.

Just as we did for sets of desirable gamble sets, we call the set of desirable gambles D_1 *at most as informative as* set of desirable gambles D_2 if $D_1 \subseteq D_2$. Here too, the partially ordered set $(\overline{\mathscr{D}}, \subseteq)$ of coherent sets of desirable gamble is a complete meet-semilattice. This implies that if a partially specified set $A \subseteq \mathscr{L}$ can be coherently extended—in other words, if $\mathbf{D}(A) := \{D \in \overline{\mathscr{D}} : A \subseteq D\} \neq \emptyset$, in which case we will call A *consistent*—there is a unique least informative such extension $\mathrm{cl}_{\overline{\mathscr{D}}}(A) := \bigcap \mathbf{D}(A)$:

Theorem 6 ([12, Theorem 1]) *Consider any assessment $A \subseteq \mathscr{L}$. Then A is consistent if and only if $\mathscr{L}_{\leq 0} \cap \mathrm{posi}(A) = \emptyset$. If this is the case, then $\mathrm{cl}_{\overline{\mathscr{D}}}(A) = \mathrm{posi}(\mathscr{L}_{>0} \cup A)$.*

Here, we used the posi operator on $\mathscr{D}$: $\mathrm{posi}(A) := \left\{\sum_{k=1}^{m} \lambda_k f_k : m \in \mathbb{N}, f_{1:m} \in A^m, \lambda_{1:m} > 0\right\}$ for all $A \subseteq \mathscr{L}$.

Theorem 6 implies that the smallest coherent—called *vacuous*—set of desirable gamble set is $D_{\mathrm{v}} := \mathscr{L}_{>0}$.

Given a set of desirable gamble sets K, its binary behaviour is summarized in the set of desirable gambles $D_K := \{f \in \mathscr{L} : \{f\} \in K\}$; D_K contains the gambles f that form desirable gamble singletons $\{f\} \in K$.

Conversely, given a coherent set of desirable gambles D, there might be multiple coherent K that imply the same binary choices D_K that are reflected in D: the non-empty collection $\{K \in \overline{\mathscr{K}} : D_K = D\}$ may have more than one element. However, it always contains one unique smallest element, which we call $K_D := \{A \in \mathscr{Q} : A \cap D \neq \emptyset\}$ [see [23, Proposition 5]]. De Bock and De Cooman [8, Proposition 8] show that K_D is coherent if and only if D is. We call any set of desirable gamble sets K *binary* if there is a set of desirable gambles D such that $K = K_D$. The smallest coherent—called *vacuous*—set of desirable gamble sets is binary, and given by $K_{\mathrm{v}} = K_{D_{\mathrm{v}}} = \{A \in \mathscr{Q} : A \cap \mathscr{L}_{>0} \neq \emptyset\}$.

De Bock and De Cooman [7] establish an important representation result for coherent sets of desirable gamble sets. They show that any coherent set of desirable gamble sets K can be represented by a collection $\mathbf{D}$ of coherent sets of desirable gambles:[2]

Theorem 7 (Representation [8, Theorem 9]) *Any set of desirable gamble sets K is coherent if and only if there is a non-empty set $\mathbf{D} \subseteq \overline{\mathscr{D}}$ of coherent sets of desirable gambles such that $K = \bigcap\{K_D : D \in \mathbf{D}\}$. We then say that* $\mathbf{D}$ represents *K. Moreover, K's largest representing set is $\mathbf{D}(K) := \{D \in \overline{\mathscr{D}} : K \subseteq K_D\}$.*

Note that $\mathbf{D}(K)$ is an *isotonic* set: if $D_1 \in \mathbf{D}(K)$ and $D_1 \subseteq D_2$, then $D_2 \in \mathbf{D}(K)$, for any D_1 and D_2 in $\overline{\mathscr{D}}$.

Example 1 *Let us give an example of a coherent set of desirable gamble sets. Consider an arbitrary non-empty (possibly non-convex) collection $\mathscr{M} \subseteq \Sigma_{\mathscr{X}} := \{p \in \mathscr{L}(\mathscr{X})_{\geq 0} : \sum_{x \in \mathscr{X}} p(x) = 1\}$ of probability mass functions on $\mathscr{X}$, called a* credal set.[3] *Let us associate with it the* E-*admissible [16, 19][4] choice function $C_{\mathscr{M}}(A) = \bigcup_{p \in \mathscr{M}}\{f \in A : (\forall g \in A) E_p(f) \geq E_p(g) \text{ and } g \not> f\}$:[5] given any gamble set A, an option f is admissible precisely when there is some p in $\mathscr{M}$ such that f has highest p-expectation in A. The set of desirable gamble sets $K_{\mathscr{M}}$ that corresponds to this, is given by $K_{\mathscr{M}} = \{A \in \mathscr{Q} : 0 \notin C_{\mathscr{M}}(\{0\} \cup A)\} = \{A \in \mathscr{Q} : A \cap \mathscr{L}_{>0} \neq \emptyset \text{ or } (\forall p \in \mathscr{M})(\exists f \in A) E_p(f) > 0\}$. $K_{\mathscr{M}}$ is the set of desirable gamble sets that corresponds to the E-admissible choice rule based on $\mathscr{M}$. Let us show that $K_{\mathscr{M}}$ is a coherent set of desirable gamble sets. One way to obtain this result is by checking that it satisfies all the rationality requirements from Definition 2, which is a cumbersome task. Thanks to Theorem 7 there is a much more elegant way to obtain this: we claim that $K_{\mathscr{M}}$ is represented by the non-empty $\{D_p : p \in \mathscr{M}\} \subseteq \overline{\mathscr{D}}$, and is therefore coherent. Here $D_p := \{f \in \mathscr{L} : f \in \mathscr{L}_{>0} \text{ or } E_p(f) > 0\}$ is the set of gambles that either have a positive p-expectation or are positive.*

Lemma 8 *$K_{\mathscr{M}}$ is represented by $\{D_p : p \in \mathscr{M}\}$.*

Jasper De Bock and Gert de Cooman showed us via private communication that Theorem 7 also allows for a simpler expression for the natural extension:

Theorem 9 (Due to De Bock & De Cooman) *An assessment $\mathscr{A} \subseteq \mathscr{Q}$ is consistent if and only if there is some D in $\overline{\mathscr{D}}$ such that $\mathscr{A} \subseteq K_D$. In that case $\mathrm{cl}_{\overline{\mathscr{K}}}(\mathscr{A}) = \bigcap\{K_D : D \in \overline{\mathscr{D}} \text{ and } \mathscr{A} \subseteq K_D\}$.*

3. Conditioning

Suppose that we have a belief model about X, be it a coherent set of desirable gamble sets on $\mathscr{X}$ or a coherent set of desirable gambles on $\mathscr{X}$, or—less generally—a set of probability mass functions on X. (We can think of a precise probability as a singleton.) When new information becomes available in the form of 'X assumes a value in some (non-empty) subset E of X', we can take this into account by conditioning our belief model on E.

We will let any event, except for the (trivially) impossible event $\emptyset$, serve as a conditioning event. We collect the allowed conditioning events in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}) := \{E \subseteq \mathscr{X} : E \neq \emptyset\}$. For any E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$ and any gamble f on E, we let its multiplication $\mathbb{I}_E f$ denote the gamble on $\mathscr{X}$ defined by

$$(\mathbb{I}_E f)(x) := \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \tag{1}$$

for all x in $\mathscr{X}$. $\mathbb{I}_E f$ is the called-off version of f: if E does not occur, the gamble will yield 0.

2. This theorem first appeared in De Bock and De Cooman [7, Theorem 7], but we prefer their later formulation in [8, Theorem 9].
3. $\Sigma_{\mathscr{X}}$ is called the *simplex* on $\mathscr{X}$: it is the collection of all probability mass functions on $\mathscr{X}$.
4. Although Levi's notion of E-admissibility was originally concerned with convex closed sets of probability mass functions [16, Chapter 5], we impose no such requirement here on the set $\mathscr{M}$.
5. We let $E_p(f) := \sum_{x \in \mathscr{X}} p(x) f(x)$ be f's p-expectation.

Definition 10 (Conditioning) *Given any set of desirable gamble sets K on $\mathscr{X}$ and any E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$, we define the* conditional set of desirable gamble sets $K\rfloor E$ *on $\mathscr{L}(E)$ as $K\rfloor E := \{A \in \mathscr{Q}(E) \colon \mathbb{I}_E A \in K\}$, where for any A in $\mathscr{Q}(E)$ and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$, we let $\mathbb{I}_E A := \{\mathbb{I}_E g \colon g \in A\}$ be a set of called-off gambles.*

It follows at once that conditioning preserves the order: if $K_1 \subseteq K_2$ then $K_1\rfloor E \subseteq K_2\rfloor E$. This definition coincides with the usual definition for sets of desirable gambles, in the sense that $K\rfloor E = K_{D\rfloor E}$, where $D\rfloor E := \{f \in \mathscr{L}(E) \colon \mathbb{I}_E f \in D\}$ is the set of desirable gambles conditional on E. In order to elegantly work with $K\rfloor E$'s representation in terms of sets of desirable gambles, let us define $\mathbf{D}\rfloor E := \{D\rfloor E \colon D \in \mathbf{D}\}$ for any $\mathbf{D} \subseteq \mathscr{D}$.

Proposition 11 ([23, Propositions 7 and 8]) *Consider any set of desirable gamble sets K on $\mathscr{X}$ and any conditioning event E in $\mathscr{P}_{\overline{\emptyset}}$. If K is coherent, then so is $K\rfloor E$. Furthermore, $K\rfloor E$ is represented by $\mathbf{D}(K)\rfloor E$, meaning that $K\rfloor E = \bigcap\{K_D \colon D \in \mathbf{D}(K)\rfloor E\}$.*

Example 2 *Let us build on Example 1, and condition the E-admissible set of desirable gamble sets $K_{\mathscr{M}}$ on an event G in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$ such that $\sum_{x\in G} p(x) > 0$ for all p in $\mathscr{M}$. Then, for any A in $\mathscr{Q}(\mathscr{X})$, we have $A \in K_{\mathscr{M}}\rfloor G \Leftrightarrow \mathbb{I}_G A \in K_{\mathscr{M}}$, which is equivalent to the requirement that for any p in $\mathscr{M}$ there is some f in A such that $E_p(\mathbb{I}_G f) > 0$. Since $E_{p|G}(f) = E_p(\mathbb{I}_G f)/E_p(\mathbb{I}_G)$, we have that $E_p(\mathbb{I}_G f) > 0 \Leftrightarrow E_{p|G}(f) > 0$. So the conditional set of desirable gamble sets $K_{\mathscr{M}}\rfloor G$ is equal to the E-admissible set of desirable gamble sets $K_{\{p|G \colon p\in\mathscr{M}\}}$ obtained by an element-wise application of Bayes's rule on $\mathscr{M}$.*

4. Multivariate Sets of Desirable Gamble Sets

In this section, we will generalize the multivariate study of desirability by De Cooman and Miranda [11] to choice models. We will provide the linear space of gambles, on which we define our sets of desirable gamble sets, with a more complex structure: we will consider the vector space of all gambles whose domain is a Cartesian product of a finite number of finite possibility spaces. More specifically, consider n in $\mathbb{N}$ variables $X_1, \dots, X_n$ that assume values in the finite possibility spaces $\mathscr{X}_1, \dots, \mathscr{X}_n$, respectively. Belief models about these variables $X_1, \dots, X_n$ will be defined using gambles on $\mathscr{X}_1, \dots, \mathscr{X}_n$. We also consider gambles on the Cartesian product $\times_{k=1}^n \mathscr{X}_k$, giving rise to the $\prod_{k=1}^n |\mathscr{X}_k|$-dimensional linear space $\mathscr{L}(\times_{k=1}^n \mathscr{X}_k)$.

Basic Notation & Cylindrical Extension For every non-empty subset $I \subseteq \{1,\dots,n\}$ of indices, we let X_I be the tuple of variables that takes values in $\mathscr{X}_I := \times_{r\in I} \mathscr{X}_r$. We will denote generic elements of $\mathscr{X}_I$ as x_I or z_I, whose components are $x_i := x_I(i)$ and $z_i := z_I(i)$, for all i in I. As we did before, when $I = \{k,\dots,\ell\}$ for some k,ℓ in $\{1,\dots,n\}$ with $k \leq \ell$, we will use as a shorthand notation $X_{k:\ell} := (X_k,\dots,X_\ell)$, taking values in $\mathscr{X}_{k:\ell}$ and whose generic elements are denoted by $x_{k:\ell} := (x_k,\dots,x_\ell)$.

We assume that the variables $X_1, \dots, X_n$ are *logically independent*, meaning that for each non-empty subset I of $\{1,\dots,n\}$, x_I may assume every value in $\mathscr{X}_I$.

It will be useful for any gamble f on $\mathscr{X}_{1:n}$, any non-empty proper subset I of $\{1,\dots,n\}$ and any x_I in $\mathscr{X}_I$, to interpret the partial map $f(x_I, \bullet)$ as a gamble on $\mathscr{X}_{I^c}$, where $I^c := \{1,\dots,n\} \setminus I$. We will need a way to relate gambles on different domains:

Definition 12 (Cylindrical extension) *Given two disjoint and non-empty subsets I and I' of $\{1,\dots,n\}$ and any gamble f on $\mathscr{X}_I$, we let its* cylindrical extension *f^* to $\mathscr{X}_{I\cup I'}$ be defined by $f^*(x_I,x_{I'}) := f(x_I)$ for all x_I in $\mathscr{X}_I$ and $x_{I'}$ in $\mathscr{X}_{I'}$. Similarly, given any set of gambles $A \subseteq \mathscr{L}(\mathscr{X}_I)$, we let its* cylindrical extension *$A^* \subseteq \mathscr{L}(\mathscr{X}_{I\cup I'})$ be defined as $A^* := \{f^* \colon f \in A\}$.*

Formally, f^* belongs to $\mathscr{L}(\mathscr{X}_{I\cup I'})$ while f belongs to $\mathscr{L}(\mathscr{X}_I)$. However, f^* is completely determined by f and *vice versa*: they clearly only depend on the value of X_I, and as such, they contain the same information and correspond to the same transaction. They are therefore indistinguishable from a behavioural point of view.

Remark 13 *As in [9, 11], we will frequently use the simplifying device of* identifying *a gamble f on $\mathscr{X}_I$ with its cylindrical extension f^* on $\mathscr{X}_{I\cup I'}$, for any disjoint and non-empty subsets I and I' of the index set $\{1,\dots,n\}$. This convention allows us, for instance, to identify $\mathscr{L}(\mathscr{X}_I)$ with a subset of $\mathscr{L}(\mathscr{X}_{1:n})$, and, as another example, for any set $A \subseteq \mathscr{L}(\mathscr{X}_{1:n})$, to regard $A \cap \mathscr{L}(\mathscr{X}_I)$ as those gambles in A that depend on the value of $\mathscr{X}_I$ only. Therefore, for any event E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$ we can identify the gamble $\mathbb{I}_E$ with $\mathbb{I}_{E\times\mathscr{X}_{I^c}}$, and hence also the event E with $E \times \mathscr{X}_{I^c}$.*

Marginalization Suppose we have a set of desirable gamble sets K on $\mathscr{X}_{1:n}$ modelling an agent's beliefs about the variable $X_{1:n}$. What is the information that K contains about X_O, where O is some non-empty subset of the index set $\{1,\dots,n\}$? Marginalization captures this information.

Definition 14 (Marginalization) *Given any non-empty subset O of $\{1,\dots,n\}$ and any set of desirable gamble sets K on $\mathscr{X}_{1:n}$, its* marginal set of desirable gamble sets $\operatorname{marg}_O K$ *on $\mathscr{X}_O$ is defined as* $\operatorname{marg}_O K := \{A \in \mathscr{Q}(\mathscr{X}_O) \colon A \in K\} = K \cap \mathscr{Q}(\mathscr{X}_O)$.

It follows at once from Definition 14 that marginalization preserves the order: if $K_1 \subseteq K_2$, then $\operatorname{marg}_O K_1 \subseteq \operatorname{marg}_O K_2$. This definition coincides with the usual definition for sets of desirable gambles, in the sense that $\operatorname{marg}_O K_D = K_{\operatorname{marg}_O D}$, where $\mathrm{marg}_O D := \{f \in \mathscr{L}(\mathscr{X}_O) : f \in D\} = D \cap \mathscr{L}(\mathscr{X}_O)$. For notational convenience, we lift the marginalization operator marg_O on $\mathscr{D}$ to a version on $\mathscr{P}(\mathscr{D})$ defined by $\mathrm{marg}_O \mathbf{D} := \{\mathrm{marg}_O D : D \in \mathbf{D}\}$ for any $\mathbf{D} \subseteq \mathscr{D}$.

Proposition 15 ([23, Propositions 9 and 10]) *Consider any set of desirable gamble sets K on $\mathscr{X}_{1:n}$ and any non-empty subset O of $\{1,\dots,n\}$. If K is coherent, then so is $\mathrm{marg}_O K$. Furthermore, $\mathrm{marg}_O K$ is represented by $\mathrm{marg}_O(\mathbf{D}(K))$: $\mathrm{marg}_O K = \bigcap\{K_D : D \in \mathrm{marg}_O(\mathbf{D}(K))\}$.*

Conditioning on Variables In Section 3 we have seen how we can condition sets of desirable gamble sets on events. Here, we take a closer look at conditioning in a multivariate context.

Suppose we have a set of desirable gamble sets K on $\mathscr{X}_{1:n}$, representing an agent's beliefs about the value of $X_{1:n}$. Assume now that we obtain the information that the I-tuple of variables X_I—where I is a non-empty subset of $\{1,\dots,n\}$—assumes a value in a certain non-empty subset E_I of $\mathscr{X}_I$. There is no new information about the other variables X_{I^c}. How can we condition K on this new information?

This is a particular instance of Definition 10, with the following specifications: $\mathscr{X} = \mathscr{X}_{1:n}$ and $E = E_I \times \mathscr{X}_{I^c}$. The indicator $\mathbb{I}_E$ of the conditioning event E satisfies $\mathbb{I}_E(x_{1:n}) = \mathbb{I}_{E_I}(x_I)$ for all $x_{1:n}$ in $\mathscr{X}_{1:n}$, and taking Remark 13 into account, therefore $\mathbb{I}_E = \mathbb{I}_{E_I}$. Equation (1) defines the multiplication of a gamble f on $E_I \times \mathscr{X}_{I^c}$ with $\mathbb{I}_{E_I}$ to be a gamble $\mathbb{I}_{E_I} f$ on $\mathscr{X}_{1:n}$, given by, for all $x_{1:n}$ in $\mathscr{X}_{1:n}$:

$$\mathbb{I}_{E_I} f(x_{1:n}) = \begin{cases} f(x_{1:n}) & \text{if } x_I \in E_I \\ 0 & \text{if } x_I \notin E_I \end{cases} \tag{2}$$

and the multiplication of $\mathbb{I}_{E_I}$ with a set A of gambles on $E_I \times \mathscr{X}_{I^c}$ is the set $\mathbb{I}_{E_I} A = \{\mathbb{I}_{E_I} f : f \in A\}$ of gambles on $\mathscr{X}_{1:n}$.

Now that we have instantiated all the relevant aspects of Definition 10, we see that $K\rfloor E_I = \{A \in \mathscr{Q}(E_I \times \mathscr{X}_{I^c}) : \mathbb{I}_{E_I} A \in K\}$. Proposition 11 guarantees that $K\rfloor E_I$ is represented by $\mathbf{D}(K)\rfloor E_I = \{D\rfloor E_I : D \in \mathbf{D}(K)\}$, where in this context $D\rfloor E_I = \{f \in \mathscr{L}(E_I \times \mathscr{X}_{I^c}) : \mathbb{I}_{E_I} f \in D\}$.

The conditional set of desirable gamble sets $K\rfloor E_I$ is defined on gambles on $E_I \times \mathscr{X}_{I^c}$. However, usually—see, for instance, [6, 11]—conditioning on information about X_I results in a model on X_{I^c}. We therefore consider $\mathrm{marg}_{I^c}(K\rfloor E_I) = \{A \in \mathscr{Q}(\mathscr{X}_{I^c}) : \mathbb{I}_{E_I} A \in K\}$ as the set of desirable gamble sets that represents the conditional beliefs about X_{I^c}, given that $X_I \in E_I$. Proposition 11 guarantees the coherence of $\mathrm{marg}_{I^c}(K\rfloor E_I)$, for any coherent K.

5. Independent Natural Extension

Now that the basic operations of multivariate sets of desirable gamble sets—marginalization and conditioning—are in place, we are ready to look at a simple type of structural assessment. The assessment that we will consider, is that of *epistemic independence*, which we define to be a symmetrized version of *epistemic irrelevance*.

Definition 16 (Epistemic (subset) irrelevance) *Consider any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$. We call X_I* epistemically (subset) irrelevant *to X_O when learning about the value of X_I does not influence or change the agent's beliefs about X_O. A set of desirable gamble sets K on $\mathscr{X}_{1:n}$ is said to* satisfy epistemic subset irrelevance *of X_I to X_O when*

$$\mathrm{marg}_O(K\rfloor E_I) = \mathrm{marg}_O K \text{ for all } E_I \text{ in } \mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I). \tag{3}$$

The idea behind this definition is that observing that X_I belongs to E_I turns K into the conditioned set of desirable gamble sets $K\rfloor E_I$ on $E_I \times \mathscr{X}_{I^c}$. Then requiring that learning that X_I belongs to E_I does not affect the agent's beliefs about X_O amounts to requiring that the marginal models of K and $K\rfloor E_I$ be equal.

This definition is a generalization of De Cooman and Miranda [11]'s definition for sets of desirable gambles. Besides their use of the less expressive models of sets of desirable gambles, there is another difference: De Cooman and Miranda [11] consider epistemic *value* irrelevance, which requires the analogue of Equation (3) only for events of the form $E_I = \{x_I\}$, with $x_I \in \mathscr{X}_I$.

De Bock [6, Example 2] shows that the two notions do indeed come apart: he gives a coherent set of desirable gambles that satisfies epistemic value irrelevance of X_1 to X_2, but not epistemic subset irrelevance. Given the connection between sets of desirable gambles and sets of desirable gamble sets, this example establishes that the two notions come apart also in the context of sets of desirable gamble sets. We follow De Bock [6] in considering epistemic subset-irrelevance to be the more natural of the two irrelevance concepts, as it requires all information about the value of X_I to be irrelevant, including partial information of the form $X_I \in E_I$, and not only of the form $X_I = x_I$.

Definition 17 (Epistemic (subset) independence) *We call $X_1, \dots X_n$* epistemically (subset) independent *when learning about the values of any of them does not influence or change the agent's beliefs about the remaining ones: for any two disjoint non-empty subsets I and O of $\{1,\dots,n\}$, X_I is epistemically subset irrelevant to X_O. We call a set of desirable gamble sets K on $\mathscr{X}_{1:n}$* epistemically (subset) independent *when*

$$\mathrm{marg}_O(K\rfloor E_I) = \mathrm{marg}_O K \text{ for all } E_I \text{ in } \mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I) \tag{4}$$

for all disjoint non-empty subsets I and O of $\{1,\dots,n\}$.

Independence assessments are useful in constructing joint sets of desirable gamble sets from local ones. Suppose we have a coherent set K_ℓ of desirable gamble sets on $\mathscr{X}_\ell$, for each ℓ in $\{1,\dots,n\}$, and an assessment that the variables $X_1,\dots,X_n$ are epistemically subset independent. Then how can we combine the coherent local assessments K_ℓ and this structural independence assessment into a coherent set of desirable gamble sets on $\mathscr{X}_{1:n}$ in a way that is as conservative as possible? If we call any coherent and epistemically independent K on $\mathscr{X}_{1:n}$ that marginalizes to K_ℓ for all ℓ in $\{1,\dots,n\}$ an *independent product* of $K_1,\dots K_n$, this means we are looking for the smallest independent product, which we will call the *independent natural extension of* $K_1,\dots K_n$, after De Cooman and Miranda's [11, Theorem 19]:

Theorem 18 ([11, Theorem 19]) *The independent natural extension of the n coherent sets of desirable gambles $D_1\subseteq\mathscr{L}(\mathscr{X}_1)$, ..., $D_n\subseteq\mathscr{L}(\mathscr{X}_n)$ exists and is given by $\bigotimes_{k=1}^n D_k := \operatorname{posi}\big(\bigcup_{j=1}^n A_{1:n\setminus\{j\}\to\{j\}}\cup\mathscr{L}(\mathscr{X}_{1:n})_{>0}\big)$, where*

$$A_{1:n\setminus\{j\}\to\{j\}} := \{\mathbb{I}_E f\colon f\in D_j \text{ and } E\in\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})\} \tag{5}$$

for any j in $\{1,\dots,n\}$.

The set $A_{1:n\setminus\{j\}\to\{j\}}$ expresses all the epistemic subset independence assessments from Equation (4), which the independent natural extension must satisfy. This is a difference with the project of [11]: the original theorem [11, Theorem 19] considers the subset $A'_{1:n\setminus\{j\}\to\{j\}} := \{\mathbb{I}_{\{x\}}f\colon f\in D_j \text{ and } x\in\mathscr{X}_{1:n\setminus\{j\}}\}$,[6] expressing the epistemic *value* independence assessments. As expected, $A'_{1:n\setminus\{j\}\to\{j\}}\subset A_{1:n\setminus\{j\}\to\{j\}}$, which is consistent with the fact that epistemic value independence is a weaker requirement than epistemic subset independence, but $\operatorname{posi}(A'_{1:n\setminus\{j\}\to\{j\}}) = \operatorname{posi}(A_{1:n\setminus\{j\}\to\{j\}})$,[7] resulting in equal independent natural extensions: the smallest coherent set of desirable gambles that is epistemically value independent also is epistemically subset independent. As we will see in Theorem 20, the independent natural extension for sets of desirable gamble sets will be represented by a collection of independent natural extensions for sets of desirable gambles, and therefore the smallest independent product for sets of desirable gamble sets will also coincide for these two notions of independence. By default, we will just say 'epistemic irrelevance' and 'epistemic independence' to indicate their subset variants.

We will generalize [11, Theorem 19] to sets of desirable gamble sets. To this end, consider the following counterpart of Equation (5)

$$\mathscr{A}_{1:n\setminus\{j\}\to\{j\}} := \big\{\mathbb{I}_E A : A\in K_j \text{ and } E\in\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})\big\} \tag{6}$$

for any j in $\{1,\dots,n\}$, with which we build the following set of desirable gamble sets:

$$\bigotimes_{j=1}^n K_j := \operatorname{Rs}\left(\operatorname{Posi}\left(\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}}\cup\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0}\right)\right).$$

We will show that the independent natural extension—the smallest independent product—of K_1, ..., K_n is exactly $\bigotimes_{j=1}^n K_j$. Before we prove this, it will be useful to show the following connection between the assessments $\mathscr{A}_{1:n\setminus\{j\}\to\{j\}}$ and $A_{1:n\setminus\{j\}\to\{j\}}$ for the two types of belief models:

Proposition 19 *Consider, for each j in $\{1,\dots,n\}$, a coherent set of desirable gambles D_j on $\mathscr{X}_j$. Then $K_{\bigotimes_{j=1}^n D_j} = \bigotimes_{j=1}^n K_{D_j}$, so $K_{\bigotimes_{j=1}^n D_j}$ is the smallest element of $\overline{\mathscr{K}}$ that includes $\bigcup_{j=1}^n\{\mathbb{I}_E A : A\in K_{D_j} \text{ and } E\in\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})\}$.*

Theorem 20 (Independent natural extension) *Consider, for each j in $\{1,\dots,n\}$, a coherent set of desirable gamble sets K_j on $\mathscr{X}_j$. Then the smallest independent product of K_1, ..., K_n is given by $\bigotimes_{j=1}^n K_j$. Furthermore, $\bigotimes_{j=1}^n K_j$ is represented by $\bigotimes_{j=1}^n \mathbf{D}(K_j) := \{\bigotimes_{j=1}^n D_j\colon D_1\in\mathbf{D}(K_1),\dots,D_n\in\mathbf{D}(K_n)\}$.*

Theorem 20 establishes that the independent natural extension $\bigotimes_{j=1}^n K_j$ can be calculated using the independent natural extension for sets of desirable gambles, because $\bigotimes_{j=1}^n \mathbf{D}(K_j)$ is a representation of $\bigotimes_{j=1}^n K_j$. So $\bigotimes_{j=1}^n K_j$ is represented by a collection $\bigotimes_{j=1}^n \mathbf{D}(K_j)$ of independent natural extensions of sets of locally desirable gambles. However, this does not also imply that the largest representation $\mathbf{D}(\bigotimes_{j=1}^n K_j)$ of $\bigotimes_{j=1}^n K_j$ will only consist of such sets of desirable gambles. Since $\mathbf{D}(\bigotimes_{j=1}^n K_j)$ is an isotonic set, it will include some maximal[8] set of desirable gambles $\hat{D}$, because every coherent set of desirable gambles is dominated by a maximal one.[9] But De Cooman and Miranda [11, Proposition 23] have established that any maximal set of desirable gambles on $\mathscr{X}_{1:n}$ is not an independent natural extension of local sets of desirable gambles on $\mathscr{X}_1$, ..., $\mathscr{X}_n$ when $n=|\mathscr{X}_1|=|\mathscr{X}_2|=2$. They leave this question open for cases where $\max\{n,|\mathscr{X}_k|\colon k\in\{1,\dots,n\}\}\geq 2$. This means that $\mathbf{D}(\bigotimes_{j=1}^n K_j)$ can indeed contain sets of desirable gambles that are not the result of an independent natural extension, and, if the answer to their open question is 'yes', will *always* contain such sets of desirable gambles.

A useful property of the independent natural extension $\bigotimes_{j=1}^n D_j$ for sets of desirable gambles is that it is associative [11, Theorem 20]: $\bigotimes_{\ell\in L_1\cup L_2} D_\ell = \bigotimes_{\ell_1\in L_1} D_{\ell_1}\otimes$

6. Actually, they define $A'_{1:n\setminus\{j\}\to\{j\}}$ to be the positive linear hull of what we call $A'_{1:n\setminus\{j\}\to\{j\}}$; this difference is immaterial.

7. To see this, note that $\operatorname{posi}(A'_{1:n\setminus\{j\}\to\{j\}})\subseteq\operatorname{posi}(A_{1:n\setminus\{j\}\to\{j\}})$ because posi is a closure operator and hence monotonic. That also $\operatorname{posi}(A_{1:n\setminus\{j\}\to\{j\}})\subseteq\operatorname{posi}(A'_{1:n\setminus\{j\}\to\{j\}})$ follows by noting that $A_{1:n\setminus\{j\}\to\{j\}}\subseteq\operatorname{posi}(A'_{1:n\setminus\{j\}\to\{j\}})$ since $\mathbb{I}_E f = \sum_{x\in E}\mathbb{I}_{\{x\}}f$.

8. A maximal set of desirable gambles is an undominated element of the complete infimum-semilattice $(\overline{\mathscr{D}},\subseteq)$: they are the maximally informative coherent sets of desirable gambles. They are characterized by the property $f\in\hat{D}$ or $-f\in\hat{D}$, for any non-zero f in $\mathscr{L}$. We refer to [2, 12] for a proof of this representation and more information.

9. See Couso and Moral [2, Theorem 20] and De Cooman and Quaeghebeur [12, Theorem 3] for a proof of this statement.

$\bigotimes_{\ell_2 \in L_2} D_{\ell_2}$ for any disjoint non-empty subsets L_1 and L_2 of $\{1,\ldots,n\}$. This property allows for a modular construction of the independent natural extension. It implies that our independent natural extension $\bigotimes_{j=1}^n K_j$ is associative too:

Proposition 21 (Associativity) *Consider, for each j in $\{1,\ldots,n\}$ a coherent set of desirable gamble sets K_j on $\mathscr{X}_j$, and let L_1 and L_2 be any disjoint non-empty subsets of $\{1,\ldots,n\}$. Then $\bigotimes_{\ell \in L_1 \cup L_2} K_\ell = \bigotimes_{\ell_1 \in L_1} K_{\ell_1} \otimes \bigotimes_{\ell_2 \in L_2} K_{\ell_2}$.*

Note that Proposition 21 implies a stronger marginalization property, namely $\operatorname{marg}_O \bigotimes_{j=1}^n K_j = \bigotimes_{j \in O} K_j$ for any non-empty subset O of $\{1,\ldots,n\}$. Indeed, by considering $L_1 := O$ and $L_2 := \{1,\ldots,n\} \setminus L_1$ we find that $\bigotimes_{j=1}^n K_j = \bigotimes_{j \in O} K_j \otimes \bigotimes_{j \in L_2} K_j$, so it is the independent natural extension of two sets of desirable gamble sets $\bigotimes_{j \in O} K_j$ and $\bigotimes_{j \in L_2} K_j$. By Theorem 20 $\bigotimes_{j=1}^n K_j$ then indeed marginalizes to $\bigotimes_{j \in O} K_j$.

A Stronger Independence Requirement In Definition 17, we defined a set of desirable gamble sets K to be epistemically independent when Equation (4) holds, or, in other words, when

$$A \in K \Leftrightarrow \mathbb{I}_E A \in K, \tag{7}$$

for all disjoint non-empty subsets I and O of $\{1,\ldots,n\}$, E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$ and A in $\mathscr{Q}(\mathscr{X}_O)$. This requires that, if A is a desirable gamble set, then at least one gamble of $\{\mathbb{I}_E f \colon f \in A\} = \mathbb{I}_E A$ should be preferred to zero, and hence $\mathbb{I}_E A \in K$. But one might argue that independence—or indeed, also irrelevance—should require something stronger, namely that

$$E_A \cdot A := \{\mathbb{I}_{E_f} f \colon f \in A\} \tag{8}$$

should belong to K, for every choice of conditioning events $E_A := \{E_f \colon f \in A\} \subseteq \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$. The gamble set $E_A \cdot A$ is the result of multiplying any gamble f in A with its corresponding indicator $\mathbb{I}_{E_f}$ with $E_f \in E_A$, so $E_A \cdot A$ contains multiplications with (indicators of) *different events*, rather than then same event E. This leads to the following stronger independence requirement, which, as we shall see, has a tight connection with epistemic independence: We say that K satisfies the stronger notion of independence if

$$A \in K \Leftrightarrow E_A \cdot A \in K, \tag{9}$$

for all disjoint non-empty subsets I and O of $\{1,\ldots,n\}$, $E_A \subseteq \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$ and A in $\mathscr{Q}(\mathscr{X}_O)$. This is a valid generalization of independence in the sense that, for any epistemically independent set of desirable gambles D, the binary K_D satisfies the requirement in Equation (9), too. We this independence notion at least as compelling as epistemic independence. As E_A may contain only one event E, in which case $E_A \cdot A = \{\mathbb{I}_E f \colon f \in A\} = \mathbb{I}_E A$, this requirement implies the usual requirement of Equation (7). However, the stronger requirement Equation (9) is satisfied by the independent natural extension:

Proposition 22 *The independent natural extension $\bigotimes_{j=1}^n K_j$ of $K_1, \ldots, K_n$ satisfies the stronger requirement of Equation (9). As a consequence, any independent product of $K_1, \ldots, K_j$ includes $\bigcup_{j=1}^n \{E_A \cdot A \colon A \in K_j \text{ and } E_A \subseteq \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})\}$.*

Example 3 *Consider for each j in $\{1,\ldots,n\}$ a credal set $\mathscr{M}_j \subseteq \operatorname{int}(\Sigma_{\mathscr{X}_j})$,*[10] *and consider the* completely independent *[3, 20] credal set on $\mathscr{X}_{1:n}$ given by $\mathscr{M} := \{\prod_{j=1}^n p_j \colon p_1 \in \mathscr{M}_1, \ldots, p_n \in \mathscr{M}_n\}$. This credal will generally be non-convex.*

Let us consider the E-admissible set of desirable gamble sets $K_{\mathscr{M}}$, defined in Example 1. Since $\mathscr{M}$ marginalizes to $\mathscr{M}_1, \ldots, \mathscr{M}_n$ and satisfies independence, the set of desirable gamble sets $K_{\mathscr{M}}$, too, will marginalize to $\mathscr{M}_1, \ldots, \mathscr{M}_n$ and satisfy independence: $K_{\mathscr{M}}$ is an independent product of $K_{\mathscr{M}_1}, \ldots, K_{\mathscr{M}_n}$.

To see that it indeed does satisfy the alternative independence requirement of Equation (9), consider any A in $\mathscr{Q}(\mathscr{X}_O)$ and $G_A \subseteq \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$. Infer for any p in $\mathscr{M}$, f in A and G_f in G_A that $E_{p|G_f}(f) = E_p(\mathbb{I}_{G_f} f)/E_p(\mathbb{I}_{G_f}) = E_p(\mathbb{I}_{G_f})E_p(f)/E_p(\mathbb{I}_{G_f}) = E_p(f)$, so that the following equivalences hold $A \in K_{\mathscr{M}} \Leftrightarrow (\forall p \in \mathscr{M})(\exists f \in A) E_p(f) = E_{p|G_f}(f) > 0 \Leftrightarrow G_A \cdot A \in K_{\mathscr{M}}$, which implies that $K_{\mathscr{M}}$ indeed satisfies Equation (9).

6. Contrasting Epistemic Irrelevance with S-Irrelevance

We have chosen to investigate the independent natural extension of sets of desirable gamble sets according to the standard that we have called ‘epistemic irrelevance’, but there are numerous other notions of irrelevance we might have investigated. One particularly interesting conception of irrelevance is a notion due to Teddy Seidenfeld [22, Section 4] and recently investigated by Jasper De Bock and Gert de Cooman [1]. The basic idea is that one proposition is irrelevant to another if the agent doesn’t regard learning about the first proposition as valuable to decisions that depend only on whether the second proposition is true [1, Section 4.1]:

> “When two variables, X and Y, are ‘independent’ then it is not reasonable to spend resources in order to use the observed value of one of them, say X, to choose between options that depend solely on the value of the other variable, Y.”

To translate this into a workable definition, consider any partition $\mathscr{P}$ of X’s finite possibility space $\mathscr{X}$, and for every element E of $\mathscr{P}$, a gamble f_E on Y’s finite possibility space $\mathscr{Y}$. Then the suggested notion of irrelevance, which De Bock

10. We use the (topological) interior $\operatorname{int}(\Sigma_{\mathscr{X}_j})$ of $\Sigma_{\mathscr{X}_j}$ to make sure that every outcome in $\mathscr{X}_j$ has a (strictly) positive probability for every element of $\mathscr{M}_j$.

and De Cooman [1] term *S-irrelevance*,[11] is that an agent who judges that X is S-irrelevant to Y will be forced to disprefer the composite gamble $\sum_{E\in\mathscr{P}}\mathbb{I}_E(X)f_E(Y)-\varepsilon$, which is the result of paying ε to find out which E^* in $\mathscr{P}$ occurs—the E^* such that $X\in E^*$—in order to decide to take the gamble $f_{E^*}(Y)$, to at least one of $\{f_E\colon E\in\mathscr{P}\}$. In other words, $\{f_E-\sum_{G\in\mathscr{P}}\mathbb{I}_G f_G+\varepsilon\colon E\in\mathscr{P}\}$ is a desirable gamble set. De Bock and De Cooman show that this is equivalent to the requirement

Definition 23 ([1, Definition 9]) *We say that X is S-irrelevant to Y with respect to a coherent set of desirable gamble sets K if $\{\sum_{G\in\mathscr{P}\setminus\{E\}}\mathbb{I}_G(f_E-f_G)+\varepsilon\colon E\in\mathscr{P}\}\in K$ for all partitions $\mathscr{P}$ of $\mathscr{X}$, $f_E\in\mathscr{L}(\mathscr{Y})$ for all E in $\mathscr{P}$, and $\varepsilon\in\mathbb{R}_{>0}$. X and Y are called S-independent when X is S-irrelevant to Y and* vice versa.

When X is a binary variable, meaning that $\mathscr{X}=\{x_1,x_2\}$, this requirement reduces to $\{\mathbb{I}_{\{x_1\}}f+\varepsilon,-\mathbb{I}_{\{x_2\}}f+\varepsilon\}\in K$, for all f in $\mathscr{L}(\mathscr{Y})$ and $\varepsilon\in\mathbb{R}_{>0}$.

S-irrelevance is an intuitively very compelling standard, which raises a natural question: how is our concept of epistemic irrelevance related? It is already clear from De Bock and De Cooman [1]'s analysis of S-irrelevance that it is not entailed by our notion of epistemic irrelevance; as they note [1, Corollary 1], under suitable continuity conditions[12] S-irrelevance has the surprising consequence of being mixing, which loosely speaking implies that the set of desirable gamble sets is represented by a collection of linear prevision. Let us give an explicit example, specialized to our context.

Example 4 *Consider the independent natural extension $K_1\otimes K_2$ of two vacuous local models K_1 and K_2 on $\mathscr{X}$ and $\mathscr{Y}$, respectively. We will show that this is the vacuous K_{v} on $\mathscr{X}\times\mathscr{Y}$. Indeed, K_{v} marginalizes to K_1 and K_2, and it also satisfies the independence requirements of Equation (7):*

$$A\in K_{\mathrm{v}}\Leftrightarrow(\exists f\in A)f>0\Leftrightarrow(\exists f\in A)\mathbb{I}_E f>0\Leftrightarrow\mathbb{I}_E A\in K_{\mathrm{v}},$$

for any A in $\mathscr{Q}(\mathscr{X})$ and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{Y})$, or A in $\mathscr{Q}(\mathscr{Y})$ and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$. So K_{v} is an independent product. Since it is the smallest coherent set of desirable gamble sets, it is equal to $K_{\mathrm{v}}=K_1\otimes K_2$.

Let us show that $K_1\otimes K_2$ does not satisfy S-irrelevance, and therefore also not S-independence. To this end, assume that both $\mathscr{X}=\{x_1,x_2\}$ and $\mathscr{Y}=\{y_1,y_2\}$ are binary. Consider the gamble $f:=(f_1(x_1),f_1(x_2))=(1,-1)$, any y in $\mathscr{Y}$, and $\varepsilon:=\frac{1}{2}>0$. The gamble set $\{\mathbb{I}_{\{y\}}f+\varepsilon,-\mathbb{I}_{\{y\}}f+\varepsilon\}$ does not contain a positive gamble, and therefore belongs not to $K_1\otimes K_2$. This means that $K_1\otimes K_2$ does not satisfy S-independence between X and Y.

11. After 'Seidenfeld'.
12. They are Archimedeanity and "credible indeterminacy", which implies that all the probabilites are positive.

As far as we know, however, whether S-irrelevance entails our standard of epistemic irrelevance has not yet been shown. In the following section, we develop an example which shows that it is possible to *satisfy* S-irrelevance while *flouting* (both value and subset) epistemic irrelevance. Thus, neither S- nor epistemic irrelevance entails the other.

Violating Epistemic Irrelevance While Satisfying S-Irrelevance There are two general ways we suspect it is possible to violate epistemic subset irrelevance while satisfying S-irrelevance. A variable X fails to be epistemically irrelevant to a variable Y just in case $A\in K$ but $\mathbb{I}_E A\notin K$, or $\mathbb{I}_E A\in K$ but $A\notin K$, for some E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$ and A in $\mathscr{Q}(\mathscr{Y})$.

In essence, there is some proposition about the value of X that the agent can learn which will change their views about the preferences for some gambles that depend only on Y. The question we are interested in is whether there is a way for an agent who knows that learning E will change their views about the preferences between these gambles to not place any *real* monetary value on learning it. There are at least two ways that it occurs to us that this could happen:

- the agent thinks that there is no real value gained by using the informed strategy over merely accepting a wager without learning;
- the agent is certain that the experiment they are (not) paying for will not yield the outcome which would change their desires.

In the present paper, we leave the latter kind of case as a conjecture. In this section, we develop an example of the former. The framework of sets of desirable gamble sets (and indeed, the less expressive framework of sets of desirable gambles) is capable of representing an agent as believing that one outcome is infinitesimally more likely than another. In the multivariate arena, this raises the possibility of correlations that generate only infinitesimal change in belief. Information that generates such an infinitesimal change will not have any *real* expected value as long as the gambles that are at issue have only finite value, which is consistent with S-irrelevance; Nonetheless, learning the information *does* make an identifiable change to which gambles the agent finds desirable, and thus subset irrelevance is violated.

Example 5 *A factory produces two kinds of coins: coins that are fair (heads and tails are equally likely) and coins that are infinitesimally biased in favor of heads (heads is more likely than tails but not by any definite amount). Consider an agent who knows that a coin produced by this factory is about to be flipped; the above description of the factory is all they know. They have no beliefs about the proportion of coins of each type the factory produces, or any specific reason to believe that the coin in question is of one type or the other.*

The agent is offered the following decision problem: they can accept a wager that pays some real payout a if the coin lands heads and $-a$ if tails, they can decline the wager *(maintain the status quo, accept the zero gamble), or they can pay some small (real-valued) fee* $\varepsilon > 0$ *to learn the type of the coin, and then decide whether to accept or reject the* $(a,-a)$ *wager. Let* X *take values in* $\mathscr{X} := \{F,U\}$ *representing whether the coin is fair or unfair and let* Y *take values in* $\mathscr{Y} := \{H,T\}$ *representing whether the coin lands heads or tails. For ease of future reference, we denote* $D_1 := \{f\colon f \in \mathscr{L}(\mathscr{Y}) \text{ and } f(H)+f(T) > 0\}$ *and* $D_2 := \{f\colon f \in \mathscr{L}(\mathscr{Y}) \text{ and } (f(H)+f(T) > 0 \text{ or } (f(H)+f(T) = 0 \text{ and } f(H) > f(T)))\}$*. Note that* D_1 *and* D_2 *are coherent sets of desirable gambles.*

We model the agent's beliefs by $\mathbb{I}_{\{F\}}D_1 \cup \mathbb{I}_{\{U\}}D_2$*'s natural extension* $D := \operatorname{posi}(\mathbb{I}_{\{F\}}D_1 \cup \mathbb{I}_{\{U\}}D_2 \cup \mathscr{L}(\mathscr{X} \times \mathscr{Y})_{>0})$*.*

Lemma 24 *D is a coherent set of desirable gambles. Moreover, it is no independent product, and it therefore fails epistemic irrelevance from X to Y.*

So D—and therefore K_D, which is coherent since D is—fails to satisfy epistemic value irrelevance, and therefore also epistemic subset irrelevance. However, despite this difference in desirability dependent on learning about X, there is no positive value ε that the agent will be willing to pay to learn the bias before deciding whether to accept or reject a gamble on the outcome of the flip, so a gamble on $\mathscr{Y}$. More generally, there are no gambles f and g on $\mathscr{Y}$ such that the agent would pay to learn the bias of the coin before deciding between f and g.

To show this formally, we consider K_D and show that it satisfies S-irrelevance:

Lemma 25 *X is S-irrelevant to Y with respect to K_D.*

Thus, we have a case where X is not epistemically irrelevant to Y, but X is S-irrelevant to Y. The upshot is that there are cases where learning about a variable X makes an identifiable change to which gambles, defined only on $\mathscr{Y}$, an agent prefers, but the agent sees this difference as negligible in (real) value.

7. Discussion

Independence is an interesting concept. When we model uncertainty with precise probabilities, it seems univocal. But when we model uncertainty with imprecise probabilities, it fractures into a multiplicity of distinct concepts, including:

- complete independence for sets of probabilities [3, 20];
- independence in selection for lower previsions [10];
- strong independence for lower previsions and sets of desirable gambles [11];
- epistemic independence (value and subset) for sets of desirable gambles [17];
- epistemic h-independence for lower previsions and credal sets [6];
- S-independence for choice functions [1];

These concepts collapse in the limit, when applied to precise probabilities, but come apart in general.

Independence is also an *important* concept. For example, many have thought that when pooling expert opinions we ought to preserve unanimous judgments of independence [14, 15, 13]. Take another example: causal modelling. Causal Bayesian networks consist of a directed acyclic graph together with an appropriate probability distribution. They are popular formal tools for modelling causal relationships. Independence judgments play a key role in constructing causal Bayes nets. Missing edges between variables in the graph of a causal Bayes net indicate that those variables are causally independent of one another.

In this paper we investigated epistemic independence in the general framework of sets of desirable gamble sets. Our results indicate that the independent natural extension can be calculated by performing the eponymous operation on a representing collection of sets of desirable gambles, but this collection may be infinite.

In addition, we took some initial steps to compare epistemic independence with another attractive notion of independence proposed by Teddy Seidenfeld: S-independence. Recently, Jasper De Bock and Gert de Cooman [1, Corollary 1] showed that if an Archimedean set of desirable gamble sets renders a variable X "credibly indeterminate", then judging that X is S-irrelevant to Y forces you to choose between gambles on Y using E-admissibility. Judgments of S-irrelevance, it turns out, are much more informative than they appear at first glance.

There are still a number of open questions about epistemic independence for sets of desirable gamble sets. For example, Cozman and Seidenfeld [5] explore the notion of *layer independence* for full conditional probability measures. Cozman [4] shows that the only extant concept of independence for (non-convex) sets of probabilities that has a range of desirable graphoid properties is element-wise layer independence. It is an open question what additional structural constraints on coherent sets of desirable gamble sets are necessary and sufficient to secure the relevant graphoid properties.

As sets of desirable gamble sets generalize many of the extant imprecise-probabilistic uncertainty models, including sets of desirable gambles, lower previsions, and sets of probability mass functions, they may be expressive enough to unify some of the aforementioned independence concepts. Example 3 is one instance of this, where we show that an E-admissible set of desirable gamble sets based on a completely independent credal set is an (epistemically) independent product. We suspect Proposition 22 to be helpful in the connection with some of the other independence concepts. We intend to investigate these connections, with the hope to obtain a unifying theory.

Acknowledgments

This research is part of the Epistemic Utility for Imprecise Probability project which is funded by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 852677). We would like to thank Catrin Campbell-Moore, Teddy Seidenfeld, Jasper De Bock and Gert de Cooman for helpful discussion. We would also like to thank four anonymous reviewers for their valuable and stimulating remarks, and especially the fourth reviewer for their suggestion about the stronger notion of independence.

References

[1] Jasper De Bock and Gert de Cooman. On a notion of independence proposed by Teddy Seidenfeld, 2021. URL https://arxiv.org/abs/2102.10342.

[2] Inés Couso and Serafín Moral. Sets of desirable gambles: conditioning, representation, and precise probabilities. *International Journal of Approximate Reasoning*, 52(7), 2011. doi: 10.1016/j.ijar.2011.08.001.

[3] Fabio G. Cozman. Sets of probability distributions, independence, and convexity. *Synthese*, 186:177–200, 2012.

[4] Fabio G. Cozman. Independence for sets of full conditional probabilities, sets of lexicograpic probabilities, and sets of desirable gambles. In Fabio G. Cozman, Thierry Denœux, Sébastien Destercke, and Teddy Seidenfeld, editors, *ISIPTA '13 – Proceedings of the Eighth International Symposium on Imprecise Probability: Theories and Applications*, pages 87–97. SIPTA, 2013.

[5] Fabio G. Cozman and Teddy Seidenfeld. Independence for full conditional measures and their graphoid properties. In Jan-Willem Romeijn Benedikt Löwe, Eric Pacuit, editor, *Foundations of the Formal Sciences VI: Reasoning about Probabilities and Probabilistic Reasoning*. College Publications, 2009.

[6] Jasper De Bock. *Credal Networks under Epistemic Irrelevance: Theory and algorithms*. PhD thesis, Ghent University, 2015.

[7] Jasper De Bock and Gert de Cooman. A desirability-based axiomatisation for coherent choice functions. In S. Destercke, T. Denoeux, M.A. Gil, P. Grzergorzewski, and O. Hryniewicz, editors, *Uncertainty Modelling in Data Science: Proceedings of SMPS 2018*, pages 46–53, 2018.

[8] Jasper De Bock and Gert de Cooman. Interpreting, axiomatising and representing coherent choice functions in terms of desirability. In Jasper De Bock, Cassio P. de Campos, Gert de Cooman, Erik Quaeghebeur, and Gregory Wheeler, editors, *Proceedings of the Eleventh International Symposium on Imprecise Probabilities: Theories and Applications*, volume 103 of *Proceedings of Machine Learning Research*, pages 125–134, Thagaste, Ghent, Belgium, 03–06 Jul 2019. PMLR.

[9] Jasper De Bock, Arthur Van Camp, Márcio Alves Diniz, and Gert de Cooman. Representation theorems for partially exchangeable random variables. *Fuzzy Sets and Systems*, 284:1–30, 2016. doi: 10.1016/j.fss.2014.10.027.

[10] Luis M. de Campos and Serafín Moral. Independence concepts for convex sets of probabilities. In P. Besnard and S. Hanks, editors, *Eleventh Conference on Uncertainty in Artificial Intelligence*, pages 108–115. San Francisco, CA, 1995.

[11] Gert de Cooman and Enrique Miranda. Irrelevant and independent natural extension for sets of desirable gambles. *Journal of Artificial Intelligence Research*, 45:601–640, 2012.

[12] Gert de Cooman and Erik Quaeghebeur. Exchangeability and sets of desirable gambles. *International Journal of Approximate Reasoning*, 53(3):363–395, 2012. Special issue in honour of Henry E. Kyburg, Jr.

[13] Franz Dietrich and Christian List. Probabilistic opinion pooling. In A. Hajek and C.R. Hitchcock, editors, *Oxford Handbook of Philosophy and Probability*. Oxford University Press, Oxford, 2015.

[14] Robert Laddaga. Lehrer and the consensus proposal. *Synthese*, 36:473–477, 1977.

[15] Keith Lehrer and Carl Wagner. Probability amalgamation and the independence issue: A reply to Laddaga. *Synthese*, 55(3):339–346, 1983.

[16] Isaac Levi. *The Enterprise of Knowledge*. MIT Press, London, 1980.

[17] Serafín Moral. Epistemic irrelevance on sets of desirable gambles. *Annals of Mathematics and Artificial Intelligence*, 45:197–214, 2005. doi: 10.1007/s10472-005-9011-0.

[18] Erik Quaeghebeur. Introduction to imprecise probabilities. chapter Desirability. John Wiley & Sons, 2014.

[19] Teddy Seidenfeld. A contrast between two decision rules for use with (convex) sets of probabilities: Gamma-maximin versus e-admissibilty. *Synthese*, 140(1):69–88, 2004. doi: 10.1023/B:SYNT.0000029942.11359.8d.

[20] Teddy Seidenfeld. Conditional independence, imprecise probabilities, null-events and graph-theoretic models, 2013. Seimour Geisser Distinguished Lecture, School of Statistics, University of Minnesota, Minneapolis, MN.

[21] Teddy Seidenfeld, Mark J. Schervish, and Joseph B. Kadane. Decisions without ordering. In W. Sieg, editor, *Acting and reflecting*, volume 211 of *Synthese Library*, pages 143–170. Kluwer, Dordrecht, 1990.

[22] Teddy Seidenfeld, Mark J. Schervish, and Joseph B. Kadane. Coherent choice functions under uncertainty. *Synthese*, 172(1):157–176, 2010. doi: 10.1007/s11229-009-9470-7.

[23] Arthur Van Camp and Enrique Miranda. Modelling epistemic irrelevance with choice functions. *International Journal of Approximate Reasoning*, 125:49 – 72, 2020. ISSN 0888-613X. doi: https://doi.org/10.1016/j.ijar.2020.06.010.

[24] Peter Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.

[25] Peter Walley. Towards a unified theory of imprecise probability. *International Journal of Approximate Reasoning*, 24:125–148, 2000.

Supplementary Material: Proofs

Proof [Proof of Lemma 8] We will show that (i) $K_{\mathscr{M}} \subseteq \bigcap\{K_{D_p} : p \in \mathscr{M}\}$ and (ii) $K_{\mathscr{M}} \supseteq \bigcap\{K_{D_p} : p \in \mathscr{M}\}$. For (i), consider any A in $K_{\mathscr{M}}$, meaning that $A \cap \mathscr{L}_{>0} \neq \emptyset$ or $(\forall p \in \mathscr{M})(\exists f \in A)E_p(f) > 0$. Both cases imply that $A \cap D_p \neq \emptyset$ for every p in $\mathscr{M}$, whence indeed $A \in \bigcap\{K_{D_p} : p \in \mathscr{M}\}$. For (ii), consider any A in $\bigcap\{K_{D_p} : p \in \mathscr{M}\}$, meaning that $A \cap D_p \neq \emptyset$ for all p in $\mathscr{M}$, and hence indeed $A \in K_{\mathscr{M}}$. ∎

Proof [Proof of Proposition 19] We will show that (i) $K_{\bigotimes_{j=1}^n D_j} \subseteq \bigotimes_{j=1}^n K_{D_j}$ and (ii) $K_{\bigotimes_{j=1}^n D_j} \supseteq \bigotimes_{j=1}^n K_{D_j}$.

For (i), consider any A in $K_{\bigotimes_{j=1}^n D_j}$. Then $A \cap \bigotimes_{j=1}^n D_j \neq \emptyset$, so let $f \in A$ belong to $\bigotimes_{j=1}^n D_j$. Then $f \in \mathscr{L}(\mathscr{X}_{1:n})_{>0}$—in which case $A \in \bigotimes_{j=1}^n K_{D_j}$ by coherence—or $f \geq \sum_{k=1}^m \lambda_k f_k$ for some m in $\mathbb{N}$, $f_1, \ldots, f_m$ in $\bigcup_{j=1}^n A_{1:n\setminus\{j\}\to\{j\}}$ and m real coefficients $\lambda_{1:m} > 0$. But then, for every k in $\{1,\ldots,m\}$, the gamble set $A_k := \{f_k\}$ belongs to $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}}$. Let furthermore $\lambda_{1:m}^{f_{1:m}} := \lambda_{1:m} > 0$ for the unique—and hence all—$f_{1:m}$ in $\bigtimes_{k=1}^m A_k$. This implies that $\{\sum_{k=1}^m f_k\} = \{\sum_{k=1}^m \lambda_k^{f_{1:m}} f_k : f_{1:m} \in \bigtimes_{k=1}^m A_k\}$ belongs to $\mathrm{Posi}(\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}})$ and since $f \geq \sum_{k=1}^m f_k$, also $\{f\} \in \mathrm{Posi}(\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0})$. Since $f \in A$, we have that then indeed $A \in \bigotimes_{j=1}^n K_{D_j}$.

For (ii), consider any A in $\bigotimes_{j=1}^n K_{D_j}$. Then $A \supseteq B \setminus \mathscr{L}(\mathscr{X}_{1:n})_{\leq 0}$ for some B in $\mathrm{Posi}(\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0})$, meaning that $B = \{\sum_{k=1}^m \lambda_k^{f_{1:m}} f_k : f_{1:m} \in \bigtimes_{k=1}^n B_k\}$ for some m in $\mathbb{N}$, $B_1, \ldots, B_m$ in $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0}$ and, for every $f_{1:m}$ in $\bigtimes_{k=1}^m B_k$, m real coefficients $\lambda_{1:m}^{f_{1:m}} > 0$. For any k in $\{1,\ldots,m\}$ we have that B_k belongs to $\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0}$—in which case we call $B_k := \{g_k\}$—or $B_k = \mathbb{I}_E B'_k$ for some j in $\{1,\ldots,n\}$, E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n} \setminus \{j\})$ and B'_k in K_{D_j}, meaning that $B'_k \cap D_j \neq \emptyset$—in which case we let h_k belong to $B'_k \cap D_j$ and define $g_k := \mathbb{I}_E h_k$. Then the gamble $f := \sum_{k=1}^m \lambda_k^{g_{1:m}} g_k$ belongs to B, and all of its terms $\lambda_k^{g_{1:m}} g_k$ either are equal to 0, or belong to $\mathscr{L}(\mathscr{X}_{1:m})_{>0}$ or to $\bigcup_{j=1}^n A_{1:n\setminus\{j\}\to\{j\}}$. Since not all of these terms are equal to 0, by Theorem 18 then $f \in \bigotimes_{j=1}^n D_j$, so that B belongs to $K_{\bigotimes_{j=1}^n D_j}$, and therefore indeed so does A.
∎

Proof [Proof of Theorem 20] This proof will consist of five parts: we will subsequently show that (i) $\bigotimes_{j=1}^n K_j$ is coherent, (ii) it is represented by $\bigotimes_{j=1}^n \mathbf{D}(K_j)$, (iii) $\mathrm{marg}_\ell(\bigotimes_{j=1}^n K_j) = K_\ell$ for every ℓ in $\{1,\ldots,n\}$, (iv) $\bigotimes_{j=1}^n K_j$ is epistemically independent, and (v) $\bigotimes_{j=1}^n K_j$ is the smallest such set of desirable gamble sets. Then (i), (iii) and (iv) establish that $\bigotimes_{j=1}^n K_j$ is an independent product of $K_1, \ldots, K_n$, which is by (v) the smallest one. (ii) establishes the last claim about $\bigotimes_{j=1}^n K_j$'s representation.

For (i), to show that $\bigotimes_{j=1}^n K_j$ is coherent, we will regard $\mathscr{A} := \bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}}$ as an assessment on $\mathscr{Q}(\mathscr{X}_{1:n})$. By Theorem 9 it suffices to show that $\mathscr{A} \subseteq K_D$ for some coherent set of desirable gambles $D \subseteq \mathscr{L}(\mathscr{X}_{1:n})$—in other words, that $\mathscr{A}$ is consistent.

To this end, note already using Theorem 7 that $\mathbf{D}(K_1), \ldots, \mathbf{D}(K_n)$ all are non-empty since $K_1, \ldots, K_n$ are coherent. Consider any D_1 in $\mathbf{D}(K_1), \ldots, D_n$ in $\mathbf{D}(K_n)$, and let $D^* := \bigotimes_{j=1}^n D_j$. Then Theorem 18 implies that D^* is a coherent set of desirable gambles on $\mathscr{L}(\mathscr{X}_{1:n})$ that is epistemically independent—by which we mean that $\mathrm{marg}_O D^* = \mathrm{marg}_O(D^* \rfloor E_I)$ for all disjoint non-empty subsets I and O of $\{1,\ldots,n\}$ and E_I in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$—and marginalizes to $D_1, \ldots, D_n$. We will show that $\mathscr{A} \subseteq K_{D^*}$. To this end, consider any A in $\mathscr{A}$, meaning that there is an index j in $\{1,\ldots,n\}$ such that $A \in \mathscr{A}_{1:n\setminus\{j\}\to\{j\}}$, or, in other words, such that $A = \mathbb{I}_E B$ for some B in K_j and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$. Since D_j belongs to $\mathbf{D}(K_j)$ we have that $K_j \subseteq K_{D_j}$, and therefore $B \in K_{D_j} = K_{\mathrm{marg}_j D^*}$. Since $K_{\mathrm{marg}_j D^*} = \mathrm{marg}_j K_{D^*}$ by Proposition 15, this means that $B \in K_{D^*}$. But D^* is an epistemically independent set of desirable gambles, and it therefore satisfies $\mathrm{marg}_j(D^* \rfloor E) = \mathrm{marg}_j D^*$, or in other words, $f \in D^* \Leftrightarrow \mathbb{I}_E f \in D^*$, for any f in $\mathscr{L}(\mathscr{X}_j)$, and hence also $A = \mathbb{I}_E B \in K_{D^*}$. Since the choice of A in $\mathscr{A}$ was arbitrary, this implies that indeed $\mathscr{A} \subseteq K_{D^*}$, guaranteeing that indeed $\bigotimes_{j=1}^n K_j$ is coherent.

For (ii), since we have just proved that $\mathscr{A}$ is consistent, we know by Theorem 9 that

$$
\begin{aligned}
\bigotimes_{j=1}^n K_j &= \bigcap\{K_D : D \in \mathbf{D}(\mathscr{X}_{1:n}) \text{ and } \mathscr{A} \subseteq K_D\} \\
&= \bigcap\{K_D : D \in \mathbf{D}(\mathscr{X}_{1:n}) \text{ and } (\forall j \in \{1,\ldots,n\})\mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \subseteq K_D\} \\
&= \bigcap\{K_D : D \in \mathbf{D}(\mathscr{X}_{1:n}) \text{ and } (\forall j \in \{1,\ldots,n\}, B \in K_j, E \in \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}}))\mathbb{I}_E B \in K_D\} = \bigcap\{K_D : D \in \mathbf{D}^*\},
\end{aligned}
$$

where we defined $\mathbf{D}^* := \{D \in \mathbf{D}(\mathscr{X}_{1:n}) : (\forall j \in \{1,\ldots,n\}, B \in K_j, E \in \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}}))\mathbb{I}_E B \in K_D\}$ for the sake of brevity. This collection $\mathbf{D}^*$ has two interesting properties: it satisfies $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \subseteq K_{D^*}$ for every D^* in $\mathbf{D}^*$, as can be seen from its definition. It also satisfies for every j in $\{1,\ldots,n\}$ the inclusion $\mathrm{marg}_j \mathbf{D}^* \subseteq \mathbf{D}(K_j)$—in other words, $\mathrm{marg}_j D^* \in \mathbf{D}(K_j)$ for all D^* in $\mathbf{D}^*$. To show this last property, consider any D^* in $\mathbf{D}^*$, j in $\{1,\ldots,n\}$, and consider $E := \mathscr{X}_{1:n\setminus\{j\}} \in \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$. That D^* belongs to $\mathbf{D}^*$ implies that $B = \mathbb{I}_E B \in K_{D^*}$ for every B in K_j, and hence $K_j \subseteq K_{D^*}$.

But K_j is a set of desirable gamble sets on $\mathscr{X}_j$, so $K_j \subseteq \mathrm{marg}_j K_{D^*} = K_{\mathrm{marg}_j D^*}$, where the equality is due to Proposition 15. This implies that indeed $\mathrm{marg}_j D^* \in \mathbf{D}(K_j)$.

This part of the proof is established if we show that $\bigcap\{K_D : D \in \mathbf{D}^*\} = \bigcap\{K_D : D \in \bigotimes_{j=1}^n \mathbf{D}(K_j)\}$. We will first show that $\bigotimes_{j=1}^n \mathbf{D}(K_j) \subseteq \mathbf{D}^*$. To this end, consider any D in $\bigotimes_{j=1}^n \mathbf{D}(K_j)$—meaning that $D = \bigotimes_{j=1}^n D_j$ for some D_1 in $\mathbf{D}(K_1)$, $\dots$, D_n in $\mathbf{D}(K_n)$—and any j in $\{1,\dots,n\}$, B in K_j and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$. That D_j belongs to $\mathbf{D}(K_j)$ implies that $B \in K_{D_j}$, and Theorem 18 tells us that $\mathrm{marg}_j D = D_j$, so $B \in K_D$. But then $f \in D$ for some f in B, and since D is an epistemically independent set of desirable gambles, therefore $\mathbb{I}_E f \in D$, whence $\mathbb{I}_E B \in K_D$. This implies that $D \in \mathbf{D}^*$, and therefore, since the choice of D in $\bigotimes_{j=1}^n \mathbf{D}(K_j)$ was arbitrary, indeed $\bigotimes_{j=1}^n \mathbf{D}(K_j) \subseteq \mathbf{D}^*$, which implies that $\bigcap\{K_D : D \in \mathbf{D}^*\} \subseteq \bigcap\{K_D : D \in \bigotimes_{j=1}^n \mathbf{D}(K_j)\}$.

To establish the equality between these two intersections, it suffices to prove that also the converse set inclusion holds. To this end, consider any A in $\bigcap\{K_D : D \in \bigotimes_{j=1}^n \mathbf{D}(K_j)\}$, meaning that $A \cap \bigotimes_{j=1}^n D_j \neq \emptyset$ for all D_1 in $\mathbf{D}(K_1), \dots, D_n$ in $\mathbf{D}(K_n)$. We need to show that then $A \in \bigcap\{K_D : D \in \mathbf{D}^*\}$—or in other words, that $A \cap D^* \neq \emptyset$ for any D^* in $\mathbf{D}^*$—so consider any D^* in $\mathbf{D}^*$. We have established earlier that then $\mathrm{marg}_j D^* \in \mathbf{D}(K_j)$ for any j in $\{1,\dots,n\}$, so that $\bigotimes_{j=1}^n \mathrm{marg}_j D^*$ belongs to $\bigotimes_{j=1}^n \mathbf{D}(K_j)$ and we therefore have that $A \cap \bigotimes_{j=1}^n \mathrm{marg}_j D^* \neq \emptyset$, or in other words, that $A \in K_{\bigotimes_{j=1}^n \mathrm{marg}_j D^*}$. But we have seen in Proposition 19 that $K_{\bigotimes_{j=1}^n \mathrm{marg}_j D^*}$ is the smallest element of $\overline{\mathscr{K}}$ that includes $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}}$, and therefore, since we already have established above that $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \subseteq K_{D^*}$, we have that $K_{\bigotimes_{j=1}^n \mathrm{marg}_j D^*} \subseteq K_{D^*}$. This implies that $A \in K_{D^*}$, whence indeed $A \cap D^* \neq \emptyset$.

For (iii), consider any ℓ in $\{1,\dots,n\}$, and we will show that $\mathrm{marg}_\ell(\bigotimes_{j=1}^n K_j) = K_\ell$. We know from the second part of this proof, established above, that $\bigotimes_{j=1}^n K_j$ is represented by $\bigotimes_{j=1}^n \mathbf{D}(K_j)$, and therefore also, using Proposition 15, that $\mathrm{marg}_\ell(\bigotimes_{j=1}^n K_j)$ is represented by $\mathrm{marg}_\ell(\bigotimes_{j=1}^n \mathbf{D}(K_j))$. Infer the following chain of equalities:

$$
\begin{aligned}
\mathrm{marg}_\ell\left(\bigotimes_{j=1}^n \mathbf{D}(K_j)\right) &= \mathrm{marg}_\ell\left(\left\{\bigotimes_{j=1}^n D_j \colon D_1 \in \mathbf{D}(K_1),\dots,D_n \in \mathbf{D}(K_n)\right\}\right)\\
&= \left\{\mathrm{marg}_\ell\left(\bigotimes_{j=1}^n D_j\right) \colon D_1 \in \mathbf{D}(K_1),\dots,D_n \in \mathbf{D}(K_n)\right\}\\
&= \{D_\ell \colon D_1 \in \mathbf{D}(K_1),\dots,D_n \in \mathbf{D}(K_n)\} = \mathbf{D}(K_\ell),
\end{aligned}
$$

where the first equality follows from the definition of $\bigotimes_{j=1}^n \mathbf{D}(K_j)$, the second one from the definition above Proposition 15 of $\mathrm{marg}_O(\mathbf{D})$ for any collection $\mathbf{D}$ of sets of desirable gambles, and the third one from Theorem 18. This means that $\mathrm{marg}_\ell(\bigotimes_{j=1}^n K_j)$ is represented by $\mathbf{D}(K_\ell)$. Theorem 7 then implies that indeed $\mathrm{marg}_\ell(\bigotimes_{j=1}^n K_j) = K_\ell$.

Finally, for (iv), let $K^* \subseteq \mathscr{Q}(\mathscr{X}_{1:n})$ be the smallest independent product of K_1, $\dots$, K_n. Since K^* is epistemically independent, we have by Equation (4) in particular, for any j in $\{1,\dots,n\}$ and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$, that

$$
\mathrm{marg}_j(K^* \rfloor E) = \mathrm{marg}_j K^* = K_j,
$$

where the first equality holds because K is epistemically independent, and the second one because K^* marginalizes to K_1, $\dots$, K_n. This implies that any B in K_j should belong to $K^* \rfloor E$, and hence that $\mathbb{I}_E B \in K^*$. Since this should hold for any j in $\{1,\dots,n\}$, B in K_j, and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$, we have that $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \subseteq K^*$. Since K^* is coherent, also $\mathrm{posi}(\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0}) \subseteq K^*$. But this tells us that $\bigotimes_{j=1}^n K_i \subseteq K^*$, establishing that $\bigotimes_{j=1}^n K_i$ indeed is the smallest independent product of K_1, $\dots$, K_n. ∎

Proof [Proof of Proposition 21] By Theorem 20 $\bigotimes_{\ell\in L_1\cup L_2} K_\ell$ is represented by $\bigotimes_{\ell\in L_1\cup L_2} \mathbf{D}(K_\ell)$. Infer using the associativity of the independent natural extension for sets of desirable gambles that

$$
\begin{aligned}
\bigotimes_{\ell\in L_1\cup L_2} \mathbf{D}(K_\ell) &= \left\{\bigotimes_{\ell\in L_1\cup L_2} D_\ell \colon (\forall \ell \in L_1\cup L_2) D_\ell \in \mathbf{D}(K_\ell)\right\}\\
&= \left\{\bigotimes_{\ell_1\in L_1} D_{\ell_1} \otimes \bigotimes_{\ell_2\in L_2} D_{\ell_2} \colon (\forall \ell \in L_1\cup L_2) D_\ell \in \mathbf{D}(K_\ell)\right\}\\
&= \left\{\bigotimes_{\ell_1\in L_1} D_{\ell_1} \colon (\forall \ell_1 \in L_1) D_{\ell_1} \in \mathbf{D}(K_{\ell_1})\right\} \otimes \left\{\bigotimes_{\ell_2\in L_2} D_{\ell_2} \colon (\forall \ell_2 \in L_1\cup L_2) D_{\ell_2} \in \mathbf{D}(K_{\ell_2})\right\}\\
&= \bigotimes_{\ell_1\in L_1} \mathbf{D}(K_{\ell_1}) \otimes \bigotimes_{\ell_2\in L_2} \mathbf{D}(K_{\ell_2}),
\end{aligned}
$$

so that $\bigotimes_{\ell\in L_1\cup L_2} K_\ell$ is represented by the independent natural extension $\bigotimes_{\ell_1\in L_1}\mathbf{D}(K_{\ell_1})\otimes\bigotimes_{\ell_2\in L_2}\mathbf{D}(K_{\ell_2})$ of two coherent sets of desirable gambles $\bigotimes_{\ell_1\in L_1}\mathbf{D}(K_{\ell_1})$ and $\bigotimes_{\ell_2\in L_2}\mathbf{D}(K_{\ell_2})$. Theorem 20 then implies that indeed $\bigotimes_{\ell\in L_1\cup L_2} K_\ell = \bigotimes_{\ell_1\in L_1} K_{\ell_1}\otimes\bigotimes_{\ell_2\in L_2} K_{\ell_2}$. ■

Proof [Proof of Proposition 22] Use Theorem 20 to infer that $\bigotimes_{j=1}^n K_j$ is represented by $\bigotimes_{j=1}^n \mathbf{D}(K_j)$, so $\bigotimes_{j=1}^n K_j = \bigcap\{K_D\colon D\in\bigotimes_{j=1}^n\mathbf{D}(K_j)\}$. Note that, by Theorem 18, any D in $\bigotimes_{j=1}^n\mathbf{D}(K_j)$ satisfies

$$\mathrm{marg}_O D = \mathrm{marg}_O(D\rfloor E_I)$$

for any disjoint non-empty subsets I and O of $\{1,\dots,n\}$, and E_I in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$. Consider any A in $\mathscr{Q}(\mathscr{X}_I)$ and $E_A\subseteq\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_O)$, and infer the following equivalences

$$A\in\bigotimes_{j=1}^n K_j \Leftrightarrow \Big(\forall D\in\bigotimes_{j=1}^n\mathbf{D}(K_j)\Big)(\exists f\in A)f\in D \Leftrightarrow \Big(\forall D\in\bigotimes_{j=1}^n\mathbf{D}(K_j)\Big)(\exists f\in A)\mathbb{I}_{E_f}f\in D$$
$$\Leftrightarrow \Big(\forall D\in\bigotimes_{j=1}^n\mathbf{D}(K_j)\Big)E_A A\cap D\neq\emptyset \Leftrightarrow E_A A\in\bigotimes_{j=1}^n K_j,$$

which establishes that $\bigotimes_{j=1}^n K_j$ satisfies the stronger requirement of Equation (9).

To show that then, as a consequence, any independent product of K_1, …, K_n includes $\mathscr{A}^*_{1:n\setminus\{j\}\to\{j\}} := \bigcup_{j=1}^n\big\{E_A\cdot A\colon A\in K_j \text{ and } E_A\subseteq\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})\big\}$, it suffices to show that the smallest independent product $\bigotimes_{j=1}^n K_j$ of K_1, …, K_n includes $\mathscr{A}^*_{1:n\setminus\{j\}\to\{j\}}$. To this end, consider any j in $\{1,\dots,n\}$ and any A in K_j. Then since $\bigotimes_{j=1}^n K_j$ marginalizes to K_j, we have $A\in\bigotimes_{j=1}^n K_j$. By Equation (9) [use $O:=\{j\}$ and $I:=\{1,\dots,n\}\setminus\{j\}$], then also $E_A\cdot A\in\bigotimes_{j=1}^n K_j$ for any $E_A\subseteq\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$. Since the choice of j in $\{1,\dots,n\}$ was arbitrary, this implies that indeed $\mathscr{A}^*_{1:n\setminus\{j\}\to\{j\}}\subseteq\bigotimes_{j=1}^n K_j$. ■

Proof [Proof of Lemma 24] To show that D is coherent, it suffices by Theorem 6 to show that $\mathscr{L}_{\leq 0}\cap\mathrm{posi}(\mathbb{I}_{\{F\}}D_1\cup\mathbb{I}_{\{U\}}D_2)=\emptyset$. To this end, consider any f in $\mathrm{posi}(\mathbb{I}_{\{F\}}D_1\cup\mathbb{I}_{\{U\}}D_2)$, meaning that $f=\sum_{k=1}^m\lambda_k f_k$ for some m in $\mathbb{N}$, real coefficients $\lambda_{1:m}>0$, and gambles f_1, …, f_m in $\mathbb{I}_{\{F\}}D_1\cup\mathbb{I}_{\{U\}}D_2$. For every k in $\{1,\dots,m\}$, if f_k belongs to $\mathbb{I}_{\{F\}}D_1$ then $f_k(U,H)=f_k(U,T)=0$ and $f_k(F,H)+f_k(F,T)>0$, and if f_k belongs to $\mathbb{I}_{\{U\}}D_2$ then $f_k(F,H)=f_k(F,T)=0$ and $f_k(U,H)+f_k(U,T)>0$, or $f_k(U,H)+f_k(U,T)=0$ but then $f_k(U,H)>f_k(U,T)$. This implies that $f(\bullet,H)+f(\bullet,T)>0$ whence indeed $f\notin\mathscr{L}_{\leq 0}$.

To show that it is no independent product, let us show that $\mathrm{marg}_Y D\subset\mathrm{marg}_Y(D\rfloor\{U\})$, so that learning that the coin is unfair, results in a bigger Y-marginal than not learning anything at all. More specifically, we will show that $\mathrm{marg}_Y D=D_1$ and $\mathrm{marg}_Y(D\rfloor\{U\})=D_2$.

To show that $\mathrm{marg}_Y D\subseteq D_1$, consider any f in $\mathrm{marg}_Y D$. Then $f\in\mathscr{L}(\mathscr{Y})$ and $f\in D$, meaning that $f>0$—in which case $f\in D_1$ by its coherence—or $f\geq\sum_{k=1}^m\lambda_k f_k$ for some m in $\mathbb{N}$, real coefficients $\lambda_{1:m}>0$, and gambles f_1, …, f_m in $\mathbb{I}_{\{F\}}D_1\cup\mathbb{I}_{\{U\}}D_2$. Since f belongs to $\mathscr{L}(\mathscr{Y})$, we have that $f\geq\frac{1}{2}\sum_{x\in\mathscr{X}}\sum_{k=1}^m\lambda_k f_k(x,\bullet)$, and therefore $f(H)+f(T)\geq\frac{1}{2}\sum_{x\in\mathscr{X}}\sum_{k=1}^m\lambda_k f_k(x,H)+\frac{1}{2}\sum_{x\in\mathscr{X}}\sum_{k=1}^m\lambda_k f_k(x,T)>0$, so that indeed $f\in D_1$.

That also $\mathrm{marg}_Y D\supseteq D_1$ follows once we realise that $D_1\subseteq D_2$, whence $D\supseteq\mathrm{posi}(\mathbb{I}_{\{F\}}D_1\cup\mathbb{I}_{\{U\}}D_1\cup\mathscr{L}(\mathscr{X}\times\mathscr{Y})_{>0})=\mathrm{posi}(D_1\cup\mathscr{L}(\mathscr{X}\times\mathscr{Y})_{>0})$, which is the *cylindrical extension*[12] of D_1, a coherent set of desirable gambles that marginalizes to D_1.

To show now that conditioning on $\{U\}$ changes the marginal $\mathrm{marg}_Y(D\rfloor\{U\})$ information to D_2, let us show first that $\mathrm{marg}_Y(D\rfloor\{U\})\subseteq D_2$. This follows once we realise that $D_1\subseteq D_2$ and therefore $D\subseteq\mathrm{posi}(\mathbb{I}_{\{F\}}D_2\cup\mathbb{I}_{\{U\}}D_2\cup\mathscr{L}(\mathscr{X}\times\mathscr{Y})_{>0})=\mathrm{posi}(D_2\cup\mathscr{L}(\mathscr{X}\times\mathscr{Y})_{>0})$, which is the cylindrical extension of D_2, a coherent set of desirable gambles that marginalizes to D_2. This implies that $\mathrm{marg}_Y D\subseteq\mathrm{marg}_Y\,\mathrm{posi}(D_2\cup\mathscr{L}(\mathscr{X}\times\mathscr{Y})_{>0})=D_2$.

To show, conversely, that $\mathrm{marg}_Y(D\rfloor\{U\})\supseteq D_2$, consider any f in D_2. This implies that $\mathbb{I}_{\{U\}}f\in\mathbb{I}_{\{U\}}D_2\subseteq D$. By the conditioning rule for sets of desirable gambles

$$D\rfloor E := \{f\in\mathscr{L}(E)\colon\mathbb{I}_E f\in D\},$$

then $f\in D\rfloor\{U\}$, and since f belongs to $\mathscr{L}(\mathscr{Y})$, indeed $f\in\mathrm{marg}_Y(D\rfloor\{U\})$. ■

Proof [Proof of Lemma 25] Since Y is a binary variable, it suffices to check that

$$\{\mathbb{I}_{\{U\}}f+\varepsilon,-\mathbb{I}_{\{F\}}f+\varepsilon\}\in K_D$$

12. See De Cooman and Miranda [11, Proposition 7].

for all f in $\mathscr{L}(\mathscr{X})$ and $\varepsilon \in \mathbb{R}_{>0}$, as discussed right after Definition 23. So consider any f in $\mathscr{L}(\mathscr{X})$ and $\varepsilon \in \mathbb{R}_{>0}$; we need to show that then $\mathbb{I}_{\{F\}}f+\varepsilon$ or $-\mathbb{I}_{\{U\}}f+\varepsilon$ belongs to D. We will proceed by considering two exhaustive cases: (i) $f \in D$ and (ii) $f \notin D$.

For (i) $f \in D$ implies that $\mathbb{I}_{\{U\}}f = \mathbb{I}_{\{U\}}f(U,\bullet) \in D\rfloor\{U\}$. But in the proof of Lemma 24 we have established that $\operatorname{marg}_Y D\rfloor\{U\} = D_2$, and therefore $f(U,\bullet) \in D_2$, whence $\mathbb{I}_{\{U\}}f = \mathbb{I}_{\{U\}}f(U,\bullet) \in \mathbb{I}_{\{U\}}D_2 \subseteq D$, and therefore indeed $\mathbb{I}_{\{U\}}f+\varepsilon \in D$.

For (ii) $f \notin D$ implies that $\mathbb{I}_{\{F\}}f = \mathbb{I}_{\{F\}}f(F,\bullet) \notin D\rfloor\{F\}$. By a completely similar argument as in the proof of Lemma 24, we can establish that $\operatorname{marg}_Y D\rfloor\{F\} = D_1$, so that $\mathbb{I}_{\{F\}}f(F,\bullet) \notin D_1$. But this means that $\mathbb{I}_{\{F\}}f(F,H)+\mathbb{I}_{\{F\}}f(F,T) \leq 0$, whence $-\mathbb{I}_{\{F\}}f(F,H)+\varepsilon-\mathbb{I}_{\{F\}}f(F,T)+\varepsilon > 0$ and therefore indeed $-\mathbb{I}_{\{F\}}f+\varepsilon = -\mathbb{I}_{\{F\}}f(F,\bullet)+\varepsilon \in D_1 \subseteq D$. ■