Irrelevant natural extension for choice functions

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1 Belief model: sets of desirable gamble(s/ sets)

The definitions and theorems in this section are taken from *[Jasper De Bock & Gert de Cooman. A Desirability-Based Axiomatisation for Coherent Choice Functions, SMPS 2018]* and *[Jasper De Bock & Gert de Cooman. Interpreting, Axiomatising and Representing Coherent Choice Functions in Terms of Desirability, ISIPTA 2019]*.

Gambles The uncertain variable X takes values in the finite possibility space $\mathscr{X}$. Any real-valued function on $\mathscr{X}$ is called a **gamble**, and we collect all of them in $\mathscr{L}(\mathscr{X})$, or $\mathscr{L}$. Given two gambles f and g in $\mathscr{L}$, we say that $f \leq g$ if $(\forall x \in \mathscr{X})\, f(x) \leq g(x)$. Its strict variant $<$ on $\mathscr{L}$ is given by: $f < g \Leftrightarrow (f \leq g$ and $f \neq g)$; we collect all gambles $f > 0$ in $\mathscr{L}_{>0}$.

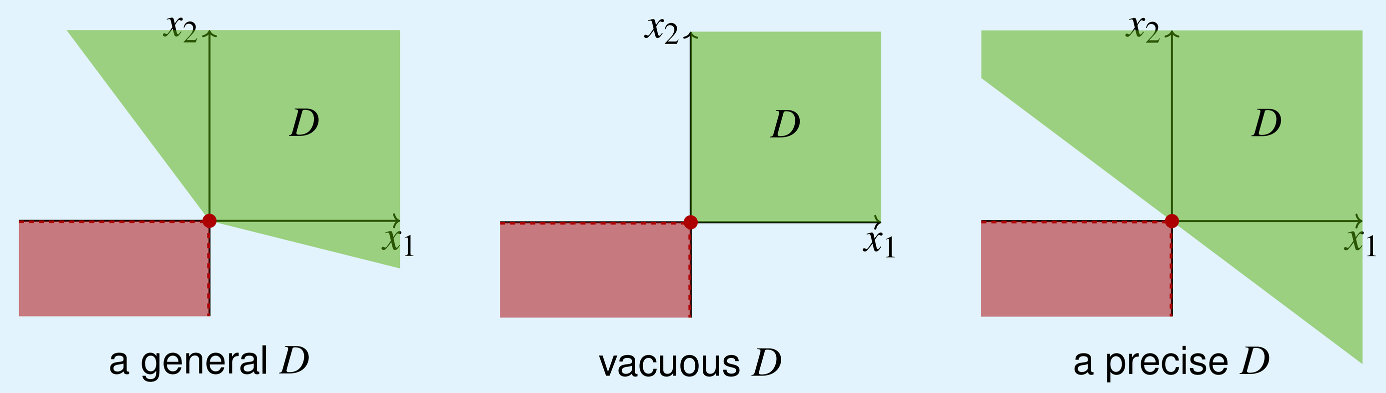
Desirability A **set of desirable gambles** $D \subseteq \mathscr{L}$ is a set of gambles that the subject prefers over 0.

$f \in D$ means: the subject prefers f over 0.

Rationality axioms We call a set of desirable gambles D **coherent** if for all gambles f and g and all real $\lambda > 0$:

D_1. $0 \notin D$; [**avoiding null gain**]
D_2. if $0 < f$ then $f \in D$; [**desiring partial gain**]
D_3. if $f \in D$ then $\lambda f \in D$; [**positive scaling**]
D_4. if $f, g \in D$ then $f + g \in D$. [**combination**]

A set of desirable gambles D is coherent if and only if it is a convex cone that includes $\mathscr{L}_{>0}$ and has nothing in common with the gambles $f \leq 0$.



a general D | vacuous D | a precise D

Sets of desirable gamble sets We define $\mathscr{Q}(\mathscr{X})$, or $\mathscr{Q}$, as the collection of **finite subsets** of $\mathscr{L}(\mathscr{X})$. A **set of desirable gamble sets** $K \subseteq \mathscr{Q}$ is a collection of sets A of gambles that contain at least one gamble $f \in A$ that is preferred over 0.

$A \in K$ means: A contains at least one gamble that the subject prefers over 0.

So a set of desirable gamble set can express more general types of uncertainty.

Rationality axioms A set of desirable gamble sets $K \subseteq \mathscr{Q}$ is called **coherent** if for all A, A_1 and A_2 in $\mathscr{Q}$, all $\{\lambda_{f,g}, \mu_{f,g} : f \in A_1, g \in A_2\} \subseteq \mathbb{R}$, and all f in $\mathscr{L}$:

K_0. $\emptyset \notin K$;
K_1. $A \in K \Rightarrow A \setminus \{0\} \in K$;
K_2. $\{f\} \in K$, for all f in $\mathscr{L}_{>0}$;
K_3. if $A_1, A_2 \in K$ and if, for all f in A_1 and g in A_2, $(\lambda_{f,g}, \mu_{f,g}) > 0$, then

$$\{\lambda_{f,g} f + \mu_{f,g} g : f \in A_1, g \in A_2\} \in K;$$

K_4. if $A_1 \in K$ and $A_1 \subseteq A_2$ then $A_2 \in K$, for all A_1 and A_2 in $\mathscr{Q}$.

Here $\lambda_{1:n} := (\lambda_1, \dots, \lambda_n) > 0$ means $\lambda_i \geq 0$ for all i, and $\lambda_j > 0$ for at least one j.

Natural extension An **assessment** $\mathscr{A} \subseteq \mathscr{Q}$ is a collection of gamble sets that the subject finds desirable, meaning that the subject's set of desirable gamble sets K must satisfy $\mathscr{A} \subseteq K$. It is called **consistent** when it can be extended to a coherent set of desirable gamble sets.

Theorem *[Jasper De Bock & Gert de Cooman, SMPS 2018, Theorem 10]* Consider any assessment $\mathscr{A} \subseteq \mathscr{Q}$. Then $\mathscr{A}$ is consistent when $\emptyset \notin \mathscr{A}$ and $\{0\} \notin \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$. If this is the case, the smallest coherent extension of $\mathscr{A}$—which is called its **natural extension**—is given by $\mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}))$.

Here we used the set $\mathscr{L}^{\mathrm{s}}(\mathscr{X})_{>0} := \{\{f\} : f \in \mathscr{L}(\mathscr{X})_{>0}\}$—often denoted simply by $\mathscr{L}^{\mathrm{s}}_{>0}$ when it is clear what the possibility space $\mathscr{X}$ is—and the following two operations on $\mathscr{P}(\mathscr{Q})$:

$$\mathrm{Rs}(K) := \{A \in \mathscr{Q} : (\exists B \in K) B \setminus \mathscr{L}_{\leq 0} \subseteq A\}$$

$$\mathrm{Posi}(K) := \left\{ \left\{ \sum_{k=1}^{n} \lambda_k^{f_{1:n}} f_k : f_{1:n} \in \bigtimes_{k=1}^{n} A_k \right\} : n \in \mathbb{N}, A_1, \dots, A_n \in K, \left(\forall f_{1:n} \in \bigtimes_{k=1}^{n} A_k \right) \lambda_{1:n}^{f_{1:n}} > 0 \right\}$$

for all K in $\mathscr{P}(\mathscr{Q})$.

Connection with choice functions A set of desirable gamble sets K is a convenient representation of a **choice function** C, which is a map $\mathscr{Q} \setminus (\emptyset) \to \mathscr{Q}$ such that $A \mapsto C(A) \subseteq A$. They are linked by

$$A - \{f\} \in K \Leftrightarrow f \notin C(A \cup \{f\}), \text{ for all } A \text{ in } \mathscr{Q} \text{ and } f \text{ in } \mathscr{L}.$$

So, every result about sets of desirable gamble sets translates to choice functions.

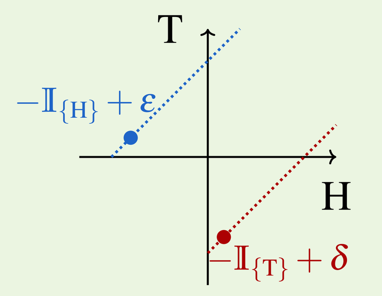
Connection with desirability Given a set of desirable gamble sets K, its corresponding set of desirable gambles D_K consists of the singleton sets in K: $D_K := \{f \in \mathscr{L} : \{f\} \in K\}$. If K is coherent, then so is D_K.

Conversely, given a coherent set of desirable gambles D, there are generally multiple corresponding coherent sets of desirable gamble sets K, the smallest of which is given by $K_D := \{A \in \mathscr{Q} : A \cap D \neq \emptyset\}$.

2 Example

Coin with two identical sides Consider a coin with two identical sides of unknown type: either both sides are heads (H) or tails (T).

Assessment Observe that:



If both sides are tails, the gamble $-\mathbb{I}_{\{H\}} + \varepsilon = (-1 + \varepsilon, \varepsilon)$ is preferred to 0, for every $\varepsilon > 0$.

If both sides are heads, the gamble $-\mathbb{I}_{\{T\}} + \delta = (\delta, -1 + \delta)$ is preferred to 0, for every $\delta > 0$.

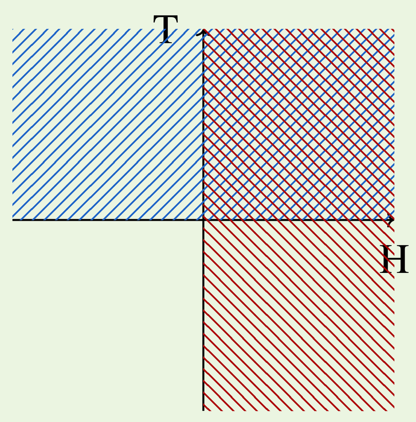
Therefore, the set $\{-\mathbb{I}_{\{T\}} + \varepsilon, -\mathbb{I}_{\{H\}} + \delta\}$ contains a gamble that is preferred to 0. So $\mathscr{A} := \{\{-\mathbb{I}_{\{T\}} + \varepsilon, -\mathbb{I}_{\{H\}} + \delta\} : \varepsilon, \delta > 0\}$ is the assessment.

Consistency Is the assessment $\mathscr{A}$ consistent? If so, then we can consider its natural extension. To this end, we calculate $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$. We find that

$$\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}) = \mathrm{Rs}(\{\{f,g\} : f, g \in \mathscr{L}_{\not\leq 0} \text{ and } (f(\mathrm{T}), g(\mathrm{H})) > 0\}). \tag{1}$$

Therefore, since $\emptyset \notin \mathscr{A}$ by its definition, and clearly $\{0\} \notin \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$, the assessment $\mathscr{A}$ is consistent.

Natural extension Since $\mathrm{Rs}(\mathrm{Rs}(A)) = \mathrm{Rs}(A)$ for any gamble set A, the natural extension $K := \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}))$ is given by Equation (1) above. This means that a gamble set A belongs to K if and only if A contains a gamble f in the blue hatched area and a gamble g in the red hatched area.



Set of desirable gambles These gambles f and g may be equal, and then $f = g$ belongs to $\mathscr{L}_{>0}$. Therefore the corresponding set of desirable gambles D_K is the vacuous set $\mathscr{L}_{>0}$: sets of desirable gambles are incapable of distinguishing between this belief, and a vacuous belief. Sets of desirable gamble sets can make this distinction.

3 Conditioning

The subject's beliefs about the uncertain variable X, taking values in $\mathscr{X}$, is described by a coherent set of desirable gamble sets K on $\mathscr{X}$.

Assume there is new information: X assumes a value in a non-empty subset E of $\mathscr{X}$.

How can this new information be taken into account?

Definition For any event (non-empty subset of $\mathscr{X}$) E, we define the **conditional set of desirable gamble sets** $K\rfloor E$ as

$K\rfloor E := \{A \in \mathscr{Q}(E) : \mathbb{I}_E A \in K\}$, where $\mathbb{I}_E A \in K := \{\mathbb{I}_E f : f \in A\}$, so that $\mathbb{I}_E A$ is a set of gambles on $\mathscr{X}$.

Note that $(\mathbb{I}_E f)(x)$ equals $f(x)$ if $x \in E$ and 0 if $x \notin E$.

Conditioning preserves coherence, and reduces to the usual definition for desirability.

4 Multivariate sets of desirable gamble sets

Setting We have two uncertain variables X and Y, taking values in the finite possibility spaces $\mathscr{X}$ and $\mathscr{Y}$ respectively. From here on, the set of all gambles on $\mathscr{X} \times \mathscr{Y}$ is denoted by $\mathscr{L}$. This is heavily inspired on *[Gert de Cooman & Enrique Miranda, Irrelevant and independent natural extension for sets of desirable gambles, JAIR 2012]*.

Cylindrical extension of gambles Let f be a gamble on $\mathscr{X}$. Its **cylindrical extension** f^* is given by

$$f^*(x, y) := f(x) \text{ for all } x \text{ in } \mathscr{X} \text{ and } y \text{ in } \mathscr{Y}.$$

f^* belongs to $\mathscr{L}$. Similarly, for any set A of gambles on $\mathscr{X}$, we let $A^* := \{f^* : f \in A\}$, and for any set of gamble sets K on $\mathscr{X}$, we let $K^* := \{A^* : A \in K\}$ be the corresponding set on $\mathscr{X} \times \mathscr{Y}$.

Marginalisation Given a set of desirable gamble sets K on $\mathscr{X} \times \mathscr{Y}$, its **marginal** $\mathrm{marg}_X K$ on $\mathscr{X}$ is

$$\mathrm{marg}_X K := \{A \in \mathscr{Q}(\mathscr{X}) : A \in K\} = K \cap \mathscr{Q}(\mathscr{X}).$$

Weak extension of sets of desirable gamble sets Let K be a coherent set of desirable gamble sets on $\mathscr{X}$.

What is the smallest coherent set of desirable gamble sets on $\mathscr{X} \times \mathscr{Y}$ that marginalises to K?

Proposition The least informative coherent set of desirable gamble sets on $\mathscr{X} \times \mathscr{Y}$ that marginalises to K is given by $\mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup K^*))$. It is called the **weak extension** of K.

Definition (Epistemic irrelevance) We say that X is **epistemically irrelevant** to Y when learning about the value of X does not influence our beliefs about Y. A set of desirable gamble sets K on $\mathscr{X} \times \mathscr{Y}$ satisfies epistemic irrelevance of X to Y if $\mathrm{marg}_Y(K\rfloor E) = \mathrm{marg}_Y K$ for all non-empty $E \subseteq \mathscr{X}$.

Irrelevant natural extension Let K be a coherent set of desirable gamble sets on $\mathscr{Y}$.

What is the smallest coherent set of desirable gamble sets on $\mathscr{X} \times \mathscr{Y}$ that marginalises to K and satisfies epistemic irrelevance of X to Y?

Theorem (Irrelevant natural extension) The smallest coherent set of desirable gamble sets on $\mathscr{X} \times \mathscr{Y}$ that marginalises to K and satisfies epistemic irrelevance of X to Y is given by

$$\mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}^{\mathrm{irr}}_{X \to Y})), \text{ where the assessment } \mathscr{A}^{\mathrm{irr}}_{X \to Y} \text{ is } \{\mathbb{I}_E A : A \in K \text{ and } E \subseteq \mathscr{X} \text{ and } E \neq \emptyset\}.$$