Irrelevant natural extension for choice functions

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Abstract

We consider coherent choice functions under the recent axiomatisation proposed by De Bock and de Cooman that guarantees a representation in terms of binary preferences, and we discuss how to define conditioning in this framework. In a multivariate context, we propose a notion of marginalisation, and its inverse operation called weak (cylindrical) extension. We combine this with our definition of conditioning to define a notion of irrelevance, and we obtain the irrelevant natural extension in this framework: the least informative choice function that satisfies a given irrelevance assessment.

Keywords: Choice functions, coherence, sets of desirable gambles, natural extension, conditioning, epistemic irrelevance.

1. Introduction

Consider two random variables X_1 and X_2, a belief model about X_2, and an assessment that X_1 is irrelevant to X_2—meaning that learning about the value of X_1 does not influence our beliefs about X_2. What is the least informative joint belief model about X_1 and X_2 that satisfies this irrelevance assessment and that marginalises to the given belief model about X_2? This belief model is called the "irrelevant natural extension". Having an expression for the irrelevant natural extension is important for inference purposes, as such extensions appear frequently in the context of credal networks [5, 6, 8].

In the framework of sets of desirable gambles, an expression for the natural extension was established by de Cooman and Miranda [11]. In this paper, we extend their result to choice functions.

Choice functions are related to the fundamental problem in decision theory: how to make a choice from within a set of available (uncertain) options. In their seminal book, von Neumann and Morgenstern [22] provided an axiomatisation of choice based on a pairwise comparisons between options, which has since received much attention, for instance by Rubin [16] who generalised this idea and proposed a theory of choice functions based on choices between more than two elements. One of the aspects of Rubin's theory [16] is that, between any pair of options, the subject either prefers one of them or is indifferent between them, so two options are never incomparable. However, for instance when the available information does not allow for a complete comparison of the options, the subject may be undecided between two options without being indifferent between them; this will for instance typically be the case when there is little relevant information available. This is one of the motivations for a theory of imprecise probabilities [23], where incomparability and indifference are distinguished. With this idea, Kadane et al. [13] and Seidenfeld et al. [18] generalised Rubin's axioms to allow for incomparability.

In this paper, we consider choice functions under the axiomatisation of De Bock and de Cooman [9], which generalises the one by Seidenfeld et al. [18]'s theory in that it does not have an Archimedean axiom. One of the main advantages of the axiomatisation in De Bock and de Cooman [9] above the earlier work by Van Camp in [19] is that it guarantees a representation in terms of pairwise choice.

In Section 2, we recall the axiomatisation of coherent choice functions in [9] and the connection with pairwise choice. Next, in Section 3, we introduce our conditioning rule for choice functions, and show how it relates with the existing conditioning rule for sets of desirable gambles. We use this definition to define a notion of irrelevance in Section 4, from which we derive a formula for the irrelevant natural extension. Some additional comments are gathered in Section 5. Due to the space limitations, proofs have been omitted.

2. Sets of desirable gamble sets & sets of desirable gambles

Consider a finite possibility space $\mathscr{X}$ in which a random variable X takes values. We denote by $\mathscr{L}(\mathscr{X})$ the set of all gambles—real-valued functions—on $\mathscr{X}$, often denoted by $\mathscr{L}$ when the it is clear from the context what the possibility space is. We attach the following interpretation to gambles. $f(X)$ is an uncertain reward: if the actual outcome turns out to be x in $\mathscr{X}$, then the subject's capital changes by $f(x)$. For any two gambles f and g, we write $f \leq g$ when $f(x) \leq g(x)$ for all x in $\mathscr{X}$, and we write $f < g$ when $f \leq g$ and $f \neq g$. We identify a real constant α with the (constant) gamble that maps every element of $\mathscr{X}$ to α. We collect all the non-negative gambles—the gambles f for which $f \geq 0$—in the set $\mathscr{L}(\mathscr{X})_{\geq 0}$ (often denoted by $\mathscr{L}_{\geq 0}$) and the positive ones—the gambles f for which $f > 0$—in $\mathscr{L}(\mathscr{X})_{>0}$ (often denoted by $\mathscr{L}_{>0}$). Similarly, we write $f \not\leq g$ when $f(x) > g(x)$ for some x in $\mathscr{X}$, and collect all the gambles f for which $f \not\leq 0$ in the set $\mathscr{L}(\mathscr{X})_{\not\leq 0}$ (often denoted by $\mathscr{L}_{\not\leq 0}$).

We denote by $\mathscr{Q}(\mathscr{L}(\mathscr{X}))$ the set of all finite subsets of $\mathscr{L}(\mathscr{X})$—also denoted by $\mathscr{Q}$ when the set of gambles $\mathscr{L}(\mathscr{X})$ is clear from the context. Elements of $\mathscr{Q}$ are the *gamble sets*. We define two special subsets of $\mathscr{Q}$: the collection $\mathscr{Q}_{\overline{\emptyset}} := \mathscr{Q} \setminus \{\emptyset\}$ of non-empty gamble sets, and the collection $\mathscr{Q}_0 := \{A \in \mathscr{Q} \colon 0 \in A\} \subseteq \mathscr{Q}_{\overline{\emptyset}}$ of gamble sets that include the status quo 0.

2.1. Sets of desirable gamble sets

A subject can state his preferences by specifying his rejected gambles from within every gamble set:

Definition 1 (Rejection function) *A rejection function R on $\mathscr{L}(\mathscr{X})$ is a map $R\colon \mathscr{Q}_{\overline{\emptyset}}(\mathscr{L}(\mathscr{X})) \to \mathscr{Q}(\mathscr{L}(\mathscr{X}))\colon A \mapsto R(A)$ with the property that $R(A) \subseteq A$.*

Equivalent to the notion of a rejection function R is that of a choice function C, which identifies the set $C(A) := A \setminus R(A)$ of non-rejected or chosen options from every gamble set A.

We focus our attention to the special subclass of *coherent* rejection functions, that describe the beliefs of a rational subject:

Definition 2 (Coherent rejection function) *We call a rejection function R* coherent *if for all A, A_1 and A_2 in $\mathscr{Q}_{\overline{\emptyset}}$, all $\{\lambda_{f,g}, \mu_{f,g} \colon f \in A_1, g \in A_2\} \subseteq \mathbb{R}$, and all f and g in $\mathscr{L}$:*

- R_0. *$R(A) \neq A$;*
- R_1. *if $f < g$ then $f \in R(\{f,g\})$;*
- R_2. *if $A_1 \subseteq R(A_2)$ and $A_2 \subseteq A$ then $A_1 \subseteq R(A)$;*
- R_3. *if $0 \in R(A_1)$ and $0 \in R(A_2)$ and if, for all f in A_1 and g in A_2, $(\lambda_{f,g}, \mu_{f,g}) > 0$, then*
$$0 \in R(\{\lambda_{f,g} f + \mu_{f,g} g \colon f \in A_1, g \in A_2\});$$
- R_4. *$f \in R(A)$ if and only if $f + g \in R(A + \{g\})$.*

In this definition, we let $A_1 + A_2 := \{f + g \colon f \in A_1, g \in A_2\}$ be the Minkowski addition of two gamble sets A_1 and A_2, and we define $(\lambda_1, \ldots, \lambda_n) > 0 \Leftrightarrow ((\forall i \in \{1, \ldots, n\})\lambda_i \geq 0$ and $(\exists i \in \{1, \ldots, n\})\lambda_i > 0)$.

These rationality requirements were introduced by De Bock and de Cooman [9] as a modification of the one considered in Van Camp's PhD dissertation [19] in order to guarantee a representation of coherent rejection functions in terms of sets of desirable gambles. In turn, Van Camp's representation is based on—after a necessary translation from horse lotteries to options that are represented by elements of a real linear space, such as gambles—Seidenfeld et al's [18]. The work of Seidenfeld et al is particularly important because they were the first to introduce *imprecise* choice functions—that distinguish between indifference and incomparability—in Reference [13] and proved a representation result in terms of probabilities in Reference [18].

The rationality requirements of Definition 2 are very similar to those of Seidenfeld et al [18]. There are, however, some differences: (i) [18] considers a strictly weaker version of Axiom R_1; (ii) they use an additional Archimedean axiom that ensures representation in terms of probabilities rather than non-Archimedean structures such as sets of desirable gambles; and (iii) they impose a mixing axiom that rules out maximality as a decision rule. Note that both Seidenfeld et al's [18] and our coherent choice functions obey Aizerman's condition, commonly written as

$$\text{if } A_1 \subseteq R(A_2) \text{ and } A \subseteq A_1 \text{ then } A_1 \setminus A \subseteq R(A_2 \setminus A),$$

for all A, A_1, A_2 in $\mathscr{Q}$. In our setting this is a consequence of Axioms R_2 and R_3.

De Bock and de Cooman [9] established a useful equivalent representation to rejection functions, namely that of a *set of desirable gamble sets*:

Definition 3 (Set of desirable gamble sets) *A set of desirable gamble sets K on $\mathscr{L}(\mathscr{X})$ is a subset of $\mathscr{Q}(\mathscr{L}(\mathscr{X}))$. We collect all the sets of desirable gamble sets in $\mathbf{K} := \mathscr{P}(\mathscr{Q})$.*

The idea is that the set of desirable gamble sets K collects all the gamble sets that contain at least one gamble that our subject strictly prefers over the status quo represented by 0, the gamble that will leave your capital unchanged whatever the outcome. A set of desirable gamble sets K is linked with a rejection function R as follows:

$$(\forall A \in \mathscr{Q})(\forall f \in \mathscr{L}) f \in R(A \cup \{f\}) \Leftrightarrow A - \{f\} \in K. \quad (1)$$

De Bock and de Cooman [9] gave an axiomatisation of *coherent* sets of desirable gamble sets—sets of desirable gamble sets of rational subjects. We refer to their article for a justification of their axioms:

Definition 4 (Coherent set of desirable gamble sets) *A set of desirable gamble sets $K \subseteq \mathscr{Q}$ is called* coherent *if for all A, A_1 and A_2 in $\mathscr{Q}$, all $\{\lambda_{f,g}, \mu_{f,g} \colon f \in A_1, g \in A_2\} \subseteq \mathbb{R}$, and all f in $\mathscr{L}$.*

- K_0. *$\emptyset \notin K$;*
- K_1. *$A \in K \Rightarrow A \setminus \{0\} \in K$;*
- K_2. *$\{f\} \in K$, for all f in $\mathscr{L}_{>0}$;*
- K_3. *if $A_1, A_2 \in K$ and if, for all f in A_1 and g in A_2, $(\lambda_{f,g}, \mu_{f,g}) > 0$, then*
$$\{\lambda_{f,g} f + \mu_{f,g} g \colon f \in A_1, g \in A_2\} \in K;$$
- K_4. *if $A_1 \in K$ and $A_1 \subseteq A_2$ then $A_2 \in K$, for all A_1 and A_2 in $\mathscr{Q}$.*

We collect all the coherent sets of desirable gamble sets in the collection $\overline{\mathbf{K}}(\mathscr{X})$, often simply denoted by $\overline{\mathbf{K}}$ when it is clear from the context what the possibility space $\mathscr{X}$ is.

Given any rejection function R and any set of desirable gamble sets K that are linked through Equation (1), we have that R is coherent if and only if K is.

Given two sets of desirable gamble sets K_1 and K_2, we follow De Bock and de Cooman [9] in calling K_1 *at most as informative as* K_2 if $K_1 \subseteq K_2$. The resulting partially ordered set $(\mathbf{K}, \subseteq)$ is a complete lattice where intersection serves the role of infimum, and union that of supremum. Furthermore De Bock and de Cooman [9, Theorem 8] show that the partially ordered set $(\overline{\mathbf{K}}, \subseteq)$ of coherent sets of desirable gamble sets is a complete meet-semilattice: given an arbitrary family $\{K_i \colon i \in I\} \subseteq \overline{\mathbf{K}}$, its infimum $\inf\{K_i \colon i \in I\} = \bigcap_{i\in I} K_i$ is a coherent set of desirable gamble sets. This allows for conservative reasoning: it makes it possible to extend a partially specified set of desirable gamble sets to the most conservative—least informative—coherent one that includes it. This procedure is called *natural extension*:

Definition 5 ([9, Definition 9]) *For any assessment $\mathscr{A} \subseteq \mathscr{Q}$, we let $\overline{\mathbf{K}}(\mathscr{A}) \coloneqq \{K \in \overline{\mathbf{K}} \colon \mathscr{A} \subseteq K\}$. We call the assessment $\mathscr{A}$* consistent *if $\overline{\mathbf{K}}(\mathscr{A}) \neq \emptyset$, and we then call $\mathrm{cl}_{\overline{\mathbf{K}}}(\mathscr{A}) \coloneqq \bigcap \overline{\mathbf{K}}(\mathscr{A})$ the* natural extension *of $\mathscr{A}$.*

One of the main results of De Bock and de Cooman [9] is their expression for the natural extension:

Theorem 6 ([9, Theorem 10]) *Consider any assessment $\mathscr{A} \subseteq \mathscr{Q}$. Then $\mathscr{A}$ is consistent if and only if $\emptyset \notin \mathscr{A}$ and $\{0\} \notin \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$. Moreover, if $\mathscr{A}$ is consistent, then $\mathrm{cl}_{\overline{\mathbf{K}}}(\mathscr{A}) = \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}))$.*

Here we used the set $\mathscr{L}^{\mathrm{s}}(\mathscr{X})_{>0} \coloneqq \{\{f\} \colon f \in \mathscr{L}(\mathscr{X})_{>0}\}$—often denoted simply by $\mathscr{L}^{\mathrm{s}}_{>0}$ when it is clear from the context what the possibility space $\mathscr{X}$ is—and the following two operations on $\mathbf{K}$ defined by

$$\mathrm{Rs}(K) \coloneqq \{A \in \mathscr{Q} \colon (\exists B \in K) B \setminus \mathscr{L}_{\leq 0} \subseteq A\}$$

$$\mathrm{Posi}(K) \coloneqq \Bigg\{ \Bigg\{ \sum_{k=1}^{n} \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \bigtimes_{k=1}^{n} A_k \Bigg\} \colon n \in \mathbb{N}, \\ A_1, \dots, A_n \in K, \Bigg(\forall f_{1:n} \in \bigtimes_{k=1}^{n} A_k\Bigg) \lambda_{1:n}^{f_{1:n}} > 0 \Bigg\}$$

for all K in $\mathbf{K}$. As usual, we use the short-hand notation $f_{1:n} \coloneqq (f_1, \dots, f_n)$ for any sequence $(f_1, \dots, f_n)$.

For arbitrary sets of desirable gamble sets K, we have $K \subseteq \mathrm{Rs}(K)$,[1] and $K \subseteq \mathrm{Posi}(K)$,[2] and therefore $K \subseteq \mathrm{Rs}(\mathrm{Posi}(K))$. For coherent sets of desirable gamble sets K however, Theorem 6 implies that $K = \mathrm{Rs}(K) = \mathrm{Posi}(K) = \mathrm{Rs}(\mathrm{Posi}(K))$.

1. To see this, note that $B \setminus \mathscr{L}_{\leq 0} \subseteq B$ for every B in K.
2. To see this, it suffices to choose $n \coloneqq 1$, $A_1 \coloneqq A \in K$, and $\lambda_{1:1}^{f_{1:1}} \coloneqq 1$ for all $f_{1:1}$ in $\bigtimes_{k=1}^{1} A_1 = A$ in the definition of the Posi operator.

In our earlier work [21, Theorem 1], we have found expressions for the characterisation of consistency and the natural extension of rejection functions. The previous result was obtained in a slightly more general setting: instead of requiring Axiom R_3, we required two strictly weaker axioms. For any given assessment $\mathscr{A}$, the resulting natural extension is therefore a less informative—more conservative—rejection function that the one determined by $\mathrm{cl}_{\overline{\mathbf{K}}}(\mathscr{A})$. However, this setting was too general to obtain a representation in terms of binary preferences, as our counterexample in [21, Section 5.1] shows. As proved by De Bock and de Cooman [9, Theorem 7], the current axiomatisation does guarantee representation in terms of sets of desirable gambles.

In order to illustrate Theorem 6, consider the following example, which we will also use in Section 2.3 as an example of a non-binary set of desirable gamble sets.

Example 1 *Consider the situation where you have a coin with two identical sides of unknown type: either both sides are heads* (H) *or tails* (T)*. The random variable X that represents the outcome of a coin flip assumes a value in the finite possibility space $\mathscr{X} = \{\mathrm{H}, \mathrm{T}\}$. This assessment is important for inference purposes: for instance, in a sequence of outcomes of successive flips from this coin, observing one of the outcomes immediately fixes all the other outcomes. As we will see in the follow-up of this example in Section 2.3, this situation cannot be modelled using sets of desirable gambles in a satisfactory way: we need to use a set of desirable gamble sets instead.*

How do we translate this situation into an assessment $\mathscr{A}$? Since either both sides are heads or tails, we take this to mean that at least one of the gambles $-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon$ or $-\mathbb{I}_{\{\mathrm{T}\}} + \delta$, are preferred to 0*, for any real $\varepsilon > 0$ and $\delta > 0$. Therefore, we let the assessment that reflects this situation be given by*[3]

$$\mathscr{A} \coloneqq \{\{-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon, -\mathbb{I}_{\{\mathrm{T}\}} + \delta\} \colon \varepsilon, \delta \in \mathbb{R}_{>0}\}. \tag{2}$$

To show that this assessment is consistent, we first find an expression for $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$:

Lemma 7 *For the assessment $\mathscr{A}$ of Equation (2), we have*

$$\begin{aligned} &\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}) \\ &= \{A \in \mathscr{Q} \colon ((\exists h_1, h_2 \in A)(h_1(\mathrm{T}) > 0 \text{ and } h_2(\mathrm{H}) > 0)) \\ &\qquad\qquad \text{or } A \cap \mathscr{L}^{\mathrm{s}}_{>0} \neq \emptyset\} \\ &= \mathrm{Rs}(\{\{h_1, h_2\} \colon h_1, h_2 \in \mathscr{L}_{\not\leq 0} \text{ and } (h_1(\mathrm{T}), h_2(\mathrm{H})) > 0\}). \end{aligned}$$

To show that our assessment is consistent, by Theorem 6 we need to show that $\emptyset \notin \mathscr{A}$—which is clearly true—and $\{0\} \notin \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$. So we focus on showing that $\{0\} \notin \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$. Using Lemma 7 we know that $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$ consists of the supersets of gamble sets

3. We denote by $\mathbb{R}_{>0}$ the set of (strictly) positive real numbers.

$\{h_1, h_2\}$ where none of h_1 and h_2 are equal to 0, so we find immediately that indeed $\{0\} \notin \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$. Therefore $\mathscr{A}$ is a consistent assessment, and by Theorem 6 its natural extension is given by the coherent set of desirable gamble sets $\mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}))$. What is this $\mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}))$? Thanks to Lemma 7, this can be easily found: since $\mathrm{Rs}(\mathrm{Rs}(K)) = \mathrm{Rs}(K)$ for any gamble set K, we immediately find that the natural extension of $\mathscr{A}$ is $\mathrm{Rs}(\{\{h_1, h_2\} \colon h_1, h_2 \in \mathscr{L}_{\not\leq 0} \text{ and } (h_1(\mathrm{T}), h_2(\mathrm{H})) > 0\})$; this is thus the smallest coherent set of desirable gamble sets that corresponds to our belief that the coin has two identical sides of unknown type. ♦

2.2. Sets of desirable gambles

Since we have taken to mean "$A \in K$" that A contains at least one gamble that is desirable, the singleton elements of K play an important role: if $\{f\} \in K$, then the gamble f is desirable. Given a set of desirable gamble sets, we collect in

$$D_K := \{f \in \mathscr{L} \colon \{f\} \in K\} \tag{3}$$

the gambles that are considered desirable, and call it the *set of desirable gambles based on K*.

Sets of desirable gambles can therefore be seen as special sets of desirable gamble sets. In the recent years, there has been much interest in sets of desirable gambles on its own, without reference to sets of desirable gamble sets or choice functions (see for instance [1, Chapter 1] or [3, 14, 17]). A set of desirable gambles D is simply a subset of $\mathscr{L}$; we collect in $\mathbf{D} := \mathscr{P}(\mathscr{L})$ all the sets of desirable gambles. We focus on the special subclass of *coherent* sets of desirable gambles:

Definition 8 (Coherent set of desirable gambles) *A set of desirable gambles D is called* coherent *if for all f and g in $\mathscr{L}$, and λ and μ in $\mathbb{R}$:*

D$_1$. $0 \notin D$;
D$_2$. $\mathscr{L}_{>0} \subseteq D$;
D$_3$. *if $f, g \in D$ and $(\lambda, \mu) > 0$, then $\lambda f + \mu g \in D$.*

We collect all the coherent sets of desirable gambles in $\overline{\mathbf{D}}(\mathscr{X})$, often simply denoted by $\overline{\mathbf{D}}$ when it is clear from the context what the possibility space $\mathscr{X}$ is.

Just as we did for sets of desirable gamble sets, we call the set of desirable gambles D_1 *at most as informative as* set of desirable gambles D_2 if $D_1 \subseteq D_2$. The partially ordered set $(\overline{\mathbf{D}}, \subseteq)$ is a complete meet-semilattice. The natural extension is defined in a similar way as for sets of desirable gamble sets: an assessment $A \subseteq \mathscr{L}$ is called *consistent* if $\overline{\mathbf{D}}(A) := \{D \in \overline{\mathbf{D}} \colon A \subseteq D\}$ is non-empty. If this is the case, $\mathrm{cl}_{\overline{\mathbf{D}}}(A) := \bigcap \overline{\mathbf{D}}(A)$ is called the *natural extension* of A. The expression for the natural extension is remarkably similar to the one in Theorem 6:

Theorem 9 ([12, Theorem 1]) *Consider any assessment $A \subseteq \mathscr{L}$. Then A is consistent if and only if $0 \notin \mathrm{posi}(\mathscr{L}_{>0} \cup A)$. Moreover, in that case $\mathrm{cl}_{\overline{\mathbf{D}}}(A) = \mathrm{posi}(\mathscr{L}_{>0} \cup A)$.*

In this theorem, we used the operation posi on $\mathbf{D}$:

$$\mathrm{posi}(A) := \left\{\sum_{k=1}^{n} \lambda_k f_k \colon n \in \mathbb{N}, f_1, \ldots, f_n \in A, \lambda_{1:n} > 0\right\},$$

for all $A \subseteq \mathscr{L}$.

2.3. Connection between sets of desirable gamble sets and sets of desirable gambles

Given a set of desirable gamble sets K, its corresponding set of desirable gambles D_K is uniquely given by Equation (3), and it is coherent if K is [9, Proposition 6]. On the other hand, a coherent set of desirable gambles D may have multiple sets of desirable gamble sets corresponding to it by Equation (3), in the sense that the collection

$$\overline{\mathbf{K}}_D := \{K \in \overline{\mathbf{K}} \colon D_K = D\}$$

may have more than one element. However, there is always a unique least informative one:

Proposition 10 *Given a coherent set of desirable gambles D, the infimum $\inf \overline{\mathbf{K}}_D$ of its compatible coherent sets of desirable gamble sets is the coherent set of desirable gamble sets $K_D := \{A \in \mathscr{Q} \colon A \cap D \neq \emptyset\}$.*

The coherent sets of desirable gamble sets of the form K_D with $D \in \overline{\mathbf{D}}$, are particularly important. Since they are completely determined by pairwise comparison (of gambles in D with 0), they are called *binary*. De Bock and de Cooman [9] established a representation result of coherent sets of desirable gamble sets, in terms of binary ones:

Theorem 11 ([9, Theorem 7]) *Every coherent set of desirable gamble sets K is dominated by at least one binary set of desirable gamble sets: $\overline{\mathbf{D}}(K) := \{D \in \overline{\mathbf{D}} \colon K \subseteq K_D\}$ is non-empty. Moreover, $K = \bigcap\{K_D : D \in \overline{\mathbf{D}}(K)\}$.*

This theorem generalises the important representation result of Seidenfeld et al. [18, Theorem 4] to a non-Archimedean setting, where the atoms that fulfil the representation are now coherent sets of desirable gambles, rather than (Archimedean) probability mass functions. In order to obtain their result, Seidenfeld et al. needed two additional axioms: an Archimedean one, guaranteeing an appropriate level of continuity, and a mixing axiom, which renders Walley–Sen maximality incompatible with coherent choice functions. De Bock and de Cooman [9] let go of these two axioms, and were able to prove the general representation Theorem 11. Additionally, they also considered the effect of adding Seidenfeld et al. [13, 18]'s mixing axiom, while still abstaining from Archimedeanity. With this additional axiom, they have established a more specialised representation in terms of *lexicographic* sets of desirable gambles, which are exactly [2, 20] the sets of desirable gambles that correspond to lexicographic probability systems (with no non-trivial Savage-null events).

Example 2 *We continue our previous Example 1, where we derived the coherent set of desirable gamble sets* $K := \mathrm{Rs}(\{\{h_1,h_2\}\colon h_1,h_2 \in \mathscr{L}_{\not\leq 0} \text{ and } (h_1(\mathrm{T}),h_2(\mathrm{H})) > 0\})$ *that corresponds to our belief that the coin has two identical sides of unknown type. In this example we wonder whether this can be retrieved using binary comparisons: is K a binary set of desirable gamble sets? If K was a binary set of desirable gamble sets, then* $K = K_D$ *for the set of desirable gambles* $D := D_K$*, as is shown in [9, Proposition 5].*

So let us first find D_K*. Using Equation (3), we find that*

$$\begin{aligned} D_K &= \{f \in \mathscr{L}\colon \{f\} \in K\} \\ &= \{f \in \mathscr{L}\colon (\exists h_1,h_2 \in \{f\}) \\ &\qquad\qquad h_1,h_2 \in \mathscr{L}_{\not\leq 0}, (h_1(\mathrm{H}),h_2(\mathrm{T})) > 0\} \\ &= \{f \in \mathscr{L}\colon f \in \mathscr{L}_{\not\leq 0}, f > 0\} = \mathscr{L}_{>0}, \end{aligned}$$

so $D_K = \mathscr{L}_{>0}$ *is the least informative coherent set of desirable gambles, also called the* vacuous *set of desirable gambles: using pairwise comparisons only, we cannot distinguish the current situation with a vacuous belief. This shows why sets of desirable gambles cannot model this in a satisfactory way. Since* $K_{D_K} = \{A \in \mathscr{Q}\colon A \cap \mathscr{L}_{>0} \neq \emptyset\}$ *does not contain the gamble set* $\{-\mathbb{I}_{\{\mathrm{H}\}} + \frac{1}{4}, -\mathbb{I}_{\{\mathrm{T}\}} + \frac{1}{4}\}$ *while K does, this also shows that K is a non-binary set of desirable gamble sets.*

How can we represent this K? In other words, what is the representing set $\overline{\mathbf{D}}(K)$ *of desirable gambles from Theorem 11? To find this set, consider first the two special coherent sets of desirable gambles*

$$\begin{aligned} D_{\mathrm{H}} &:= \{f \in \mathscr{L}\colon f(\mathrm{H}) > 0\} \cup \mathscr{L}_{>0} \\ D_{\mathrm{T}} &:= \{f \in \mathscr{L}\colon f(\mathrm{T}) > 0\} \cup \mathscr{L}_{>0} \end{aligned}$$

which correspond to (practical) certainty about H *and* T*, respectively. Indeed, if the subject is certain about* H*, then any gamble that yields a positive gain when* H *occurs, however small, will be desirable. Actually, in very much the same way as in Example 1,* D_{H} *and* D_{T} *can be retrieved as the natural extensions of the consistent assessment* $A_{\mathrm{H}} := \{-\mathbb{I}_{\{\mathrm{T}\}} + \varepsilon\colon \varepsilon \in \mathbb{R}_{>0}\}$ *and* $A_{\mathrm{T}} := \{-\mathbb{I}_{\{\mathrm{H}\}} + \delta\colon \delta \in \mathbb{R}_{>0}\}$*, respectively.*

To find $\overline{\mathbf{D}}(K)$*, we need to find all the coherent sets of desirable gambles D such that* $K \subseteq K_D$*. So consider any A in K. This implies that there is a subset* $\{h_1,h_2\} \subseteq A$ *such that* $h_1,h_2 \in \mathscr{L}_{\not\leq 0}$ *and* $(h_1(\mathrm{T}),h_2(\mathrm{H})) > 0$*. Then* $A \cap D_{\mathrm{H}} \neq \emptyset$ *and* $A \cap D_{\mathrm{T}} \neq \emptyset$*, so* $A \in K_{D_{\mathrm{H}}}$ *and* $A \in K_{D_{\mathrm{T}}}$*. Therefore* $D_{\mathrm{H}}, D_{\mathrm{T}} \in \overline{\mathbf{D}}(K)$*. But* D_{H} *and* D_{T} *are the only elements of* $\overline{\mathbf{D}}(K)$*.*[4] *So we find by Theorem 11 that* $K = K_{D_{\mathrm{H}}} \cap K_{D_{\mathrm{T}}}$*.*

This is therefore an example of a conceptually easy type of belief that cannot be modelled by sets of desirable gambles—and therefore also not by credal sets or lower previsions—in a satisfactory way, but can if non-binary sets of desirable gamble sets are used instead. ⧫

3. Conditioning

Consider a variable X that assumes values in a non-empty possibility space $\mathscr{X}$. Suppose that we have a belief model about X, be it a coherent set of desirable gamble sets on $\mathscr{L}$ or a coherent set of desirable gambles on $\mathscr{X}$, or—less general—just a single probability mass function on X, or a set of them. When new information becomes available, in the form of 'X assumes a value in some (non-empty) subset E of X', we can take this into account by conditioning our belief model on E.

For some of these belief models, such as (sets of) probability mass functions, conditioning on events of probability zero can be problematic, because, roughly speaking, Bayes's Rule typically requires to divide by zero in these situations. However, working with sets of desirable gambles is one way of overcoming this problem. In this section, we will see why, and explain that sets of desirable gamble sets do not suffer from this problem either.

We will let any event, except for the (trivially) impossible event $\emptyset$, serve as a conditioning event. We collect the allowed conditioning events in

$$\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}) := \{E \subseteq \mathscr{X}\colon E \neq \emptyset\}.$$

We will first review how conditioning is done using sets of desirable gambles (see [12] for more details). After that, we will introduce conditional sets of desirable gamble sets, and study the connection between both cases. Given the discussion in Section 2.3, this immediately translates to rejection functions and choice functions as well.

There are multiple equivalent definitions for conditional sets of desirable gambles. Most of them, for instance those in [4, 15, 23, 24] result in a conditional set of desirable gambles on $\mathscr{X}$. However, we find it more useful and convenient that a conditional model is defined on gambles on E, rather than on $\mathscr{X}$, because, after getting to know that E occurs, the possibility space becomes effectively E.

Definition 12 ([12, Equation (17)]) *Consider any set of desirable gambles* $D \subseteq \mathscr{L}(\mathscr{X})$ *and any conditioning event* E *in* $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$*, we define the* conditional set of desirable gambles $D\rfloor E \subseteq \mathscr{L}(E)$ *as*

$$D\rfloor E := \{f \in \mathscr{L}(E)\colon \mathbb{I}_E f \in D\}.$$

4. To see this, assume *ex absurdo* that another coherent set of desirable gambles D belongs to $\overline{\mathbf{D}}(K)$, so $K \subseteq K_D$. This would imply that $-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon, -\mathbb{I}_{\{\mathrm{T}\}} + \delta \notin D$ for some ε and δ in $\mathbb{R}_{>0}$. But the gamble set $\{-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon, -\mathbb{I}_{\{\mathrm{T}\}} + \delta\}$ belongs to K, a contradiction.

In this definition, we let for any E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$ and any gamble f on E its multiplication $\mathbb{I}_E f$ denote the gamble on $\mathscr{X}$ defined by

$$(\mathbb{I}_E f)(x) := \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \tag{4}$$

for all x in $\mathscr{X}$. Note that, for any gambles f and g on E, we have $f \neq g \Leftrightarrow \mathbb{I}_E f \neq \mathbb{I}_E g$, and, as a consequence, $f < g \Leftrightarrow \mathbb{I}_E f < \mathbb{I}_E g$.

It was proved by de Cooman and Quaghebeur [12, Proposition 8] that conditioning preserves coherence: if D is a coherent set of desirable gambles, then so is $D\rfloor E$, for any E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$. This explains why sets of desirable gambles do not suffer from conditioning on events of probability zero: $D\rfloor E$ is well-defined and coherent for every conditioning event E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$, even if E has probability zero according to some, or all, of the probabilities induced by D.

For sets of desirable gamble sets, conditioning can be defined using the same simple underlying ideas:

Definition 13 (Conditioning) *Given any set of desirable gamble sets K and any conditioning event E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$, we define the* conditional set of desirable gamble sets $K\rfloor E$ *on $\mathscr{L}(E)$ as*

$$K\rfloor E := \{A \in \mathscr{Q}(\mathscr{L}(E)) \colon \mathbb{I}_E A \in K\},$$

where for any A in $\mathscr{Q}(\mathscr{L}(E))$ and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$, we let $\mathbb{I}_E A := \{\mathbb{I}_E g \colon g \in A\}$ be a set of gambles on $\mathscr{X}$.

Proposition 14 *Consider any set of desirable gamble sets K on $\mathscr{L}(\mathscr{X})$ and any conditioning event E in $\mathscr{P}_{\overline{\emptyset}}$. If K is coherent, then so is $K\rfloor E$.*

Is Definition 13 a suitable definition of conditioning? One of the useful properties our definition has, is that it preserves coherence, as shown in Proposition 14, and therefore sets of desirable gamble sets also do not suffer from conditioning on events of probability zero. But does it also generalise the Definition 12 of conditional sets of desirable gambles, or in other words, does Definition 13 reduce to the Definition 12 of conditioning sets of desirable gambles when only considering binary choice? Of course, to investigate this, we must keep in mind the connection between sets of desirable gamble sets and sets of desirable gambles, explained in Section 2.3.

For our two conditioning rules—the one in Definition 12 for sets of desirable gambles and the one in Definition 13 for sets of desirable gamble sets—to be a match, we must prove that: (i) the conditioning rule for sets of desirable gamble sets reverts to the known conditioning rule for the corresponding sets of desirable gambles, and (ii) in the special case of purely binary choice, the conditioning for sets of desirable gamble sets coincides with the conditioning rule for desirability. Mathematically, (i) means that $D_K\rfloor E = D_{K\rfloor E}$ for any coherent set of desirable gamble sets K and conditioning event E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$, and (ii) means that $K_D\rfloor E = K_{D\rfloor E}$, for any coherent set of desirable gambles D, and any conditioning event E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$. The next proposition guarantees that both these conditions are satisfied:

Proposition 15 *Consider any coherent set of desirable gamble sets K, any coherent set of desirable gambles D, and any conditioning event E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X})$. Then $D_K\rfloor E = D_{K\rfloor E}$ and $K_D\rfloor E = K_{D\rfloor E}$. Furthermore, $K\rfloor E = \bigcap\{K_{D\rfloor E} \colon D \in \overline{\mathbf{D}}(K)\}$.*

The last statement of Proposition 15 guarantees that the conditional set of desirable gamble sets $K\rfloor E$ can be retrieved by conditioning every element of K's representing set $\overline{\mathbf{D}}(K)$ from Theorem 11.

4. Multivariate sets of desirable gamble sets

In this section, we will generalise the concepts of marginalisation, weak (cylindrical) extension and irrelevant natural extension introduced by de Cooman and Miranda for sets of desirable gambles [11] to choice models. We will provide the linear space of gambles, on which we define our sets of desirable gamble sets, with a more complex structure: we will consider the vector space of all gambles whose domain is a Cartesian product of a finite number of finite possibility spaces. More specifically, consider n in $\mathbb{N}$ variables $X_1, \dots, X_n$ that assume values in the finite possibility spaces $\mathscr{X}_1, \dots, \mathscr{X}_n$, respectively. Belief models about these variables $X_1, \dots, X_n$ will be defined on gambles on $\mathscr{X}_1, \dots, \mathscr{X}_n$. We may also consider gambles on the Cartesian product $\times_{k=1}^n \mathscr{X}_k$, giving rise to the $\prod_{k=1}^n |\mathscr{X}_k|$-dimensional linear space $\mathscr{L}(\times_{k=1}^n \mathscr{X}_k)$.

4.1. Basic notation & cylindrical extension

For every non-empty subset $I \subseteq \{1, \dots, n\}$ of indices, we let X_I be the tuple of variables that takes values in $\mathscr{X}_I := \times_{r \in I} \mathscr{X}_r$. We will denote generic elements of $\mathscr{X}_I$ as x_I or z_I, whose components are $x_i := x_I(i)$ and $z_i := z_I(i)$, for all i in I. As we did before, when $I = \{k, \dots, \ell\}$ for some k, ℓ in $\{1, \dots, n\}$ with $k \leq \ell$, we will use as a short-hand notation $X_{k:\ell} := X_{\{k,\dots,\ell\}}$, taking values in $\mathscr{X}_{k:\ell} := \mathscr{X}_{\{k,\dots,\ell\}}$ and whose generic elements are denoted by $x_{k:\ell} := x_{\{k,\dots,\ell\}} = (x_k, \dots, x_\ell)$.

We assume that the variables $X_1, \dots, X_n$ are *logically independent*, meaning that for each non-empty subset I of $\{1, \dots, n\}$, x_I may assume every value in $\mathscr{X}_I$.

It will be useful for any gamble f on $\mathscr{X}_{1:n}$, any non-empty proper subset I of $\{1, \dots, n\}$ and any x_I in $\mathscr{X}_I$, to interpret the partial map $f(x_I, \bullet)$ as a gamble on $\mathscr{X}_{I^c}$, where $I^c := \{1, \dots, n\} \setminus I$. Likewise, for any set A of gambles on $\mathscr{X}_{1:n}$, we let $A(x_I, \bullet) := \{f(x_I, \bullet) \colon f \in A\}$ be a corresponding set of gambles on $\mathscr{X}_I$.

We will need a way to relate gambles on different domains:

Definition 16 (Cylindrical extension) *Given two disjoint and non-empty subsets I and I' of $\{1,\dots,n\}$ and any gamble f on $\mathscr{X}_I$, we let its* cylindrical extension *f^* to $\mathscr{X}_{I\cup I'}$ be defined by*

$$f^*(x_I,x_{I'}) := f(x_I) \text{ for all } x_I \text{ in } \mathscr{X}_I \text{ and } x_{I'} \text{ in } \mathscr{X}_{I'}.$$

Similarly, given any set of gambles $A \subseteq \mathscr{L}(\mathscr{X}_I)$, we let its cylindrical extension *$A^* \subseteq \mathscr{L}(\mathscr{X}_{I\cup I'})$ be defined as $A^* := \{f^*\colon f \in A\}$.*

Formally, f^* belongs to $\mathscr{L}(\mathscr{X}_{I\cup I'})$ while f belongs to $\mathscr{L}(\mathscr{X}_I)$. However, f^* is completely determined by f and vice versa: they clearly only depend on the value of X_I, and as such, they contain the same information and correspond to the same transaction. They are therefore indistinguishable from a behavioural point of view.

Remark 17 *As in [10, 11], we will frequently use the simplifying device of* identifying *a gamble f on $\mathscr{L}(\mathscr{X}_I)$ with its cylindrical extension f^* on $\mathscr{L}(\mathscr{X}_{I\cup I'})$, for any disjoint and non-empty subsets I and I' of the index set $\{1,\dots,n\}$. This convention allows us for instance to identify $\mathscr{L}(\mathscr{X}_I)$ with a subset of $\mathscr{L}(\mathscr{X}_{1:n})$, and, as another example, for any set $A \subseteq \mathscr{L}(\mathscr{X}_{1:n})$, to regard $A \cap \mathscr{L}(\mathscr{X}_I)$ as those gambles in A that depend on the value of $\mathscr{X}_I$ only. Therefore, for any event E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$ we can identify the gamble $\mathbb{I}_E$ with $\mathbb{I}_{E\times\mathscr{X}_{I^c}}$, and hence also the event E with $E \times \mathscr{X}_{I^c}$. This device for instance also allows us to write, for any f on $\mathscr{X}_I$ and g on $\mathscr{X}_{I\cup I'}$, that $f \leq g \Leftrightarrow (\forall x_I \in \mathscr{X}_I, x_{I'} \in \mathscr{X}_{I'}) f(x_I) \leq g(x_I,x_{I'})$.* ♦

4.2. Marginalisation and weak extension

Suppose we have a set of desirable gamble sets K on $\mathscr{L}(\mathscr{X}_{1:n})$ modelling a subject's beliefs about the variable $X_{1:n}$. What is the information that K contains about X_O, where O is some non-empty subset of the index set $\{1,\dots,n\}$? Finding this information can be done through marginalisation.

Definition 18 (Marginalisation) *Given any non-empty subset O of $\{1,\dots,n\}$ and any set of desirable gamble sets K on $\mathscr{L}(\mathscr{X}_{1:n})$, its* marginal set of desirable gamble sets *$\mathrm{marg}_O K$ on $\mathscr{L}(\mathscr{X}_O)$ is defined as*

$$\begin{aligned}\mathrm{marg}_O K :=& \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O))\colon A \in K\}\\ =& K \cap \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)).\end{aligned}$$

We use the simplifying device of Remark 17 of identifying A with a subset of $\mathscr{L}(\mathscr{X}_{1:n})$. Without resorting to this convention, we can characterise $\mathrm{marg}_O K$ as:

$$(\forall A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)))\, A \in \mathrm{marg}_O K \Leftrightarrow A^* \in K.$$

It follows at once from Definition 18 that marginalisation preserves the order: if $K_1 \subseteq K_2$, then $\mathrm{marg}_O K_1 \subseteq \mathrm{marg}_O K_2$, for all sets of desirable gamble sets K_1 and K_2 on $\mathscr{L}(\mathscr{X}_{1:n})$. Marginalisation also preserves coherence:

Proposition 19 *Consider any set of desirable gamble sets K on $\mathscr{L}(\mathscr{X}_{1:n})$ and any non-empty subset O of $\{1,\dots,n\}$. If K is coherent, then so is $\mathrm{marg}_O K$.*

Let us compare with sets of desirable gambles. De Cooman and Miranda [11] defined, for any non-empty subset O of $\{1,\dots,n\}$ and any set of desirable gambles D, its *marginal set of desirable gambles* $\mathrm{marg}_O D$ on $\mathscr{L}(\mathscr{X}_O)$ as

$$\mathrm{marg}_O D := \{f \in \mathscr{L}(\mathscr{X}_O)\colon f \in D\} = D \cap \mathscr{L}(\mathscr{X}_O).$$

Let us ascertain that the definition of marginalisation reduces, in the case of binary choice, to the one for sets of desirable gambles:

Proposition 20 *Consider any non-empty subset O of $\{1,\dots,n\}$, any set of desirable gamble sets K on $\mathscr{L}(\mathscr{X}_{1:n})$, and any set of desirable gambles $D \subseteq \mathscr{L}(\mathscr{X}_{1:n})$. Then*

$$\mathrm{marg}_O D_K = D_{\mathrm{marg}_O K} \text{ and } \mathrm{marg}_O K_D = K_{\mathrm{marg}_O D}.$$

Furthermore, $\mathrm{marg}_O K = \bigcap\{K_{\mathrm{marg}_O D}\colon D \in \overline{\mathbf{D}}(K)\}$.

The last statement of this proposition guarantees that the marginal set of desirable gamble sets $\mathrm{marg}_O K$ can be retrieved by marginalising every element of K's representing set $\overline{\mathbf{D}}(K)$.

Now that marginalisation is in place, and that we know that it coincides with the eponymous concept for sets of desirable gambles in the case of pairwise choice, we are ready to look for some kind of inverse operation to it. Suppose we have a coherent set of desirable gamble sets K_O on $\mathscr{L}(\mathscr{X}_O)$ modelling a subject's belief about X_O, where O is a non-empty subset of $\{1,\dots,n\}$. We want to extend K_O to a coherent set of desirable gamble sets on $\mathscr{L}(\mathscr{X}_{1:n})$ that represents the same beliefs. So we are looking for a coherent set of desirable gamble sets K on $\mathscr{L}(\mathscr{X}_{1:n})$ such that $\mathrm{marg}_O K = K_O$ and that is as least informative as possible. It it exists, then we call K the *weak extension* of K_O.

We study this notion of weak extension in more detail. Given a non-empty subset O of $\{1,\dots,n\}$ and a coherent set of desirable gamble sets K_O on $\mathscr{L}(\mathscr{X}_O)$, an assessment based on it that is important for the weak extension, is

$$\mathscr{A}^{1:n}_{K_O} := \{A^*\colon A \in K_O\} \subseteq \mathscr{Q}(\mathscr{L}(\mathscr{X}_{1:n})).$$

To make clear that $\mathscr{A}^{1:n}_{K_O}$ is a collection of sets of gambles on $\mathscr{X}_{1:n}$, we made the cylindrical extension explicit by writing A^*. Using our simplifying device of Remark 17 however, we can equivalently write $\mathscr{A}^{1:n}_{K_O} = K_O$—and we will do this throughout—and interpret it as a collection of sets of gambles on $\mathscr{X}_{1:n}$, and therefore as an assessment for sets of desirable gamble sets on $\mathscr{L}(\mathscr{X}_{1:n})$.

It turns out that the weak extension always exists:

Proposition 21 (Weak extension) *Consider any non-empty subset O of $\{1,\dots,n\}$ and any coherent set of desirable gamble sets K_O on $\mathscr{L}(\mathscr{X}_O)$. Then the least informative coherent set of desirable gamble sets on $\mathscr{L}(\mathscr{X}_{1:n})$ that marginalises to K_O is given by*

$$\mathrm{ext}_{1:n}(K_O) := \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0} \cup \mathscr{A}^{1:n}_{K_O})),$$

and it satisfies $\mathrm{marg}_O(\mathrm{ext}_{1:n}(K_O)) = K_O$.

How is this result connected with the weak extension for sets of desirable gambles? De Cooman and Miranda [11, Proposition 7] show that, given any non-empty subset O of $\{1,\dots,n\}$ and any coherent set of desirable gambles $D_O \subseteq \mathscr{L}(\mathscr{X}_O)$, its weak extension $\mathrm{ext}^{\mathbf{D}}_{1:n}(D_O) \subseteq \mathscr{L}(\mathscr{X}_{1:n})$—the least informative coherent set of desirable gambles on $\mathscr{X}_{1:n}$ that marginalises to D_O—exists and is given by $\mathrm{ext}^{\mathbf{D}}_{1:n}(D_O) := \mathrm{posi}(\mathscr{L}(\mathscr{X}_{1:n})_{>0} \cup D_O)$. We show that the weak extension $\mathrm{ext}_{1:n}(K_O)$ of a coherent set of desirable gamble sets K_O on $\mathscr{L}(\mathscr{X}_O)$ can also be retrieved by taking the weak extension of every element of K_O's representing set $\overline{\mathbf{D}}(K_O)$ from Theorem 11:

Proposition 22 *Consider any non-empty subset O of $\{1,\dots,n\}$ and any coherent set of desirable gamble sets K_O on $\mathscr{L}(\mathscr{X}_O)$. Then*

$$\mathrm{ext}_{1:n}(K_O) = \bigcap\{K_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D_O)} \colon D_O \in \overline{\mathbf{D}}(K_O)\}.$$

4.3. Conditioning on variables

In Section 3 we have seen how we can condition sets of desirable gamble sets on events. Here, we take a closer look at conditioning in a multivariate context.

Suppose we have a set of desirable gamble sets K_n on $\mathscr{L}(\mathscr{X}_{1:n})$, representing a subject's beliefs about the value of $X_{1:n}$. Assume now that we obtain the information that the I-tuple of variables X_I—where I is a non-empty subset of $\{1,\dots,n\}$—assumes a value in a certain non-empty subset E_I of $\mathscr{X}_I$—so E_I belongs to $\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$. There is no new information about the other variables X_{I^c}. How can we condition K_n using this new information?

This is a particular instance of Definition 13, with the following specifications: $\mathscr{X} = \mathscr{X}_{1:n}$ and $E = E_I \times \mathscr{X}_{I^c}$. The indicator $\mathbb{I}_E$ of the conditioning event E satisfies $\mathbb{I}_E(x_{1:n}) = \mathbb{I}_{E_I}(x_I)$ for all $x_{1:n}$ in $\mathscr{X}_{1:n}$, and taking Remark 17 into account, therefore $\mathbb{I}_E = \mathbb{I}_{E_I}$. Equation (4) defines the multiplication of a gamble f on $E_I \times \mathscr{X}_{I^c}$ with $\mathbb{I}_{E_I}$ to be a gamble $\mathbb{I}_{E_I} f$ on $\mathscr{X}_{1:n}$, given by, for all $x_{1:n}$ in $\mathscr{X}_{1:n}$:

$$\mathbb{I}_{E_I} f(x_{1:n}) = \begin{cases} f(x_{1:n}) & \text{if } x_I \in E_I \\ 0 & \text{if } x_I \notin E_I \end{cases} \tag{5}$$

and the multiplication of $\mathbb{I}_{E_I}$ with a set A of gambles on $E_I \times \mathscr{X}_{I^c}$ is the set $\mathbb{I}_{E_I} A = \{\mathbb{I}_{E_I} f \colon f \in A\}$ of gambles on $\mathscr{X}_{1:n}$.

Now that we have instantiated all the relevant aspects of Definition 13, we are ready to find the conditional set of desirable gamble sets $K_n \rfloor E_I$, given a joint set of desirable gamble sets K_n on $\mathscr{L}(\mathscr{X}_{1:n})$:

$$K_n \rfloor E_I = \{A \in \mathscr{Q}(\mathscr{L}(E_I \times \mathscr{X}_{I^c})) \colon \mathbb{I}_{E_I} A \in K_n\}.$$

The conditional set of desirable gamble sets $K_n \rfloor E_I$ is defined on gambles on $E_I \times \mathscr{X}_{I^c}$. However, usually—see, for instance, [6, 11]—conditioning on information about X_I results in a model on X_{I^c}. We therefore consider

$$\mathrm{marg}_{I^c}(K_n \rfloor E_I) = \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_{I^c})) \colon \mathbb{I}_{E_I} A \in K_n\}$$

as the set of desirable gamble sets that represents the conditional beliefs about X_{I^c}, given that $X_I \in E_I$. In this context, the multiplication $\mathbb{I}_{E_I} f$ of $\mathbb{I}_{E_I}$ and a gamble f in A is defined through Equation (5):

$$\mathbb{I}_{E_I} f(x_{1:n}) = \begin{cases} f(x_{I^c}) & \text{if } x_I \in E_I \\ 0 & \text{if } x_I \notin E_I \end{cases} \text{ for all } x_{1:n} \text{ in } \mathscr{X}_{1:n}.$$

Note that, in the particular case of conditioning on a singleton—say, $E_I = \{x_I\}$ for some x_I in $\mathscr{X}_I$—the set $K_n \rfloor x_I$ of desirable gamble sets[5] is on $\mathscr{L}(\{x_I\} \times \mathscr{X}_{I^c})$. Every gamble f on $\{x_I\} \times \mathscr{X}_{I^c}$ can be uniquely identified with a gamble $f(x_I, \bullet)$ on $\mathscr{X}_{I^c}$, and therefore $\{x_I\} \times \mathscr{X}_{I^c}$ can be identified with $\mathscr{X}_{I^c}$. Therefore the resulting set of desirable gamble sets $K_n \rfloor x_I$ can be identified with its marginal $\mathrm{marg}_{I^c}(K_n \rfloor x_I)$.

Propositions 14 and 19, guarantee the coherence of $\mathrm{marg}_{I^c}(K_n \rfloor E_I)$, for any coherent K_n.

As is the case for desirability ([11, Proposition 9]), the order of marginalisation and conditioning can be reversed, under some conditions:

Proposition 23 *Consider any coherent set of desirable gamble sets K_n on $\mathscr{L}(\mathscr{X}_{1:n})$, any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, and any E_I in $\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$. Then*

$$\mathrm{marg}_O(K_n \rfloor E_I) = \mathrm{marg}_O((\mathrm{marg}_{I \cup O} K_n) \rfloor E_I).$$

4.4. Irrelevant natural extension

Now that the basic operations of multivariate sets of desirable gamble sets—marginalisation, weak extension and conditioning—are in place, we are ready to look at a simple type of structural assessment. The assessment that we will consider, is that of *epistemic irrelevance*.

Definition 24 (Epistemic (subset)-irrelevance) *Consider any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$. A set of desirable gamble sets K_n on*

5. Actually, since the conditioning event is $\{x_I\}$, we should write $K_n \rfloor \{x_I\}$ rather than $K_n \rfloor x_I$, but since no confusion can arise, and for notational simplicity, we will use the latter notation. A similar choice has been made by de Cooman and Miranda in [11].

$\mathscr{L}(\mathscr{X}_{1:n})$ *is said to satisfy epistemic irrelevance of* X_I *to* X_O *when*

$$\mathrm{marg}_O(K_n \rfloor E_I) = \mathrm{marg}_O K_n \text{ for all } E_I \text{ in } \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I).$$

The idea behind this definition is that observing that X_I belongs to E_I turns K_n into the conditioned set of desirable gamble sets $K_n \rfloor E_I$ on $\mathscr{L}(E_I \times \mathscr{X}_{I^c}) \supseteq \mathscr{L}(\mathscr{X}_O)$, so requiring that learning that X_I belongs to E_I does not affect the subject's beliefs about X_O, amounts to requiring that the marginal models of K_n and $K_n \rfloor E_I$ should be equal.

Epistemic irrelevance can be reformulated in an interesting and slightly different manner:

Proposition 25 *Consider any coherent set of desirable gamble sets* K_n *on* $\mathscr{L}(\mathscr{X}_{1:n})$*, and any disjoint and non-empty subsets I and O of* $\{1,\ldots,n\}$*. Then the following statements are equivalent:*

(i) $\mathrm{marg}_O(K_n \rfloor E_I) = \mathrm{marg}_O K_n$ *for all* E_I *in* $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$*;*
(ii) $A \in K_n \Leftrightarrow \mathbb{I}_{E_I} A \in K_n$*, for all A in* $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ *and* E_I *in* $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$*.*

Epistemic irrelevance assessments are useful in constructing sets of desirable gamble sets on larger domains from other ones on smaller domains. Suppose we have a set of desirable gamble sets K_O on $\mathscr{L}(\mathscr{X}_O)$, and an assessment that X_I is epistemically irrelevant to X_O, where I and O are disjoint and non-empty subsets of $\{1,\ldots,n\}$. How can we combine K_O and this irrelevance assessment into a coherent set of desirable gamble sets on $\mathscr{L}(\mathscr{X}_{I\cup O})$, or more generally, on $\mathscr{L}(\mathscr{X}_{1:n})$? We want this new set of desirable gamble sets furthermore to be as *least informative* as possible.

The following set will play a crucial role:

$$\mathscr{A}^{\mathrm{irr}}_{I\to O} := \{\mathbb{I}_{E_I} A_O : A_O \in K_O \text{ and } E_I \in \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)\} \tag{6}$$

which we will interpret as an assessment on $\mathscr{L}(\mathscr{X}_{I\cup O})$.

Theorem 26 (Irrelevant natural extension) *Consider any disjoint and non-empty subsets I and O of* $\{1,\ldots,n\}$*, and any coherent set of desirable gamble sets* K_O *on* $\mathscr{L}(\mathscr{X}_O)$*. The least informative coherent set of desirable gamble sets on* $\mathscr{L}(\mathscr{X}_{1:n})$ *that marginalises to* K_O *and that satisfies epistemic irrelevance of* X_I *to* X_O *is given by* $\mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O) := \mathrm{ext}_{1:n}(K^{\mathrm{irr}}_{I\cup O})$*, where*

$$K^{\mathrm{irr}}_{I\cup O} := \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\mathrm{irr}}_{I\to O})).$$

Furthermore,

$$\mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O) = \bigcap\{K_{\mathrm{ext}^{\mathbf{D}}_{1:n}(D)} : D \in \overline{\mathbf{D}}(K^{\mathrm{irr}}_{I\cup O})\}.$$

The final statement of this theorem guarantees that the irrelevant natural extension that marginalises to a set of desirable gamble sets K_O can be retrieved by extending every element of $K^{\mathrm{irr}}_{I\cup O}$'s representing set $\overline{\mathbf{D}}(K^{\mathrm{irr}}_{I\cup O})$ from Theorem 11.

5. Conclusions

We have studied the irrelevant natural extension in the framework of choice functions. To define this, we introduced conditioning and marginalisation in this framework. We related our definitions and results with the existing definitions and results in the framework of sets of desirable gambles, and showed that they match with each other. The results in this paper are important because they are a first step for establishing a theory of credal networks with choice functions. Besides their generality, such credal networks would have the advantage that the local models are easy to elicit: choice functions can be assessed directly from a subject, simply by collecting the gambles she rejects from within any given set of gambles.

However, one important issue in this respect is the lack of an expression for the *independent* natural extension for choice functions. The independent natural extension is a symmetric version of the irrelevant natural extension: if X_I is independent to X_O, then both X_I is irrelevant to X_O, and *vice versa*. A possible future goal is to investigate the independent natural extension in this framework. We expect the representation result of De Bock and de Cooman [9, Theorem 7] to be crucial for this. A first step in this direction, would be to establish the following representation of the irrelevant natural extension:

$$\mathrm{ext}^{\mathrm{irr}}_{1:n}(K_O) = \bigcap\{K_{\mathrm{ext}^{\overline{\mathbf{D}},\mathrm{irr}}_{1:n}(D_O)} : D_O \in \overline{\mathbf{D}}(K_O)\}$$

where $\mathrm{ext}^{\overline{\mathbf{D}},\mathrm{irr}}_{1:n}(D_O)$ is the irrelevant natural extension for sets of desirable gambles, established in [11]. This is a conjecture of us, based on some preliminary insight, but we have no proof as of yet.

In addition, it would also be interesting to consider other, intermediate notions of irrelevance and independence, such as the notion of *subset irrelevance* considered in [7], and more generally, the compatibility of choice functions with other structural assessments, such as weak and strong invariance.

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Appendix A. Proofs

Proof [Proof of Lemma 7] For the sake of brevity, we denote

$$K^1_{\mathrm{H,T}} := \{A \in \mathscr{Q} \colon ((\exists h_1, h_2 \in A)(h_1(\mathrm{T}) > 0 \text{ and } h_2(\mathrm{H}) > 0)) \text{ or } A \cap \mathscr{L}^{\mathrm{s}}_{>0} \neq \emptyset\},$$

$$K^2_{\mathrm{H,T}} := \mathrm{Rs}(\{\{h_1, h_2\} \colon h_1, h_2 \in \mathscr{L}_{\not\leq 0}, (h_1(\mathrm{T}), h_2(\mathrm{H})) > 0\}).$$

We will show (i) that $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}) \subseteq K^1_{\mathrm{H,T}}$, (ii) that $K^1_{\mathrm{H,T}} \subseteq K^2_{\mathrm{H,T}}$, and (iii) that $K^2_{\mathrm{H,T}} \subseteq \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$.

For (i)—to show that $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}) \subseteq K^1_{\mathrm{H,T}}$—consider any gamble set A in $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$. This means that there are n in $\mathbb{N}$, A_1, ..., A_n in $\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}$, and, for all $f_{1:n}$ in $\times_{k=1}^n A_k$, coefficients $\lambda_{1:n}^{f_{1:n}} > 0$, such that $A = \left\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\right\}$. Without loss of generality, assume that $A_1, \dots, A_\ell \in \mathscr{A}$ and $A_{\ell+1}, \dots, A_n \in \mathscr{L}^{\mathrm{s}}_{>0}$ for some ℓ in $\{0, \dots, n\}$. Therefore, we may denote, also without loss of generality, $A_1 = \{-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon_1, -\mathbb{I}_{\{\mathrm{T}\}} + \delta_1\}$, ..., $A_\ell = \{-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon_\ell, -\mathbb{I}_{\{\mathrm{T}\}} + \delta_\ell\}$, $A_{\ell+1} = \{g_{\ell+1}\}$, ..., $A_n = \{g_n\}$, where ε_1, δ_1, ..., ε_ℓ, δ_ℓ are elements of $\mathbb{R}_{>0}$ and $g_{\ell+1}$, ..., g_n elements of $\mathscr{L}_{>0}$. If $\ell = 0$ or $\lambda_{1:\ell}^{f_{1:n}} = 0$ for some $f_{1:n}$ in $\times_{k=1}^n A_k$—and therefore necessarily $\lambda_{\ell+1:n}^{f_{1:n}} > 0$—then we have that $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = \sum_{k=\ell+1}^n \lambda_k^{f_{1:n}} g_k$ is a gamble in $\mathscr{L}_{>0}$, so we find that $\left\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\right\} = A \cap \mathscr{L}^{\mathrm{s}}_{>0} \neq \emptyset$. If, on the other hand, $\ell \geq 1$ and $\lambda_{1:\ell}^{f_{1:n}} \neq 0$—and hence $\lambda_{1:\ell}^{f_{1:n}} > 0$—for every $f_{1:n}$ in $\times_{k=1}^n A_k$, then for the two sequences of gambles $f^{\mathrm{H}}_{1:n} = (f^{\mathrm{H}}_1, \dots, f^{\mathrm{H}}_n) := (-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon_1, \dots, -\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon_\ell, g_{\ell+1}, \dots, g_n)$ and $f^{\mathrm{T}}_{1:n} = (f^{\mathrm{T}}_1, \dots, f^{\mathrm{T}}_n) := (-\mathbb{I}_{\{\mathrm{T}\}} + \delta_1, \dots, -\mathbb{I}_{\{\mathrm{T}\}} + \delta_\ell, g_{\ell+1}, \dots, g_n)$ in $\times_{k=1}^n A_k$ we have that

$$\begin{aligned} h_1 := \sum_{k=1}^n \lambda_k^{f^{\mathrm{H}}_{1:n}} f^{\mathrm{H}}_k &= \sum_{k=1}^\ell \lambda_k^{f^{\mathrm{H}}_{1:n}} f^{\mathrm{H}}_k + \sum_{k=\ell+1}^n \lambda_k^{f^{\mathrm{H}}_{1:n}} g_k \\ &\geq \sum_{k=1}^\ell \lambda_k^{f^{\mathrm{H}}_{1:n}} f^{\mathrm{H}}_k \\ &= -\mathbb{I}_{\{\mathrm{H}\}} \sum_{k=1}^\ell \lambda_k^{f^{\mathrm{H}}_{1:n}} + \sum_{k=1}^\ell \lambda_k^{f^{\mathrm{H}}_{1:n}} \varepsilon_k \end{aligned}$$

and, similarly,

$$h_2 := \sum_{k=1}^n \lambda_k^{f^{\mathrm{T}}_{1:n}} f^{\mathrm{T}}_k \geq -\mathbb{I}_{\{\mathrm{T}\}} \sum_{k=1}^\ell \lambda_k^{f^{\mathrm{T}}_{1:n}} + \sum_{k=1}^\ell \lambda_k^{f^{\mathrm{T}}_{1:n}} \delta_k,$$

so $h_1(\mathrm{T}) \geq \sum_{k=1}^\ell \lambda_k^{f^{\mathrm{H}}_{1:n}} \varepsilon_k > 0$ and $h_2(\mathrm{H}) \geq \sum_{k=1}^\ell \lambda_k^{f^{\mathrm{T}}_{1:n}} \delta_k > 0$. Note that both h_1 and h_2 belong to A, so we find that $(\exists h_1, h_2 \in A)(h_1(\mathrm{T}) > 0 \text{ and } h_2(\mathrm{H}) > 0)$. Therefore indeed $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A}) \subseteq K^1_{\mathrm{H,T}}$.

For (ii)—to show that $K^1_{\mathrm{H,T}} \subseteq K^2_{\mathrm{H,T}}$—consider any gamble set A in $K^1_{\mathrm{H,T}}$. Then (a) $h_1(\mathrm{T}) > 0$ and $h_2(\mathrm{H}) > 0$ for some h_1 and h_2 in A, or (b) $A \cap \mathscr{L}^{\mathrm{s}}_{>0} \neq \emptyset$. If (a), then $h_1, h_2 \in \mathscr{L}_{\not\leq 0}$, and $(h_1(\mathrm{T}), h_2(\mathrm{H})) > 0$, so $A \in K^2_{\mathrm{H,T}}$. If (b), then $h > 0$ for some h in A, so for $h_1 := h_2 := h$ trivially $h_1, h_2 \in \mathscr{L}_{\not\leq 0}$, and $(h_1(\mathrm{T}), h_2(\mathrm{H})) = (h(\mathrm{T}), h(\mathrm{H})) > 0$, whence $A \in K^2_{\mathrm{H,T}}$. We conclude that indeed $K^1_{\mathrm{H,T}} \subseteq K^2_{\mathrm{H,T}}$.

For (iii)—to show that $K^2_{\mathrm{H,T}} \subseteq \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$—consider any gamble set A in $K^2_{\mathrm{H,T}}$. Then $A \supseteq \{h_1, h_2\} \setminus \mathscr{L}_{\leq 0} = \{h_1, h_2\}$ for some h_1 and h_2 in $\mathscr{L}_{\not\leq 0}$ such that $(h_1(\mathrm{T}), h_2(\mathrm{H})) > 0$. Without loss of generality, rename the gambles in

$$A = \{f^{\mathrm{I}}_1, \dots, f^{\mathrm{I}}_{n_{\mathrm{I}}}, f^{\mathrm{II}}_1, \dots, f^{\mathrm{II}}_{n_{\mathrm{II}}}, f^{\mathrm{III}}_1, \dots, f^{\mathrm{III}}_{n_{\mathrm{III}}}, f^{\mathrm{IV}}_1, \dots, f^{\mathrm{IV}}_{n_{\mathrm{IV}}}\},$$

with n_{I}, n_{II}, n_{III} and n_{IV} in $\{0\} \cup \mathbb{N}$ such that $n := 2n_{\mathrm{I}} + n_{\mathrm{II}} + 2n_{\mathrm{III}} + n_{\mathrm{IV}} \geq 1$, gambles f^{I}_1, ..., $f^{\mathrm{I}}_{n_{\mathrm{I}}}$ in the positive quadrant $\mathscr{L}_{>0}$, gambles f^{II}_1, ..., $f^{\mathrm{II}}_{n_{\mathrm{II}}}$ in the second quadrant $\mathscr{L}_{\mathrm{II}} := \{f \in \mathscr{L} \colon f(\mathrm{H}) < 0 < f(\mathrm{T})\}$, gambles f^{III}_1, ..., $f^{\mathrm{III}}_{n_{\mathrm{III}}}$ in the negative quadrant $\mathscr{L}_{\leq 0}$, and gambles f^{IV}_1, ..., $f^{\mathrm{IV}}_{n_{\mathrm{IV}}}$ in the fourth quadrant $\mathscr{L}_{\mathrm{IV}} := \{f \in \mathscr{L} \colon f(\mathrm{T}) < 0 < f(\mathrm{H})\}$. We must show that A belongs to $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup \mathscr{A})$. To this end, we will construct n gamble sets A_1, ..., A_n and, for every $f_{1:n}$ in $\times_{k=1}^n A_k$, coefficients $\lambda_{1:n}^{f_{1:n}} > 0$ such that $A = \left\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\right\}$.

Let $A_1 := \{g_1\} \in \mathscr{L}^{\mathrm{s}}_{>0}$, ..., $A_{n_{\mathrm{I}}} := \{g_{n_{\mathrm{I}}}\} \in \mathscr{L}^{\mathrm{s}}_{>0}$. We consider the additional n_{I} gamble sets $A_{n_{\mathrm{I}}+1} := \dots := A_{2n_{\mathrm{I}}} := \{-\mathbb{I}_{\{\mathrm{H}\}} + 1, -\mathbb{I}_{\{\mathrm{T}\}} + 1\} \in \mathscr{A}$, in order to have enough freedom in selecting the coefficients $\lambda_{1:n}^{f_{1:n}} > 0$ later on. For every i in $\{1, \dots, n_{\mathrm{II}}\}$, let $A_{2n_{\mathrm{I}}+i} := \{-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon_i, -\mathbb{I}_{\{\mathrm{T}\}} + \delta\} \in \mathscr{A}$ with $\varepsilon_i := \frac{f^{\mathrm{II}}_i(\mathrm{T})}{f^{\mathrm{II}}_i(\mathrm{T}) - f^{\mathrm{II}}_i(\mathrm{H})} > 0$ and $\delta := \frac{f^{\mathrm{IV}}_1(\mathrm{H})}{f^{\mathrm{IV}}_1(\mathrm{H}) - f^{\mathrm{IV}}_1(\mathrm{T})} > 0$ if $n_{\mathrm{IV}} \geq 1$, otherwise $\delta := 1$. For every i in $\{1, \dots, n_{\mathrm{III}}\}$, if $f^{\mathrm{III}}_i \neq 0$, let $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+i} := \{-\mathbb{I}_{\{\mathrm{H}\}} + \frac{1}{4}, -\mathbb{I}_{\{\mathrm{T}\}} + 1\} \in \mathscr{A}$ and $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+i} := \{-\mathbb{I}_{\{\mathrm{H}\}} + 1, -\mathbb{I}_{\{\mathrm{T}\}} + \frac{1}{4}\} \in \mathscr{A}$; if $f^{\mathrm{III}}_i = 0$, let $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+i} := A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+i} := \{-\mathbb{I}_{\{\mathrm{H}\}} + \frac{1}{2}, -\mathbb{I}_{\{\mathrm{T}\}} + \frac{1}{2}\} \in \mathscr{A}$. For every i in $\{1, \dots, n_{\mathrm{IV}}\}$, let $A_{2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+i} := \{-\mathbb{I}_{\{\mathrm{H}\}} + 1, -\mathbb{I}_{\{\mathrm{T}\}} + \delta_i\} \in \mathscr{A}$ with $\delta_i := \frac{f^{\mathrm{IV}}_i(\mathrm{H})}{f^{\mathrm{IV}}_i(\mathrm{H}) - f^{\mathrm{IV}}_i(\mathrm{T})} > 0$.

The set $\times_{k=1}^n A_k$ contains $2^{n-n_{\mathrm{I}}} = 2^{n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+n_{\mathrm{IV}}}$ sequences. Each such sequence $f_{1:n}$ is characterised by a choice of f_i in the binary set A_i—which we will denote by $\{g^{\mathrm{H}}_i, g^{\mathrm{T}}_i\}$, where g^{H}_i is the gamble in A_i of the form $-\mathbb{I}_{\{\mathrm{H}\}} + \varepsilon$ and g^{T}_i the gamble in A_i of the form $-\mathbb{I}_{\{\mathrm{T}\}} + \delta$—, for every i in $\{n_{\mathrm{I}} + 1, \dots, n\}$. For the first n_{I} entries $f_{1:n_{\mathrm{I}}}$ of $f_{1:n}$ we have no choice but to chose $f_{1:n_{\mathrm{I}}} = g_{1:n_{\mathrm{I}}}$, since $\times_{k=1}^{n_{\mathrm{I}}} A_k$ is the singleton $\{g_{1:n_{\mathrm{I}}}\}$.

For any sequence $f_{1:n}$ in $\times_{k=1}^n A_k$, define n real coefficients $\lambda_{1:n}^{f_{1:n}}$ as follows:

- Situation (a): If there is an i in $\{2n_{\mathrm{I}} + 1, \dots, 2n_{\mathrm{I}} + n_{\mathrm{II}}\}$ such that

$$(f_{2n_{\mathrm{I}}+1}, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}},$$

$$f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,f_n) = (g^{\mathrm{T}}_{2n_{\mathrm{I}}+1},\dots,g^{\mathrm{T}}_{i-1},g^{\mathrm{H}}_i,g^{\mathrm{T}}_{i+1},\dots,g^{\mathrm{T}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}, g^{\mathrm{H}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,g^{\mathrm{H}}_n)$$

or, in other words, such that $f_i = g^{\mathrm{H}}_i$, $(\forall k \in \{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}\setminus\{i\})f_k = g^{\mathrm{T}}_k$, and $(\forall k \in \{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1,\dots,n\})f_k = g^{\mathrm{H}}_k$, then let

$$\lambda^{f_{1:n}}_i := f^{\mathrm{II}}_j(\mathrm{T}) - f^{\mathrm{II}}_j(\mathrm{H}) > 0 \text{ for } j := i-2n_{\mathrm{I}},$$
$$\lambda^{f_{1:n}}_k := 0 \text{ for all } k \text{ in } \{1,\dots,n\}\setminus\{i\}.$$

- Situation (b): If there is an i in $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+1,\dots,n\}$ such that

$$(f_{2n_{\mathrm{I}}+1},\dots,f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}, f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_n) = (g^{\mathrm{T}}_{2n_{\mathrm{I}}+1},\dots,g^{\mathrm{T}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}, g^{\mathrm{H}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,g^{\mathrm{H}}_{i-1},g^{\mathrm{T}}_i,g^{\mathrm{H}}_{i+1},\dots,g^{\mathrm{H}}_n),$$

or, in other words, such that $f_i = g^{\mathrm{T}}_i$, $(\forall k \in \{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\})f_k = g^{\mathrm{T}}_k$, and $(\forall k \in \{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1,\dots,n\}\setminus\{i\})f_k = g^{\mathrm{H}}_k$, then let

$$\lambda^{f_{1:n}}_i := f^{\mathrm{IV}}_j(\mathrm{H}) - f^{\mathrm{IV}}_j(\mathrm{T}) > 0 \quad \text{for } j := i-2n_{\mathrm{I}}-n_{\mathrm{II}}-2n_{\mathrm{III}},$$
$$\lambda^{f_{1:n}}_k := 0 \text{ for all } k \text{ in } \{1,\dots,n\}\setminus\{i\}.$$

- Situation (c): If there is an i in $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}$ such that

$$(f_{2n_{\mathrm{I}}+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}, f_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,f_{n_{\mathrm{III}}+i-1},f_{n_{\mathrm{III}}+i},f_{n_{\mathrm{III}}+i+1},\dots,f_n) = (g^{\mathrm{T}}_{2n_{\mathrm{I}}+1},\dots,g^{\mathrm{T}}_{i-1},g^{\mathrm{H}}_i,g^{\mathrm{T}}_{i+1},\dots,g^{\mathrm{T}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}}, g^{\mathrm{H}}_{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1},\dots,g^{\mathrm{H}}_{n_{\mathrm{III}}+i-1},g^{\mathrm{T}}_{n_{\mathrm{III}}+i},g^{\mathrm{H}}_{n_{\mathrm{III}}+i+1},\dots,g^{\mathrm{H}}_n),$$

or, in other words, such that $f_i = g^{\mathrm{H}}_i$, $f_{n_{\mathrm{III}}+i} = g^{\mathrm{T}}_{n_{\mathrm{III}}+i}$, $(\forall k \in \{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}\setminus\{i\})f_k = g^{\mathrm{T}}_k$ and $(\forall k \in \{2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}+1,\dots,n\}\setminus\{n_{\mathrm{III}}+i\})f_k = g^{\mathrm{H}}_k$, then let

$$\lambda^{f_{1:n}}_i := \lambda^{f_{1:n}}_{n_{\mathrm{III}}+i} := 1 \text{ if } f^{\mathrm{III}}_{i-2n_{\mathrm{I}}+n_{\mathrm{II}}} = 0,$$
$$\lambda^{f_{1:n}}_i := -\frac{1}{2}(3f^{\mathrm{III}}_j(\mathrm{H}) + f^{\mathrm{III}}_j(\mathrm{T})) > 0 \text{ and}$$
$$\lambda^{f_{1:n}}_{n_{\mathrm{III}}+i} := -\frac{1}{2}(f^{\mathrm{III}}_j(\mathrm{H}) + 3f^{\mathrm{III}}_j(\mathrm{T})) > 0 \quad \text{for } j := i-2n_{\mathrm{I}}-n_{\mathrm{II}} \text{ and if } f^{\mathrm{III}}_j \neq 0,$$
$$\lambda^{f_{1:n}}_k := 0 \text{ for all } k \text{ in } \{1,\dots,n\}\setminus\{i,n_{\mathrm{III}}+i\}.$$

- Situation (d): If none of the Situations (a), (b) nor (c) apply, and if there is an i in $\{n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}\}$ such that

$$(f_{n_{\mathrm{I}}+1},\dots,f_{i-1},f_i,f_{i+1},\dots,f_{2n_{\mathrm{I}}}) = (g^{\mathrm{T}}_{n_{\mathrm{I}}+1},\dots,g^{\mathrm{T}}_{i-1},g^{\mathrm{H}}_i,g^{\mathrm{T}}_{i+1},\dots,g^{\mathrm{T}}_{2n_{\mathrm{I}}}),$$

or, in other words, such that $f_i = g^{\mathrm{H}}_i$ and $(\forall k \in \{n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}\}\setminus\{i\})f_k = g^{\mathrm{T}}_k$, then let

$$\lambda^{f_{1:n}}_{i-n_{\mathrm{I}}} := 1,$$
$$\lambda^{f_{1:n}}_k := 0 \text{ for all } k \text{ in } \{1,\dots,n\}\setminus\{i-n_{\mathrm{I}}\}.$$

- Situation (e1): If $A\cap\mathscr{L}_{>0} \neq \emptyset$—so $n_{\mathrm{I}} \geq 1$—and none of the Situations (a), (b), (c) nor (d) apply, then let

$$\lambda^{f_{1:n}}_1 := 1 \text{ and } \lambda^{f_{1:n}}_{2:n} := 0.$$

- Situation (e2): If $A\cap\mathscr{L}_{>0} = \emptyset$—so $n_{\mathrm{II}} \geq 1$ and $n_{\mathrm{IV}} \geq 1$ because $(h_1(\mathrm{T}),h_2(\mathrm{H})) > 0$—and none of the Situations (a), (b), (c) nor (d) apply, then let, with $i := 2n_{\mathrm{I}}+1$,

$$\lambda^{f_{1:n}}_i := f^{\mathrm{II}}_1(\mathrm{T}) - f^{\mathrm{II}}_1(\mathrm{H}) > 0 \text{ if } f_i = g^{\mathrm{H}}_i,$$
$$\lambda^{f_{1:n}}_i := f^{\mathrm{IV}}_1(\mathrm{H}) - f^{\mathrm{IV}}_1(\mathrm{T}) > 0 \text{ if } f_i = g^{\mathrm{T}}_i,$$
$$\lambda^{f_{1:n}}_k := 0 \text{ or all } k \text{ in } \{1,\dots,n\}\setminus\{i\}.$$

In this way, we have defined coefficients $\lambda^{f_{1:n}}_{1:n} > 0$ for every $f_{1:n}$ in $\times_{k=1}^n A_k$. It only remains to show, with our choices of $\lambda^{f_{1:n}}_{1:n} > 0$, that $A = \{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$.

We first prove that $A \subseteq \{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$. To show that $f^{\mathrm{II}}_j \in \{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$ for every j in $\{1,\dots,n_{\mathrm{II}}\}$, consider any $f_{1:n}$ in $\times_{k=1}^n A_k$ such that $f_{1:n}$ satisfies the conditions of Situation (a) for $i := j+2n_{\mathrm{I}}$, which is then an element of $\{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}\}$. Then $\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k = (f^{\mathrm{II}}_j(\mathrm{T}) - f^{\mathrm{II}}_j(\mathrm{H}))g^{\mathrm{H}}_i = (f^{\mathrm{II}}_j(\mathrm{T}) - f^{\mathrm{II}}_j(\mathrm{H}))\big(-\mathbb{I}_{\{\mathrm{H}\}} + \frac{f^{\mathrm{II}}_i(\mathrm{T})}{f^{\mathrm{II}}_i(\mathrm{T})-f^{\mathrm{II}}_i(\mathrm{H})}\big) = (f^{\mathrm{II}}_j(\mathrm{H}) - f^{\mathrm{II}}_j(\mathrm{T}))\mathbb{I}_{\{\mathrm{H}\}} + f^{\mathrm{II}}_j(\mathrm{T}) = f^{\mathrm{II}}_j$, so indeed $f^{\mathrm{II}}_j \in \{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$.

To show that $f^{\mathrm{IV}}_j \in \{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$ for every j in $\{1,\dots,n_{\mathrm{IV}}\}$, consider any $f_{1:n}$ in $\times_{k=1}^n A_k$ such that $f_{1:n}$ satisfies the conditions of Situation (b) for $i := j+2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}$, which is then an element of $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+2n_{\mathrm{III}}+1,\dots,n\}$. Then $\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k = (f^{\mathrm{IV}}_j(\mathrm{H}) - f^{\mathrm{IV}}_j(\mathrm{T}))g^{\mathrm{T}}_i = (f^{\mathrm{IV}}_j(\mathrm{H}) - f^{\mathrm{IV}}_j(\mathrm{T}))\big(-\mathbb{I}_{\{\mathrm{T}\}} + \frac{f^{\mathrm{IV}}_i(\mathrm{H})}{f^{\mathrm{IV}}_i(\mathrm{H})-f^{\mathrm{IV}}_i(\mathrm{T})}\big) = (f^{\mathrm{IV}}_j(\mathrm{T}) - f^{\mathrm{IV}}_j(\mathrm{H}))\mathbb{I}_{\{\mathrm{T}\}} + f^{\mathrm{IV}}_j(\mathrm{H}) = f^{\mathrm{IV}}_j$, so indeed $f^{\mathrm{IV}}_j \in \{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$.

To show that $f^{\mathrm{III}}_j \in \{\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$ for every j in $\{1,\dots,n_{\mathrm{III}}\}$, consider any $f_{1:n}$ in $\times_{k=1}^n A_k$ such that $f_{1:n}$ satisfies the conditions of Situation (c) for $i := j+2n_{\mathrm{I}}+n_{\mathrm{II}}$, which is then an element of $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}$. Then $\sum_{k=1}^n \lambda^{f_{1:n}}_k f_k = -\frac{1}{2}(3f^{\mathrm{III}}_j(\mathrm{H}) + f^{\mathrm{III}}_j(\mathrm{T}))g^{\mathrm{H}}_i - \frac{1}{2}(f^{\mathrm{III}}_j(\mathrm{H}) + 3f^{\mathrm{III}}_j(\mathrm{T}))g^{\mathrm{T}}_i = -\frac{1}{2}(3f^{\mathrm{III}}_j(\mathrm{H}) + f^{\mathrm{III}}_j(\mathrm{T}))\big(-\mathbb{I}_{\{\mathrm{H}\}} + \frac{1}{4}\big) - \frac{1}{2}(f^{\mathrm{III}}_j(\mathrm{H}) +$

$3f_j^{\mathrm{III}}(\mathrm{T}))\big(-\mathbb{I}_{\{\mathrm{T}\}}+\frac{1}{4}\big) = f_j^{\mathrm{III}}(\mathrm{H})\mathbb{I}_{\{\mathrm{H}\}} + f_j^{\mathrm{III}}(\mathrm{T})\mathbb{I}_{\{\mathrm{T}\}} = f_j^{\mathrm{III}}$ if $f_j^{\mathrm{III}} \neq 0$, and $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = g_i^{\mathrm{H}} + g_i^{\mathrm{T}} = -\mathbb{I}_{\{\mathrm{H}\}} + \frac{1}{2} - \mathbb{I}_{\{\mathrm{T}\}} + \frac{1}{2} = 0 = f_j^{\mathrm{III}}$ if $f_j^{\mathrm{III}} = 0$, so indeed $f_j^{\mathrm{III}} \in \{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$.

To show that $f_j^{\mathrm{I}} \in \{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$ for every j in $\{1,\dots,n_{\mathrm{I}}\}$, consider any $f_{1:n}$ in $\times_{k=1}^n A_k$ such that $f_{1:n}$ satisfies the conditions of Situation (d) for $i := j + n_{\mathrm{I}}$, which is then an element of $\{n_{\mathrm{I}} + 1,\dots,2n_{\mathrm{I}}\}$. Then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = g_j = f_j^{\mathrm{I}}$, so indeed $f_j^{\mathrm{I}} \in \{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$.

We finally show, conversely, that $A \supseteq \{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$. Consider any f in $\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$. Then $f = \sum_{k=1}^n \lambda_k^{f_{1:n}} f_k$ for some $f_{1:n}$ in $\times_{k=1}^n A_k$. If this $f_{1:n}$ satisfies the conditions of Situation (a) for some i in $\{2n_{\mathrm{I}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}\}$, then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = f_j^{\mathrm{II}}$ for $j := i - 2n_{\mathrm{I}}$, as shown above, so $f \in A$. If $f_{1:n}$ satisfies the conditions of Situation (b) for some i in $\{2n_{\mathrm{I}} + n_{\mathrm{II}} + 2n_{\mathrm{III}} + 1,\dots,n\}$, then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = f_j^{\mathrm{IV}}$ for $j := i - 2n_{\mathrm{I}} - n_{\mathrm{II}} - 2n_{\mathrm{III}}$, as shown above, so $f \in A$. If $f_{1:n}$ satisfies the conditions of Situation (c) for some i in $\{2n_{\mathrm{I}}+n_{\mathrm{II}}+1,\dots,2n_{\mathrm{I}}+n_{\mathrm{II}}+n_{\mathrm{III}}\}$, then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = f_j^{\mathrm{III}}$ for $j := i - 2n_{\mathrm{I}} - n_{\mathrm{II}}$, as shown above, so $f \in A$. If $f_{1:n}$ satisfies the conditions of Situation (d) for some i in $\{n_{\mathrm{I}} + 1,\dots,2n_{\mathrm{I}}\}$, then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = f_j^{\mathrm{I}}$ for $j := i - n_{\mathrm{I}}$, as shown above, so $f \in A$. The only other possibility is that $f_{1:n}$ satisfies the conditions of Situation (e1) or (e2), depending on whether or not $A \cap \mathscr{L}_{>0} \neq \emptyset$. If $A \cap \mathscr{L}_{>0} \neq \emptyset$ (so Situation (e1)), then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = f_1^{\mathrm{I}}$, which is an element of A since $n_{\mathrm{I}} \geq 1$, so $f \in A$. If $A \cap \mathscr{L}_{>0} = \emptyset$ (so Situation (e2)), then $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = (f_1^{\mathrm{II}}(\mathrm{T}) - f_1^{\mathrm{II}}(\mathrm{H}))\big(-\mathbb{I}_{\{\mathrm{H}\}} + \frac{f_1^{\mathrm{II}}(\mathrm{T})}{f_1^{\mathrm{II}}(\mathrm{T})-f_i^{\mathrm{II}}(\mathrm{H})}\big) = f_1^{\mathrm{II}}$ or $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k = (f_1^{\mathrm{IV}}(\mathrm{H}) - f_1^{\mathrm{IV}}(\mathrm{T}))\big(-\mathbb{I}_{\{\mathrm{H}\}} + \frac{f_1^{\mathrm{IV}}(\mathrm{H})}{f_1^{\mathrm{IV}}(\mathrm{H})-f_i^{\mathrm{IV}}(\mathrm{T})}\big) = f_1^{\mathrm{IV}}$, which both belong to A since $n_{\mathrm{II}} \geq 1$ and $n_{\mathrm{IV}} \geq 1$, so $f \in A$. There are no other possibilities, so we conclude that indeed $A \supseteq \{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\}$. ∎

Proof [Proof of Proposition 10] By definition, the least informative coherent set of desirable gamble sets that includes $\{\{f\} \colon f \in D\}$ is the natural extension $\mathrm{cl}_{\overline{\mathbf{K}}}(\mathscr{A}_D)$ of the assessment $\mathscr{A}_D := \{\{f\} \colon f \in D\}$.

Let us first show that $\mathscr{A}_D$ is consistent. By Theorem 6, we need to show that $\emptyset \notin \mathscr{A}_D$ and $\{0\} \notin \mathrm{Posi}(\mathscr{L}_{>0}^{\mathrm{s}} \cup \mathscr{A}_D) = \mathrm{Posi}(\mathscr{A}_D)$, where the equality follows from the fact that $\mathscr{L}_{>0}^{\mathrm{s}} \subseteq \mathscr{A}_D$ by Axiom D_2. By definition, $\emptyset \notin \mathscr{A}_D$, so it remains to prove that $\{0\} \notin \mathrm{Posi}(\mathscr{A}_D)$. To this end, consider any singleton $\{g\}$ in $\mathrm{Posi}(\mathscr{A}_D)$. There are n in $\mathbb{N}$, A_1, …, A_n in $\mathscr{A}_D$, and, for all $f_{1:n}$ in $\times_{k=1}^n A_k$, coefficients $\lambda_{1:n}^{f_{1:n}} > 0$, such that $\Big\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\Big\} = \{g\}$. Since the entries of any sequence $f_{1:n}$ in $\times_{k=1}^n A_k$ belong to $\mathscr{L}_{>0} \cup D$, so does $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k$, by repeated application of Axiom D_3. So $g \in D \cup \mathscr{L}_{>0}$. By Axiom D_1, $0 \notin D$, whence indeed $g \neq 0$.

We now know that $\mathscr{A}_D$ is consistent, so by Theorem 6, its natural extension $\mathscr{A}_D$ is equal to $\mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$, since we already know that $\mathscr{L}_{>0}^{\mathrm{s}} \subseteq \mathscr{A}_D$. Let us show that $K_D = \mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$; we prove (i) $K_D \subseteq \mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$ and (ii) $K_D \supseteq \mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$. For (i), consider any A in K_D, so $A \cap D \neq \emptyset$, and therefore $f \in A$ for some f in D. This tells us that $\{f\} \in \mathscr{A}_D$. Since $K \subseteq \mathrm{Posi}(K)$ for any K in $\mathbf{K}$, we find that $\{f\} \in \mathrm{Posi}(\mathscr{A}_D)$. Therefore, any superset of $\{f\}$—and in particular indeed the set A—will belong to $\mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$.

Let us now show that (ii) $K_D \supseteq \mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$. To this end, consider any A in $\mathrm{Rs}(\mathrm{Posi}(\mathscr{A}_D))$. Then, by the definition of the Rs operator, there is some B in $\mathrm{Posi}(\mathscr{A}_D)$ such that $B \setminus \mathscr{L}_{\leq 0} \subseteq A$. This means that there are n in $\mathbb{N}$, A_1, …, A_n in $\mathscr{L}_{>0}^{\mathrm{s}} \cup \mathscr{A}_D$, and, for all $f_{1:n}$ in $\times_{k=1}^n A_k$, coefficients $\lambda_{1:n}^{f_{1:n}} > 0$, such that $\Big\{\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k \colon f_{1:n} \in \times_{k=1}^n A_k\Big\} = B$. Since the entries of any sequence $f_{1:n}$ in $\times_{k=1}^n A_k$ belong to $\mathscr{L}_{>0} \cup D$, so does $\sum_{k=1}^n \lambda_k^{f_{1:n}} f_k$, by repeated application of Axiom D_3. So we find $B \subseteq D \cup \mathscr{L}_{>0} = D$, where the equality is a consequence of Axiom D_2, and hence $B \setminus \mathscr{L}_{\leq 0} \subseteq D$. So $A \cap D \neq \emptyset$, and therefore indeed $A \in \mathbf{K}_D$.

We finish the proof by showing that K_D is indeed compatible with D, or, in other words, that $D_{K_D} = D$. Indeed, infer that $D_{K_D} = \{f \in \mathscr{L} \colon \{f\} \in K_D\} = \{f \in \mathscr{L} \colon \{f\} \cap D \neq \emptyset\} = D$. ∎

Proof [Proof of Proposition 14] For Axiom K_0, consider any A in $K\rfloor E$. Then $\mathbb{I}_E A \in K$, whence $\mathbb{I}_E A \neq \emptyset$ since K satisfies Axiom K_0. Therefore indeed $A \neq \emptyset$.

For Axiom K_1, consider any A in $K\rfloor E$. Then $\mathbb{I}_E A \in K$, whence $\mathbb{I}_E A \setminus \{0\} \in K$ since K satisfies Axiom K_1. Since $\mathbb{I}_E f \neq 0 \Leftrightarrow f \neq 0$ for any gamble f on E, we find that $\mathbb{I}_E(A \setminus \{0\}) \in K$, whence indeed $A \setminus \{0\} \in K\rfloor E$.

For Axiom K_2, consider any f in $\mathscr{L}(E)_{>0}$. Then $\mathbb{I}_E f \in \mathscr{L}(\mathscr{X})_{>0}$, whence by Axiom K_2 $\{\mathbb{I}_E f\} \in K$. Therefore indeed $\{f\} \in K\rfloor E$.

For Axiom K_3, consider any A_1 and A_2 in $K\rfloor E$, and, for any f in A_1 and g in A_2, any $(\lambda_{f,g}, \mu_{f,g}) > 0$. Then $\mathbb{I}_E A_1 \in K$ and $\mathbb{I}_E A_2 \in K$, whence by Axiom K_3 $\{\lambda_{f,g} f + \mu_{f,g} g \colon f \in \mathbb{I}_E A_1, g \in \mathbb{I}_E A_2\} = \{\lambda_{f,g} \mathbb{I}_E f + \mu_{f,g} \mathbb{I}_E g \colon f \in A_1, g \in A_2\} = \mathbb{I}_E \{\lambda_{f,g} f + \mu_{f,g} g \colon f \in A_1, g \in A_2\} \in K$, where we identified $(\lambda_{\mathbb{I}_E f, \mathbb{I}_E g}, \mu_{\mathbb{I}_E f, \mathbb{I}_E g})$ with $(\lambda_{f,g}, \mu_{f,g})$, for any f in A_1 and g in A_2. Therefore indeed $\{\lambda_{f,g} f + \mu_{f,g} g \colon f \in A_1, g \in A_2\} \in K\rfloor E$.

For Axiom K_4, consider any A_1 in $K\rfloor E$ and any A_2 in $\mathscr{Q}$ such that $A_1 \subseteq A_2$. Then $\mathbb{I}_E A_1 \in K$ and $\mathbb{I}_E A_1 \subseteq \mathbb{I}_E A_2$, whence by Axiom K_4 $\mathbb{I}_E A_2 \in K$. Therefore indeed $A_2 \in R\rfloor E$. ∎

Proof [Proof of Proposition 15] For the first statement, consider any f in $\mathscr{L}(E)$, and infer the following chain of equivalences:

$$f \in D_K\rfloor E \Leftrightarrow \mathbb{I}_E f \in D_K \Leftrightarrow \{\mathbb{I}_E f\} \in K \Leftrightarrow \{f\} \in K\rfloor E \Leftrightarrow f \in D_{K\rfloor E},$$

where the first equivalence follows from Definition 12, the second one and the last one are due to Equation (3), and the third one follows from Definition 13.

For the second statement, consider any A in $\mathscr{Q}(\mathscr{L}(E))$ and the following chain of equivalences:

$$\begin{aligned} A \in K_D\rfloor E &\Leftrightarrow \mathbb{I}_E A \in K_D \Leftrightarrow \mathbb{I}_E A \cap D \neq \emptyset \\ &\Leftrightarrow (\exists f \in A)\mathbb{I}_E f \in D \\ &\Leftrightarrow A \cap D\rfloor E \neq \emptyset \Leftrightarrow A \in K_{D\rfloor E}, \end{aligned}$$

where the first equivalence follows from Definition 13, the second one and the last one are due to Proposition 10, and the fourth one follows from Definition 12.

We now turn to the last statement. By Theorem 11 we have that $K = \bigcap\{K_D : D \in \overline{\mathbf{D}}(K)\}$, implying that $A \in K \Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D$, for any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}))$. Therefore in particular, for any A in $\mathscr{Q}(\mathscr{L}(E))$,

$$\begin{aligned} A \in K\rfloor E \Leftrightarrow \mathbb{I}_E A \in K &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))\mathbb{I}_E A \in K_D \\ &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D\rfloor E \\ &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_{D\rfloor E} \\ &\Leftrightarrow A \in \bigcap\{K_{D\rfloor E} : D \in \overline{\mathbf{D}}(K)\}, \end{aligned}$$

where the first and third equivalences follow from Definition 13, and the fourth one follows from the already established second statement of this proposition. Therefore indeed $K\rfloor E = \bigcap\{K_{D\rfloor E} : D \in \overline{\mathbf{D}}(K)\}$. ■

Proof [Proof of Proposition 19] The result follows immediately, once we realise that $A_1 \neq \emptyset \Leftrightarrow A_1^* \neq \emptyset$, $f > 0 \Leftrightarrow f^* > 0$, $\lambda f + \mu g \in A_1 \Leftrightarrow \lambda f^* + \mu g^* \in A_1^*$, and $A_1 \subseteq A_2 \Leftrightarrow A_1^* \subseteq A_2^*$, for all f in $\mathscr{L}(\mathscr{X}_O)$ whose cylindrical extension is f^*, all A_1 and A_2 in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ whose cylindrical extensions are A_1^* and A_2^*, and all λ in μ in $\mathbb{R}$ such that $(\lambda, \mu) > 0$. ■

Proof [Proof of Proposition 20] For the first statement, observe that indeed

$$\begin{aligned} \mathrm{marg}_O D_K &= \{f \in \mathscr{L}(\mathscr{X}_O) : f \in D_K\} \\ &= \{f \in \mathscr{L}(\mathscr{X}_O) : \{f\} \in K\} \\ &= \{f \in \mathscr{L}(\mathscr{X}_O) : \{f\} \in \mathrm{marg}_O K\} = D_{\mathrm{marg}_O K}, \end{aligned}$$

where the second and last equalities follow from Equation (3), and the third one follows from Definition 18.

For the second statement, observe that

$$\begin{aligned} &\mathrm{marg}_O K_D \\ &= \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) : A \in K_D\} \\ &= \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) : A \cap D \neq \emptyset\} \\ &= \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) : A \cap \mathrm{marg}_O D \neq \emptyset\} \\ &= \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) : A \in K_{\mathrm{marg}_O D}\} = K_{\mathrm{marg}_O D}, \end{aligned}$$

where the first equality follows from Definition 18 and the second and penultimate equalities follow from Proposition 10.

We now turn to the last statement. By Theorem 11 we have that $K = \bigcap\{K_D : D \in \overline{\mathbf{D}}(K)\}$, implying that $A \in K \Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D$, for any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_{1:n}))$. Therefore in particular, for any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$,

$$\begin{aligned} A \in \mathrm{marg}_O K \Leftrightarrow A \in K &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_D \\ &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in \mathrm{marg}_O K_D \\ &\Leftrightarrow (\forall D \in \overline{\mathbf{D}}(K))A \in K_{\mathrm{marg}_O D} \\ &\Leftrightarrow A \in \bigcap\{K_{\mathrm{marg}_O D} : D \in \overline{\mathbf{D}}(K)\}, \end{aligned}$$

where the first and third equivalences follow from Definition 18, and the fourth one follows from the already established second statement of this proposition. Therefore indeed $\mathrm{marg}_O K = \bigcap\{K_{\mathrm{marg}_O D} : D \in \overline{\mathbf{D}}(K)\}$. ■

Proof [Proof of Proposition 21] We will first show that any coherent set of desirable gamble sets K' on $\mathscr{L}(\mathscr{X}_{1:n})$ that marginalises to K_O must be at least as informative as $\mathrm{ext}_{1:n}(K_O)$. To establish this, since K' marginalises to K_O, note that $A \in K_O \Leftrightarrow A \in K'$, for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$. Therefore, in particular, $A \in K_O \Rightarrow A \in K'$ for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$, so $K_O \subseteq K'$. This implies that indeed $\mathrm{ext}_{1:n}(K_O) = \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0} \cup K_O)) \subseteq \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0} \cup K')) = K'$, where the final equality holds because K' is coherent.

So we already know that any coherent set of desirable gamble sets that marginalises to K_O must be at least as informative as $\mathrm{ext}_{1:n}(K_O)$. It therefore suffices to prove that $\mathrm{ext}_{1:n}(K_O)$ is coherent and that it marginalises to K_O. To show that $\mathrm{ext}_{1:n}(K_O) = \mathrm{Rs}(\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0} \cup \mathscr{A}^{1:n}_{K_O}))$ is coherent, by Theorem 6 it suffices to show that K_O is a consistent assessment—that is, to show that $\emptyset \notin \mathscr{A}^{1:n}_{K_O}$ and $\{0\} \notin \mathrm{Posi}(\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0} \cup \mathscr{A}^{1:n}_{K_O})$. That this is indeed the case follows from the coherence of $K_O = \mathscr{A}^{1:n}_{K_O}$.

The proof is therefore complete if we can show that $\mathrm{marg}_O(\mathrm{ext}_{1:n}(K_O)) = K_O$. Since for any A in K_O it is obvious that both $A \in \mathrm{ext}_{1:n}(K_O)$ and $A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$, we see immediately that $K_O \subseteq \mathrm{marg}_O(\mathrm{ext}_{1:n}(K_O))$, so we concentrate on proving the converse inclusion. Consider any A in $\mathrm{marg}_O(\mathrm{ext}_{1:n}(K_O))$, meaning that both $A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and $A \in \mathrm{ext}_{1:n}(K_O)$. That $A \in \mathrm{ext}_{1:n}(K_O)$ implies that $B \setminus \mathscr{L}_{\leq 0} \subseteq A$ for some B in $\mathrm{Posi}(\mathscr{L}^{\mathrm{s}}_{>0} \cup K_O)$. Then there are m in $\mathbb{N}$, $A_1, \ldots, A_m$ in $\mathscr{L}^{\mathrm{s}}_{>0} \cup K_O$, and coefficients $\lambda^{f_{1:m}}_{1:m} > 0$ for all $f_{1:m}$ in $\times_{k=1}^m A_k$ such that $B = \left\{\sum_{k=1}^m \lambda_k^{f_{1:m}} f_k : f_{1:m} \in \times_{k=1}^m A_k\right\}$. Without loss of generality, assume that $A_1, \ldots, A_\ell \in K_O$ and $A_{\ell+1}, \ldots, A_m \in \mathscr{L}^{\mathrm{s}}_{>0}$ for some ℓ in $\{0, \ldots, m\}$. Consider the special subset $P :=$

$\{f_{1:m} \in \times_{k=1}^m A_k \colon \lambda_{1:\ell}^{f_{1:m}} = 0\}$ of $\times_{k=1}^m A_k$. If $P \neq \emptyset$, then for every element $g_{1:m}$ of P we have that $\sum_{k=1}^m \lambda_k^{g_{1:m}} g_k > 0$, so $B \cap \mathscr{L}_{>0} \neq \emptyset$. Since $B \setminus \mathscr{L}_{\leq 0} \subseteq A$, also $A \cap \mathscr{L}_{>0}(\mathscr{X}_O) \neq \emptyset$, whence $A \in K_O$ by coherence [more specifically, by Axioms K_2 and K_4]. Therefore, assume that $P = \emptyset$, and define the coefficients

$$\mu_k^{f_{1:m}} := \begin{cases} \lambda_k^{f_{1:m}} & \text{if } k \leq \ell \\ 0 & \text{if } k \geq \ell + 1 \end{cases}$$

for all $f_{1:m}$ in $\times_{k=1}^m A_k$ and k in $\{1, \dots, m\}$. Because $P = \emptyset$, for every $f_{1:m}$ in $\times_{k=1}^m A_k$ we have that $\mu_{1:\ell}^{f_{1:m}} = \lambda_{1:\ell}^{f_{1:m}} > 0$. Also, for every $f_{1:m}$ in $\times_{k=1}^m A_k$ and $k \geq \ell + 1$, the coefficient $\mu_k^{f_{1:m}}$ equals 0, so we identify $\mu_{1:\ell}^{f_{1:\ell}}$ with $\mu_{1:\ell}^{f_{1:m}}$. Then every element of $\left\{\sum_{k=1}^m \mu_k^{f_{1:m}} f_k \colon f_{1:m} \in \times_{k=1}^m A_k\right\} = \left\{\sum_{k=1}^\ell \mu_k^{f_{1:\ell}} f_k \colon f_{1:\ell} \in \times_{k=\ell+1}^m A_k\right\} \in \mathrm{Posi}(K_O) = K_O$ is dominated by an element of B. Therefore, by Lemma 27 below $B \in K_O$, whence by coherence, indeed also $A \in K_O$. ■

Lemma 27 *Consider any coherent set of desirable gamble sets K and any gamble sets A and B in $\mathscr{Q}$. If $A \in K$ and $(\forall f \in A)(\exists g \in B) f \leq g$, then $B \in K$.*

Proof Let $A := \{f_1, \dots, f_m\}$ for some m in $\mathbb{N}$, and denote the finite possibility space $\mathscr{X} = \{x_1, \dots, x_\ell\}$ for some ℓ in $\mathbb{N}$. Since $(\forall f \in A)(\exists g \in B) f \leq g$, we have that B is a superset of

$$\begin{aligned} B' :=& \left\{f_1 + \sum_{k=1}^\ell \mu_{k,1} \mathbb{I}_{\{x_k\}}, \dots, f_m + \sum_{k=1}^\ell \mu_{k,m} \mathbb{I}_{\{x_k\}}\right\} \\ =& \left\{f_j + \sum_{k=1}^\ell \mu_{k,j} \mathbb{I}_{\{x_k\}} \colon j \in \{1, \dots, m\}\right\} \end{aligned}$$

for some $\mu_{k,j} \geq 0$ for all k in $\{1, \dots, \ell\}$ and j in $\{1, \dots, m\}$. Use the definition of the Posi operator, with $A_1 := \{\mathbb{I}_{\{x_1\}}\} \in K, \dots, A_\ell := \{\mathbb{I}_{\{x_\ell\}}\} \in K$, $A_{\ell+1} := A \in K$, and for all $f_{1:\ell+1}^j := (\mathbb{I}_{\{x_1\}}, \mathbb{I}_{\{x_2\}}, \dots, \mathbb{I}_{\{x_\ell\}}, f_j) \in \times_{k=1}^{\ell+1} A_k$, let $\lambda_{1:\ell+1}^{f_{1:\ell+1}^j} := (\mu_{1,j}, \mu_{2,j}, \dots, \mu_{\ell,j}, 1) > 0$, to infer that

$$\begin{aligned} &\left\{\sum_{k=1}^{\ell+1} \lambda_k^{f_{1:\ell+1}^j} f_k^j \colon f_{1:\ell+1}^j \in \times_{k=1}^{\ell+1} A_k\right\} \\ &\quad = \left\{f_j + \sum_{k=1}^\ell \mu_{k,j} \mathbb{I}_{\{x_k\}} \colon j \in \{1, \dots, m\}\right\} = B' \end{aligned}$$

belongs to $\mathrm{Posi}(K)$. Because $B \supseteq B'$, we have that $B \in \mathrm{Rs}(\mathrm{Posi}(K))$. But since $K = \mathrm{Rs}(\mathrm{Posi}(K))$ by coherence, we infer that indeed $B \in K$. ■

Proof [Proof of Proposition 22] By Theorem 11 we have that $K_O = \bigcap\{K_{D_O} \colon D_O \in \overline{\mathbf{D}}(K_O)\}$, implying that $A \in K_O \Leftrightarrow (\forall D_O \in \overline{\mathbf{D}}(K_O)) A \in K_{D_O}$, for any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$. Therefore, for any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$,

$$\begin{aligned} A \in \mathrm{ext}_{1:n}(K_O) &\Leftrightarrow A \in K_O \\ &\Leftrightarrow (\forall D_O \in \overline{\mathbf{D}}(K_O)) A \in K_D \\ &\Leftrightarrow (\forall D_O \in \overline{\mathbf{D}}(K_O)) A \cap D_O \neq \emptyset \\ &\Leftrightarrow (\forall D_O \in \overline{\mathbf{D}}(K_O)) A \cap \mathrm{ext}_{1:n}^{\mathbf{D}}(D_O) \neq \emptyset \\ &\Leftrightarrow A \in \bigcap\{K_{\mathrm{ext}_{1:n}^{\mathbf{D}}(D_O)} \colon D \in \overline{\mathbf{D}}(K_O)\}, \end{aligned}$$

where the first equivalence holds because $\mathrm{ext}_{1:n}(K_O)$ marginalises to K_O, the third one because of Proposition 10, and the fourth one because $\mathrm{ext}_{1:n}^{\mathbf{D}}(D_O)$ marginalises to D_O. So indeed $\mathrm{ext}_{1:n}(K_O) = \bigcap\{K_{\mathrm{ext}_{1:n}^{\mathbf{D}}(D_O)} \colon D_O \in \overline{\mathbf{D}}(K_O)\}$. ■

Proof [Proof of Proposition 23] Consider the following chain of equalities:

$$\begin{aligned} &\mathrm{marg}_O(K_n \rfloor E_I) \\ &\quad = \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) \colon A \in K_n \rfloor E_I\} \\ &\quad = \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) \colon \mathbb{I}_{E_I} A \in K_n\} \\ &\quad = \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) \colon \mathbb{I}_{E_I} A \in \mathrm{marg}_{I \cup O} K_n\} \\ &\quad = \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) \colon A \in (\mathrm{marg}_{I \cup O} K_n) \rfloor E_I\} \\ &\quad = \mathrm{marg}_O((\mathrm{marg}_{I \cup O} K_n) \rfloor E_I), \end{aligned}$$

where the third equality holds because $\mathbb{I}_{E_I} A$ is a set of gambles on $\mathscr{X}_{I \cup O}$. ■

Proof [Proof of Proposition 25] To show that (i) implies (ii), consider any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and E_I in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$, and recall the following equivalences:

$$\begin{aligned} A \in K_n &\Leftrightarrow A \in \mathrm{marg}_O(K_n \rfloor E_I) && \text{by Definition 18 and (i)} \\ &\Leftrightarrow A \in K_n \rfloor E_I && \text{by Definition 18} \\ &\Leftrightarrow \mathbb{I}_{E_I} A \in K_n && \text{by Definition 13.} \end{aligned}$$

To show that (ii) implies (i), consider any E_I in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$, and recall the following equalities:

$$\begin{aligned} \mathrm{marg}_O(K_n \rfloor E_I) &= \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) \colon A \in K_n \rfloor E_I\} \\ &= \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) \colon \mathbb{I}_{E_I} A \in K_n\} \\ &= \{A \in \mathscr{Q}(\mathscr{L}(\mathscr{X}_O)) \colon A \in K_n\} \\ &= \mathrm{marg}_O K_n, \end{aligned}$$

where the first and last equalities follow from Definition 18, the second one from Definition 13, and the third one from (ii). ■

Proof [Proof of Theorem 26] We will first show that any coherent set of desirable gamble sets K' on $\mathscr{L}(\mathscr{X}_{1:n})$ that marginalises to K_O and that satisfies epistemic irrelevance of X_I to X_O must be at least as informative as $\mathrm{ext}_{1:n}^{\mathrm{irr}}(K_O)$. To this end, consider any B in $\mathscr{A}_{I \to O}^{\mathrm{irr}}$.

Then $B = \mathbb{I}_{E_I}A$ for some E_I in $\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$ and A in K_O. Since K' marginalises to K_O, infer that $A \in K'$. Furthermore, since K' satisfies epistemic irrelevance of X_I to X_O, by Proposition 25 also $B = \mathbb{I}_{E_I}A \in K'$. We conclude that $B \in \mathscr{A}^{\text{irr}}_{I\to O} \Rightarrow B \in K' \Leftrightarrow B \in \text{marg}_{I\cup O}K'$, for every B in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_{I\cup O}))$, so $\mathscr{A}^{\text{irr}}_{I\to O} \subseteq \text{marg}_{I\cup O}K'$. This implies that $\text{Rs}(\text{Posi}(\mathscr{L}^{\text{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\text{irr}}_{I\to O})) \subseteq \text{Rs}(\text{Posi}(\text{marg}_{I\cup O}K')) = \text{marg}_{I\cup O}K'$, where the equality follows from the fact that $\text{marg}_{I\cup O}K'$ is coherent by Proposition 19. Then $\text{ext}^{\text{irr}}_{1:n}(K_O) = \text{ext}_{1:n}(\text{Rs}(\text{Posi}(\mathscr{L}^{\text{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\text{irr}}_{I\to O}))) \subseteq \text{ext}_{1:n}(\text{marg}_{I\cup O}K')$ and since by Proposition 21 $\text{ext}_{1:n}(\text{marg}_{I\cup O}K')$ is the least informative coherent set of desirable gamble sets on $\mathscr{L}(\mathscr{X}_{1:n})$ that marginalises to $\text{marg}_{I\cup O}K'$, we have that $\text{ext}_{1:n}(\text{Rs}(\text{Posi}(\text{marg}_{I\cup O}K'))) \subseteq K'$. Therefore indeed $\text{ext}^{\text{irr}}_{1:n}(K_O) \subseteq K'$.

The proof of the first statement is therefore complete if we could show that $\text{ext}^{\text{irr}}_{1:n}(K_O)$ (i) is coherent, (ii) marginalises to K_O, and (iii) satisfies epistemic irrelevance of X_I to X_O.

For (i), it suffices to show that $\emptyset \notin \mathscr{A}^{\text{irr}}_{I\to O}$ and $\{0\} \notin K^{\text{irr}}_{I\cup O} = \text{Rs}(\text{Posi}(\mathscr{L}^{\text{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\text{irr}}_{I\to O}))$:[6] indeed, if this is the case, then by Theorem 6 $K^{\text{irr}}_{I\cup O}$ is a coherent set of desirable gamble sets on $\mathscr{L}(\mathscr{X}_{I\cup O})$, and then by Proposition 21 $\text{ext}^{\text{irr}}_{1:n}(K_O)$ is a coherent set of desirable gamble sets on $\mathscr{L}(\mathscr{X}_{1:n})$. So we will show that $\emptyset \notin \mathscr{A}^{\text{irr}}_{I\to O}$ and $\{0\} \notin K^{\text{irr}}_{I\cup O}$. That $\emptyset \notin \mathscr{A}^{\text{irr}}_{I\to O}$ is clear from Equation (6) because K_O is coherent. So we focus on proving that $\{0\} \notin K^{\text{irr}}_{I\cup O}$. Assume *ex absurdo* that $\{0\} \in K^{\text{irr}}_{I\cup O}$. By Lemma 28 below we would then infer that $\left\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in \{0\}\right\} = \{0\} \in K_O$, contradicting the coherence of K_O. Therefore indeed $\{0\} \notin K^{\text{irr}}_{I\cup O}$.

For (ii), we need to show that $A \in \text{ext}^{\text{irr}}_{1:n}(K_O) \Leftrightarrow A \in K_O$ for any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$. For necessity, consider any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and assume that $A \in \text{ext}^{\text{irr}}_{1:n}(K_O)$. By Lemma 28 then $\left\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in A\right\} \in K_O$. Since A is a set of gambles on $\mathscr{X}_O$, we infer $\left\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in A\right\} = \left\{\sum_{x_I\in\mathscr{X}_I} h \colon h \in A\right\} = \left\{|\mathscr{X}_I| h \colon h \in A\right\} = |\mathscr{X}_I| A$, whence by coherence, indeed $A \in K_O$. For sufficiency, consider any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and assume that $A \in K_O$. Then $A = \mathbb{I}_{\mathscr{X}_I}A$ and $\mathscr{X}_I \in \mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$, so $A \in \mathscr{A}^{\text{irr}}_{I\to O}$. Therefore indeed $A \in \text{ext}^{\text{irr}}_{1:n}(K_O)$.

For (iii), by Proposition 25 it suffices to show that $A \in \text{ext}^{\text{irr}}_{1:n}(K_O) \Leftrightarrow \mathbb{I}_{E_I}A \in \text{ext}^{\text{irr}}_{1:n}(K_O)$, for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and E_I in $\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$. For necessity, consider any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and any E_I in $\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$, and assume that $A \in \text{ext}^{\text{irr}}_{1:n}(K_O)$. Since we just have shown that $\text{marg}_O \text{ext}^{\text{irr}}_{1:n}(K_O) = K_O$, this implies that $A \in K_O$, whence indeed $\mathbb{I}_{E_I}A \in \mathscr{A}^{\text{irr}}_{I\to O} \subseteq \text{ext}^{\text{irr}}_{1:n}(K_O)$. For sufficiency, consider any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_O))$ and any E_I in $\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$, and assume that $\mathbb{I}_{E_I}A \in \text{ext}^{\text{irr}}_{1:n}(K_O)$. Since by Proposition 21 $\text{ext}^{\text{irr}}_{1:n}(K_O)$ marginalises to $K^{\text{irr}}_{I\cup O}$, this implies that $\mathbb{I}_{E_I}A \in K^{\text{irr}}_{I\cup O}$. Use Lemma 28 to infer that then $\left\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in \mathbb{I}_{E_I}A\right\} = \left\{\sum_{x_I\in\mathscr{X}_I} \mathbb{I}_{E_I}h(x_I,\bullet) \colon h \in A\right\} = \left\{|E_I| h \colon h \in A\right\} = |E_I| A \in K_O$, whence by coherence indeed $A \in K_O$.

The second statement is a direct application of Proposition 22. ∎

6. These two conditions are equivalent to $\emptyset \notin \mathscr{A}^{\text{irr}}_{I\to O}$ and $\{0\} \notin \text{Posi}(\mathscr{L}^{\text{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\text{irr}}_{I\to O})$.

Lemma 28 *Consider any disjoint and non-empty subsets I and O of $\{1,\dots,n\}$, and any coherent set of desirable gamble sets K_O on $\mathscr{L}(\mathscr{X}_O)$. Then*

$$A \in \text{Rs}(\text{Posi}(\mathscr{L}^{\text{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\text{irr}}_{I\to O})) \Rightarrow \left\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in A\right\} \in K_O,$$

for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_{I\cup O}))$.

Proof Consider any A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}_{I\cup O}))$ and assume that $A \in \text{Rs}(\text{Posi}(\mathscr{L}^{\text{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\text{irr}}_{I\to O}))$. Then $B \setminus \mathscr{L}_{\leq 0} \subseteq A$ for some B in $\text{Posi}(\mathscr{L}^{\text{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\text{irr}}_{I\to O})$, implying that $B = \left\{\sum_{k=1}^m \lambda_k^{f_{1:m}} f_k \colon f_{1:m} \in \times_{k=1}^m A_k\right\}$ for some m in $\mathbb{N}$, $A_1, \dots, A_m$ in $\mathscr{L}^{\text{s}}(\mathscr{X}_{I\cup O})_{>0} \cup \mathscr{A}^{\text{irr}}_{I\to O}$, and coefficients $\lambda_{1:m}^{f_{1:m}} > 0$ for all $f_{1:m}$ in $\times_{k=1}^m A_k$. Without loss of generality, assume that $A_1,\dots,A_\ell \in \mathscr{A}^{\text{irr}}_{I\to O}$ and $A_{\ell+1},\dots,A_m \in \mathscr{L}^{\text{s}}(\mathscr{X}_{I\cup O})_{>0}$ for some ℓ in $\{0,\dots,m\}$. Consider the special subset $P := \left\{f_{1:m} \in \times_{k=1}^m A_k \colon \lambda_{1:\ell}^{f_{1:m}} = 0\right\}$ of $\times_{k=1}^m A_k$. If $P \neq \emptyset$, then for every element $g_{1:m}$ of P we have that $\sum_{k=1}^m \lambda_k^{g_{1:m}} g_k > 0$, so $B \cap \mathscr{L}(\mathscr{X}_{I\cup O})_{>0} \neq \emptyset$, and therefore also $A \cap \mathscr{L}(\mathscr{X}_{I\cup O})_{>0} \neq \emptyset$, whence $\left\{\sum_{x_I\in\mathscr{X}_I} h(x_I,\bullet) \colon h \in A\right\} \in K_O$ by the coherence of K_O [more specifically, by Axioms K_2 and K_4]. So assume that $P = \emptyset$, and define the coefficients

$$\mu_k^{f_{1:m}} := \begin{cases} \lambda_k^{f_{1:m}} & \text{if } k \leq \ell \\ 0 & \text{if } k \geq \ell+1 \end{cases}$$

for all $f_{1:m}$ in $\times_{k=1}^m A_k$ and k in $\{1,\dots,m\}$. Because $P = \emptyset$, for every $f_{1:m}$ in $\times_{k=1}^m A_k$ we have that $\mu_{1:\ell}^{f_{1:m}} = \lambda_{1:\ell}^{f_{1:m}} > 0$. Also, for every $f_{1:m}$ in $\times_{k=1}^m A_k$ and $k \geq \ell+1$, the coefficient $\mu_k^{f_{1:m}}$ equals 0, so we identify $\mu_{1:\ell}^{f_{1:m}}$ with $\mu_{1:\ell}^{f_{1:\ell}}$. Then every element of $B' := \left\{\sum_{k=1}^m \mu_k^{f_{1:m}} f_k \colon f_{1:m} \in \times_{k=1}^m A_k\right\} = \left\{\sum_{k=1}^\ell \mu_k^{f_{1:\ell}} f_k \colon f_{1:\ell} \in \times_{k=1}^\ell A_k\right\}$ is dominated by an element of B. For every k in $\{1,\dots,\ell\}$ the gamble set A_k belongs to $\mathscr{A}^{\text{irr}}_{I\to O}$, so we may write $A_k = \mathbb{I}_{E_k}A_{O,k}$ with $E_k \in \mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$ and $A_{O,k} \in K_O$. Therefore $|A_k| = |A_{O,k}|$, and every f_k in A_k can be uniquely written as $f_k = \mathbb{I}_{E_k}g_k$ with g_k in $A_{O,k}$. So for every $f_{1:\ell}$ in $\times_{k=1}^\ell A_k$ there is a unique $g_{1:\ell}$ in $\times_{k=1}^\ell A_{O,k}$ such that $f_k = \mathbb{I}_{E_k}g_k$ for every k in

$\{1,\ldots,\ell\}$. For every $f_{1:\ell}$ in $\bigtimes_{k=1}^{\ell} A_k$ and its corresponding unique $g_{1:\ell}$ in $\bigtimes_{k=1}^{\ell} A_{O,k}$, we define $\mu_{1:\ell}^{g_{1:\ell}} := \mu_{1:\ell}^{f_{1:\ell}}$. Therefore $B' = \{\sum_{k=1}^{\ell} \mu_k^{g_{1:\ell}} \mathbb{I}_{E_k} g_k \colon g_{1:\ell} \in \bigtimes_{k=1}^{\ell} A_{O,k}\}$, and hence

$$\begin{aligned}\left\{\sum_{x_I \in \mathscr{X}_I} h(x_I, \bullet) \colon h \in B'\right\} \\ &= \left\{\sum_{x_I \in \mathscr{X}_I} \sum_{k=1}^{\ell} \mu_k^{g_{1:\ell}} \mathbb{I}_{E_k} g_k(x_I, \bullet) \colon g_{1:\ell} \in \bigtimes_{k=1}^{\ell} A_{O,k}\right\} \\ &= \left\{\sum_{k=1}^{\ell} \mu_k^{g_{1:\ell}} |E_k| g_k(x_I, \bullet) \colon g_{1:\ell} \in \bigtimes_{k=1}^{\ell} A_{O,k}\right\}\end{aligned}$$

belongs to $\text{Posi}(K_O) = K_O$. Since every element of B' is dominated by an element of B, we have that every element of $\{\sum_{x_I \in \mathscr{X}_I} h(x_I, \bullet) \colon h \in B'\}$ is dominated by an element of $\{\sum_{x_I \in \mathscr{X}_I} h(x_I, \bullet) \colon h \in B\}$, so by Lemma 27 $\{\sum_{x_I \in \mathscr{X}_I} h(x_I, \bullet) \colon h \in B\} \in K_O$. By K_4 we have that also indeed $\{\sum_{x_I \in \mathscr{X}_I} h(x_I, \bullet) \colon h \in A\} \in K_O$. ■