Modelling Indifference with Choice Functions

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Abstract

We investigate how to model indifference with choice functions. We take the coherence axioms for choice functions proposed by Seidenfeld, Schervisch and Kadane as a source of inspiration, but modify them to strengthen the connection with desirability. We discuss the properties of choice functions that are coherent under our modified set of axioms and the connection with desirability. Once this is in place, we present an axiomatisation of indifference in terms of desirability. On this we build our characterisation of indifference in terms of choice functions.

Keywords. Choice function, coherence, indifference, set of desirable gambles, maximality, E-admissibility.

1 Introduction

The language of classical probability—(probability) mass functions, say—is insufficiently versatile and powerful to describe certain aspects of beliefs, such as indecision. Imprecise probability uncertainty models, such as coherent lower previsions and coherent sets of desirable gambles, are often used to remedy this. Coherent sets of desirable gambles in particular play a crucial role in theories of conservative reasoning [16], predictive inference [10], credal networks [6], and so on. They have many advantages, such as mathematical elegance and the lack of problems for conditioning on an event with (lower) probability zero. However, they are not capable of modelling beliefs corresponding to 'or' statements, such as the belief that a coin has two equal sides of unknown type—twice heads or twice tails. It turns out such more general types of assessments can be modelled with choice functions.

To allow for incomparability, Seidenfeld, Schervisch and Kadane [23] introduce axioms for rational choice expressed by choice functions that are a weakened version of the ones suggested by Rubin [18]. We modify them slightly, in order to allow for Walley–Sen maximality [28, 26] to be coherent, and we drop their Archimedean continuity axiom to allow for a more direct connection with coherent sets of desirable gambles. We introduce our notion of coherence for choice functions in Section 2. We work with abstract vectors (called options), rather than horse lotteries or gambles: this will allow us to deal with indifference without too many mathematical difficulties, later on in this paper. Because we are interested in conservative reasoning with coherent choice functions, we introduce an 'is not more informative than' ordering, which allows us to consider the most conservative choice function compatible with an assessment as an infimum associated with this partial order.

In Section 3, we relate our theory of coherent choice functions to coherent sets of desirable options, and identify the most conservative coherent choice function compatible with a coherent set of desirable options as the one associated with Walley–Sen maximality: it selects the undominated options under the strict partial order generated by a coherent set of desirable options, and is therefore fully based on binary choice.

In Section 4, we show that there are other general classes of coherent choice functions not based on binary choice, and we relate them to each other.

An important aspect of any uncertainty theory is how it deals with indifference. Adding indifference to the picture typically reduces the complexity of the modelling effort. Also, knowing how to model indifference opens up a path towards modelling symmetry, which has many important practical applications. As an example of both aspects, the permutation symmetry that lies behind exchangeability has important applications in statistical modelling, and reduces the complexity of the modelling effort, as is exemplified by de Finetti's representation theorem [12]. Our treatment here lays the foundation for dealing with, say, exchangeability for choice functions.

In Section 5, we give an intuitive definition of indifference for choice functions that reduces to the existing account for sets of desirable gambles (options). We exhibit the power and simplicity of our definition of indifference in an interesting example.

2 Choice functions on option sets

Consider a real vector space $\mathcal{V}$, provided with the vector addition $+$ and scalar multiplication. We denote by 0 the additive identity, or null vector. For any subsets O_1 and O_2 of $\mathcal{V}$ and any λ in $\mathbb{R}$, we define $\lambda O_1 := \{\lambda u : u \in O_1\}$ and $O_1 + O_2 := \{u + v : u \in O_1, v \in O_2\}$. Elements u of $\mathcal{V}$ are intended as abstract representations of *options* amongst which a subject can express his preferences, by specifying, as we shall see below, choice functions. Mostly, options will be real-valued maps on the possibility space, also called *gambles*. We want to work with the more abstract notion of options—elements of some general vector space—because in Section 5, we will need choice functions defined on *equivalence classes* of options. These constitute a vector space—and hence are abstract options themselves—but can no longer be interpreted easily and directly as gambles.

We denote by $\mathcal{Q}(\mathcal{V})$ the set of all non-empty *finite* subsets of $\mathcal{V}$, a strict subset of the power set of $\mathcal{V}$. Elements O of $\mathcal{Q}(\mathcal{V})$ are the option sets amongst which a subject can choose his preferred options. When it is clear what vector space of options we are talking about, we will omit explicit mention of $\mathcal{V}$ and simply write $\mathcal{Q}$.

Definition 1. A *choice function* C on $\mathcal{Q}$ is a map

$$C\colon \mathcal{Q} \to \mathcal{Q} \cup \{\varnothing\} \colon O \mapsto C(O) \text{ such that } C(O) \subseteq O.$$

We collect all choice functions in the set $\mathcal{C}$.

The idea underlying this definition is that a choice function C selects the set $C(O)$ of 'best' options in the *option set* O. Our definition resembles the one commonly used in the literature [1, 23, 25], except for a not unusual restriction to *finite* option sets [13, 19, 24].

2.1 Rationality axioms

Seidenfeld et al. [23, Section 3] call a choice function C *coherent* if there is a non-empty set of probability-utility pairs $\mathcal{S}$ such that $C(O)$ is the set of options in O that maximise expected utility for some probability-utility pair in $\mathcal{S}$. They also provide an axiomatisation for this type of coherence, based on the one for binary preferences [2]. One of their axioms is an 'Archimedean' continuity condition, and another one is a convexity condition, necessary for the connection with a set of probability-utility pairs.

We prefer to define coherence directly in terms of axioms, without reference to probabilities and utilities. In such a context, we see no compelling reason to adopt an Archimedean axiom, all the more so because we are interested in establishing the connection between choice functions and Walley's sets of desirable gambles Walley [29], which violate this axiom. Furthermore, the convexity condition does not allow for choice functions that select the undominated options under some partial ordering, which is something we find natural, and shall need later on.

We will weaken their axioms in Section 2.1.2 by dropping the Archimedean condition and by replacing their convexity condition with a weaker variant. On the other hand, our second axiom is a strengthened version of theirs, needed for the conditioning we intend to discuss in a later paper.

2.1.1 Some useful definitions

We call $\mathbb{N}$ the set of all (positive) integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Also, we call $\mathbb{R}_{>0}$ the set of all (strictly) positive real numbers, and $\mathbb{R}_{\geq 0} := \mathbb{R}_{>0} \cup \{0\}$.

Given any subset O of $\mathcal{V}$, we define the *linear hull* $\operatorname{span}(O)$ as the set of all finite linear combinations of elements of O:

$$\operatorname{span}(O) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}, u_k \in O \right\} \subseteq \mathcal{V},$$

the *positive hull* $\operatorname{posi}(O)$ as the set of all positive finite linear combinations of elements of O:

$$\operatorname{posi}(O) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, u_k \in O \right\} \subseteq \mathcal{V},$$

and the *convex hull* $\operatorname{CH}(O)$ as the set of convex combinations of elements of O:

$$\operatorname{CH}(O) := \left\{ \sum_{k=1}^{n} \alpha_k u_k : n \in \mathbb{N}, \alpha_k \in \mathbb{R}_{\geq 0}, \sum_{k=1}^{n} \alpha_k = 1, u_k \in O \right\} \subseteq \mathcal{V}.$$

A subset O of $\mathcal{V}$ is called a *convex cone* if it is closed under positive finite linear combinations, i.e. if $\operatorname{posi}(O) = O$. A convex cone $\mathcal{K}$ is called *proper* if $\mathcal{K} \cap -\mathcal{K} = \{0\}$.

With any proper convex cone $\mathcal{K} \subseteq \mathcal{V}$ such that $0 \in \mathcal{K}$, we associate an ordering $\leq$ on $\mathcal{V}$, defined for all u and v in $\mathcal{V}$ as follows:

$$u \leq_{\mathcal{K}} v \Leftrightarrow v - u \in \mathcal{K} \Leftrightarrow 0 \leq_{\mathcal{K}} v - u \Leftrightarrow u - v \leq_{\mathcal{K}} 0.$$

We also write $u \geq_{\mathcal{K}} v$ for $v \leq_{\mathcal{K}} u$. The ordering $\leq_{\mathcal{K}}$ is actually a *vector ordering*: it is a partial order (reflexive, antisymmetric and transitive) that satisfies the following two characteristic properties:

$$u_1 \leq_{\mathcal{K}} u_2 \Leftrightarrow u_1 + v \leq_{\mathcal{K}} u_2 + v; \tag{1}$$

$$u_1 \leq_{\mathcal{K}} u_2 \Leftrightarrow \lambda u_1 \leq_{\mathcal{K}} \lambda u_2, \tag{2}$$

for all u_1, u_2, v in $\mathcal{V}$ and λ in $\mathbb{R}_{>0}$. Conversely, given a vector ordering $\leq$, the proper convex cone $\mathcal{K}$ from which it is derived can always be retrieved by $\mathcal{K} = \{u \in \mathcal{V} : u \geq 0\}$. When the abstract options are gambles, $\mathcal{K}$ will usually be the non-negative orthant, and the ordering $\leq$ is then pointwise. When the options are equivalence classes, as in Section 5.2, the ordering will be the induced ordering on equivalence classes, as defined in Eq. (10).

The vector space of options $\mathcal{V}$, ordered by the vector ordering $\leq_{\mathcal{K}}$, is called an *ordered vector space* $\langle \mathcal{V}, \leq_{\mathcal{K}} \rangle$. We shall refrain from explicitly mentioning the actual proper convex cone $\mathcal{K}$ we are using, and simply write $\mathcal{V}$ to mean the ordered vector space, and $\leq$ for the associated vector ordering.

Finally, with any vector ordering $\leq$, we associate the strict partial ordering $<$ as follows:

$u < v \Leftrightarrow (u \leq v$ and $u \neq v) \Leftrightarrow v - u \in \mathcal{K} \setminus \{0\}$ for all u, v in $\mathcal{V}$.

We call u *positive* if $u > 0$, and collect all positive options in the convex cone $\mathcal{V}_{>0} := \mathcal{K} \setminus \{0\}$.

2.1.2 Rationality axioms for choice functions

Definition 2. We call a choice function C on $\mathcal{Q}(\mathcal{V})$ *coherent* if for all O, O_1, O_2 in $\mathcal{Q}$, u, v in $\mathcal{V}$ and λ in $\mathbb{R}_{>0}$:

C_1. $C(O) \neq \emptyset$;
C_2. if $u < v$ then $\{v\} = C(\{u, v\})$;
C_3. a. if $C(O_2) \subseteq O_2 \setminus O_1$ and $O_1 \subseteq O_2 \subseteq O$ then $C(O) \subseteq O \setminus O_1$;
b. if $C(O_2) \subseteq O_1$ and $O \subseteq O_2 \setminus O_1$ then $C(O_2 \setminus O) \subseteq O_1$;
C_4. a. if $O_1 \subseteq C(O_2)$ then $\lambda O_1 \subseteq C(\lambda O_2)$;
b. if $O_1 \subseteq C(O_2)$ then $O_1 + \{u\} \subseteq C(O_2 + \{u\})$;
C_5. if $O \subseteq \mathrm{CH}(\{u, v\})$ then $\{u, v\} \cap C(O \cup \{u, v\}) \neq \emptyset$.[1]

We collect all coherent choice functions on $\mathcal{V}$ in the set $\bar{\mathcal{C}}$.

Parts C_3a and C_3b of Axiom C_3 are respectively known as Sen's condition α and Aizerman's condition. They are more commonly written as, respectively:

$$(O_1 \cap C(O_2) = \emptyset \text{ and } O_1 \subseteq O_2 \subseteq O) \Rightarrow O_1 \cap C(O) = \emptyset \quad (3)$$

and

$$(O_1 \cap C(O_2) = \emptyset \text{ and } O \subseteq O_1) \Rightarrow O_1 \cap C(O_2 \setminus O) = \emptyset \quad (4)$$

for all O, O_1, O_2 in $\mathcal{Q}$.

Proposition 1. *The following statements hold for any coherent choice function C:*

(i) $\lambda C(O) + \{u\} = C(\lambda O + \{u\})$ *for all O in $\mathcal{Q}$, λ in $\mathbb{R}_{>0}$ and u in $\mathcal{V}$;*
(ii) *for all u_1, u_2 in $\mathcal{V}$ such that $u_1 \leq u_2$, all O in $\mathcal{Q}$ and all v in $O \setminus \{u_1, u_2\}$:*
  a. *if $u_2 \in O$ and $v \notin C(O \cup \{u_1\})$ then $v \notin C(O)$;*
  b. *if $u_1 \in O$ and $v \notin C(O)$ then $v \notin C(\{u_2\} \cup O \setminus \{u_1\})$;*
(iii) *C is* insensitive to the omission of non-chosen options [9, Definition 11]*: $C(O') = C(O)$ for all O, O' in $\mathcal{Q}$ such that $C(O) \subseteq O' \subseteq O$;*
(iv) *$C(C(O)) = C(O)$ for all O in $\mathcal{Q}$.*

[1] This axiom is not needed to prove the results in this paper, and all results remain valid without it. We include it because it seems reasonable: the version with rational convex combinations can be derived from our other axioms, so C_5 amounts to requiring some very weak continuity. More importantly, this axiom is instrumental for the proofs of some results not included in this paper due to space limitations; because of this, we prefer to keep a unified set of axioms in all of our work in this topic.

For Bradley [3], any choice function must at least satisfy property (iv). Seidenfeld *et al.* [23] impose the two properties (ii)a and (ii)b as rationality axioms [23, Axiom 4]. Our proofs for them rely quite heavily on, amongst other things, Axiom C_2, which is a strengthened version of another of their rationality axioms. This does not imply, however, that our rationality axioms are stronger than theirs, since we have dropped their Archimedean axiom [23, Axiom 3], and replaced their convexity axiom [23, Axiom 2b] by our strictly weaker variant C_5.

2.2 The 'is not more informative than' relation

Because we are interested in *conservative reasoning* with choice functions, we look for the implications of a given assessment that are as 'uninformative' as possible. Therefore, we need some binary relation $\sqsubseteq$ on $\mathcal{C}$, having the specific interpretation of being 'not more informative than', or, in other words, 'at least as uninformative as'.

Definition 3. Given two choice functions C_1 and C_2 in $\mathcal{C}$, we call C_1 *not more informative than* C_2—and we write $C_1 \sqsubseteq C_2$—if $(\forall O \in \mathcal{Q}) C_1(O) \supseteq C_2(O)$.

This intuitive way of ordering choice functions is also used by Bradley [3], and in earlier work by the authors [27]. The underlying idea is that a choice function is more informative when it chooses more specifically, or restrictively, amongst the available options.

Since by definition $\sqsubseteq$ is a product ordering of set inclusions, the following result is immediate [5].

Proposition 2. *The structure $(\mathcal{C}; \sqsubseteq)$ is a complete lattice:*

(i) *it is a* partially ordered set*, or* poset*, meaning that the binary relation $\sqsubseteq$ on $\mathcal{C}$ is* reflexive*,* antisymmetric *and* transitive*;*
(ii) *for any subset $\mathcal{C}'$ of $\mathcal{C}$, its infimum $\inf \mathcal{C}'$ and its supremum $\sup \mathcal{C}'$ with respect to the ordering $\sqsubseteq$ exist in $\mathcal{C}$, and are given by $\inf \mathcal{C}'(O) = \bigcup_{C \in \mathcal{C}'} C(O)$ and $\sup \mathcal{C}'(O) = \bigcap_{C \in \mathcal{C}'} C(O)$ for all O in $\mathcal{Q}$.*

The idea is that $\inf \mathcal{C}'$ is the most informative model that is not more informative than any of the models in $\mathcal{C}'$, and $\sup \mathcal{C}'$ the least informative model that is not less informative than any of the models in $\mathcal{C}'$.

We also consider the poset $(\bar{\mathcal{C}}; \sqsubseteq)$, where $\bar{\mathcal{C}} \subseteq \mathcal{C}$ inherits the partial order $\sqsubseteq$ from $\mathcal{C}$.

Proposition 3. *$(\bar{\mathcal{C}}; \sqsubseteq)$ is complete infimum-semilattice: $\bar{\mathcal{C}}$ is closed under arbitrary non-empty infima, so $\inf \mathcal{C}' \in \bar{\mathcal{C}}$ for any non-empty subset $\mathcal{C}'$ of $\bar{\mathcal{C}}$.*

3 Relation with sets of desirable options

Choice functions cannot be characterised using pairwise comparison of options,[2] meaning that a binary relation on options does not uniquely determine a choice function. In this section, we study the ones that do correspond to a pairwise comparison of options.

3.1 Sets of desirable options

Sets of desirable options are a generalisation of *sets of desirable gambles*. Gambles are real-valued maps on a possibility space $\mathcal{X}$, interpreted as uncertain rewards. Such gambles can be seen as vectors in the vector space $\mathbb{R}^{\mathcal{X}}$. Here we generalise this notion by looking at a general (abstract) vector space $\mathcal{V}$ of (abstract) options, rather than gambles. We shall see that sets of desirable options amount to a pairwise comparison of options and therefore correspond to a special kind of choice functions.

A set of desirable options D is simply a subset of the vector space of options $\mathcal{V}$. We collect all sets of desirable options in the set $\mathcal{D}$. As we did for choice functions, we pay special attention to *coherent* sets of desirable options.

Definition 4. A set of desirable options D is called coherent if for all u and v in $\mathcal{V}$ and λ in $\mathbb{R}_{>0}$:

D_1. $0 \notin D$;
D_2. $\mathcal{V}_{\succ 0} \subseteq D$;
D_3. if $u \in D$ then $\lambda u \in D$;
D_4. if $u, v \in D$ then $u + v \in D$.

We collect all coherent sets of desirable options in the set $\bar{\mathcal{D}}$.

Axioms D_3 and D_4 turn coherent sets of desirable options D into cones—$\operatorname{posi}(D) = D$. They include the positive options due to Axiom D_2, and do not contain the zero option due to Axiom D_1. As an immediate consequence, their intersection with $\mathcal{V}_{\prec 0} := -\mathcal{V}_{\succ 0}$ is empty. As usual, we may associate with the cone D a strict partial order $\prec$ on $\mathcal{V}$, by letting $u \prec v \Leftrightarrow 0 \prec v - u \Leftrightarrow v - u \in D$, so $D = \{u \in \mathcal{V} : 0 \prec u\}$ [8, 16].

3.2 The 'is not more informative than' relation

As for choice functions, sets of desirable options can be ordered according to a 'not more informative than' relation.

Definition 5. Given two sets of desirable options D_1, D_2 in $\mathcal{D}$, we call D_1 *not more informative than* D_2 when $D_1 \subseteq D_2$.

Because the ordering of sets of desirable options $\subseteq$ is just set inclusion, it is a partial ordering on $\mathcal{D}$, and the poset $(\mathcal{D}; \subseteq)$ is a complete lattice, with supremum operator $\bigcup$, and infimum operator $\bigcap$.

[2] An equivalent representation of a coherent choice function C is a binary relation $\lhd$ on $\mathcal{Q}$—on *sets of options*—defined through $O_1 \lhd O_2 \Leftrightarrow O_1 \cap C(O_1 \cup O_2) = \emptyset$ for all O_1, O_2 in $\mathcal{Q}$. This binary relation $\lhd$ is a strict partial order on $\mathcal{Q}$ [14].

Proposition 4. *$(\bar{\mathcal{D}}; \subseteq)$ is a complete infimum-semilattice, or alternatively, $\bar{\mathcal{D}}$ is an intersection structure—closed under arbitrary non-empty intersections.*

Proposition 4 guarantees us that there is a unique least informative set of desirable options in $\bar{\mathcal{D}}$, called the *vacuous set of desirable options* D_{v}.

Proposition 5. *The least informative (smallest) set of desirable options D_{v} is given by $D_{\mathrm{v}} := \mathcal{V}_{\succ 0}$.*

It will be useful to also consider the maximally informative, or *maximal* coherent sets of desirable options.[3] They are the undominated elements of the complete infimum-semilattice $(\bar{\mathcal{D}}; \subseteq)$; we collect them into a set $\hat{\mathcal{D}}$:

$$\hat{\mathcal{D}} := \{D \in \bar{\mathcal{D}} : (\forall D' \in \bar{\mathcal{D}})(D \subseteq D' \Rightarrow D = D')\}.$$

First we prove a useful proposition that will allow us to characterise these maximal elements very elegantly.

Proposition 6. *Given any coherent set of desirable options D and any non-zero option $u \notin D$, then $\operatorname{posi}(D \cup \{-u\})$ is a coherent set of desirable options.*

Proposition 7. *A coherent set of desirable options D is maximal if and only if*

$$(\forall u \in \mathcal{V} \setminus \{0\})(u \in D \text{ or } -u \in D). \tag{5}$$

Proposition 8. *For any coherent set of desirable options D, its set of dominating maximal coherent sets of desirable options $\hat{\mathcal{D}}_D := \{\hat{D} \in \hat{\mathcal{D}} : D \subseteq \hat{D}\}$ is non-empty.*

Proposition 9. *$(\bar{\mathcal{D}}; \subseteq)$ is dually atomic, meaning that any coherent set of desirable options D is the infimum of its non-empty set of dominating maximal coherent sets of desirable options $\hat{\mathcal{D}}_D$: $D = \inf \hat{\mathcal{D}}_D$.*

3.3 Connection between choice functions and sets of desirable options

In this section, we establish a connection between choice functions and sets of desirable options.

Definition 6. Given a choice functions C, we say that an option v is *chosen above* some option u whenever $u \notin C(\{u, v\})$, or equivalently whenever $v \neq u$ and $\{v\} = C(\{u, v\})$. Similarly, given a set of desirable options D, we say that an option v is *preferred to* some option u whenever $v - u \in D$, or equivalently, $u \prec v$. We call a choice function C and a set of desirable options D *compatible* when

$$u \notin C(\{u, v\}) \Leftrightarrow v - u \in D \Leftrightarrow u \prec v \text{ for all } u, v \in \mathcal{V}.$$

Compatibility means that the behaviour of a choice function *restricted to pairs of options* reflects the behaviour of a

[3] The discussion in the rest of this section is based on similar discussions about sets of desirable gambles [8, 4, 17]. We repeat the details here *mutatis mutandis* to make the paper more self-contained.

set of desirable options.[4] So, a choice function C will have at most one compatible set of desirable options, whereas conversely, a set of desirable options D may have many compatible choice functions: compatibility only directly influences the behaviour of a choice function on doubletons.

3.3.1 From choice functions to desirability

We begin by studying the properties of the set of desirable options compatible with a given coherent choice function.

Proposition 10. *Given a coherent choice function C in $\bar{\mathcal{C}}$, there is a unique compatible coherent set of desirable options D_C, given by $D_C := \{u \in \mathcal{V} : 0 \notin C(\{0,u\})\}$.*

3.3.2 From desirability to choice functions

We collect in $\bar{\mathcal{C}}_D$ all the compatible coherent choice functions with the given coherent set of desirable options D:

$$\begin{aligned}\bar{\mathcal{C}}_D :=&\{C \in \bar{\mathcal{C}} : (\forall u,v \in \mathcal{V})(v \notin C(\{u,v\}) \Leftrightarrow u-v \in D)\}\\ =&\{C \in \bar{\mathcal{C}} : D_C = D\}.\end{aligned}$$

Proposition 11. *Given a coherent set of desirable options D, the infimum—most uninformative element—$\inf\bar{\mathcal{C}}_D$ of its set of compatible coherent choice functions $\bar{\mathcal{C}}_D$ is the coherent choice function C_D, defined by*

$$\begin{aligned}C_D(O) :=&\{u \in O : (\forall v \in O) v-u \notin D\}\\ =&\{u \in O : (\forall v \in O) u \nprec v\} \text{ for all } O \text{ in } \mathcal{Q}. \qquad (6)\end{aligned}$$

The coherent choice function C_D is the least informative choice function that is compatible with a coherent set of desirable options D: it is based on the binary ordering represented by D *and nothing else*. As we shall see in Proposition 17, there are other coherent choice functions C compatible with D, but they encode more information than just the binary ordering represented by D. Proposition 11 is especially interesting because it shows that the most conservative choice function based on a strict partial order of options, is the choice function based on *maximality*—the one that selects the *undominated* options under the strict partial order $\prec$ associated with a coherent set of desirable options D. Any choice function that is based on maximality under such a strict partial order is coherent.

Proposition 3 guarantees that there is a unique smallest—least informative—coherent choice function. We shall call it the *vacuous choice function*, and denoted it by C_{v}.

Proposition 12. *The vacuous choice function C_{v} is given by $C_{\mathrm{v}}(O) = C_{D_{\mathrm{v}}}(O) = \{u \in O : (\forall v \in O) u \nless v\}$ for all O in $\mathcal{Q}$. It selects from any set of options the ones that are undominated under the strict vector ordering $<$.*

Example 1. Consider, as a simple example, the case that the vector ordering is total, meaning that for any u,v in $\mathcal{V}$, either $u < v$, $v < u$ or $u = v$. It then follows from Proposition 12 that, for any coherent choice function C, $C(O) \subseteq C_{\mathrm{v}}(O) = \max O$ for all $O \in \mathcal{Q}$, where $\max O$ is the unique largest element of the finite option set O according to the strict total ordering $<$. But then Axiom C_1 guarantees that $C(O) = C_{\mathrm{v}}(O) = \max O$ for all $O \in \mathcal{Q}$, so C_{v} is the *only* coherent choice function. □

3.3.3 Properties of the relation between choice functions and desirability

Since sets of desirable options represent only pairwise comparison, and are therefore generally less expressive than choice functions, we expect that going from a choice function to a compatible set of desirable options leads to a loss of information, whereas going the opposite route does not. This is confirmed by Propositions 13 and 14, but in particular by their Corollary 15. Example 2 in Section 4 further on shows that the inequalities in these results can be strict.

Proposition 13. *Consider any set of coherent choice functions $\mathcal{C}' \subseteq \bar{\mathcal{C}}$. Then $D_{\inf\mathcal{C}'} = \inf\{D_C : C \in \mathcal{C}'\}$ and $C_{\inf\{D_C : C \in \mathcal{C}'\}} \sqsubseteq \inf\mathcal{C}'$, and therefore also $C_{D_{\inf\mathcal{C}'}} \sqsubseteq \inf\mathcal{C}'$.*

Proposition 14. *Consider any set of coherent sets of desirable options $\mathcal{D}' \subseteq \bar{\mathcal{D}}$ and any coherent set of desirable options D'. Then $D_{\inf\{C_D : D \in \mathcal{D}'\}} = \inf\mathcal{D}'$ and therefore $D_{C_{D'}} = D'$. Moreover, $C_{\inf\mathcal{D}'} \sqsubseteq \inf\{C_D : D \in \mathcal{D}'\}$.*

Corollary 15. *Consider any coherent set of desirable options $D \in \bar{\mathcal{D}}$ and any coherent choice function $C \in \bar{\mathcal{C}}$. Then $D = D_{C_D}$ and $C_{D_C} \sqsubseteq C$.*

4 Other types of coherent choice functions

There are other types of coherent choice functions than the ones '*based on maximality*', derived from a coherent set of desirable options by selecting undominated elements as in Eq. (6). For instance, any infimum of such coherent choice functions is still coherent.

Definition 7. For any set of coherent sets of desirable options $\mathcal{D}' \subseteq \bar{\mathcal{D}}$, we define the *'infimum of maximality'* choice function as $C_{\mathcal{D}'} := \inf\{C_D : D \in \mathcal{D}'\}$.

Proposition 16. *Consider any set of coherent sets of desirable options $\mathcal{D}' \subseteq \bar{\mathcal{D}}$, then $C_{\mathcal{D}'}$ is a coherent choice function.*

We now consider two special cases of these infimum of maximality choice functions. In Definition 8, we focus only on sets of *maximal* coherent sets of desirable options.

Definition 8. If $\mathcal{D}' \subseteq \hat{\mathcal{D}}$ is a set of *maximal* coherent set of desirable options, the coherent choice function $C_{\mathcal{D}'}$ is called *M-admissible*. We shall also denote it by $C^{M}_{\mathcal{D}'}$ as a reminder that the infimum is taken over maximal sets.

In particular, we can consider the M-admissible choice functions for the set $\mathcal{D}' = \hat{\mathcal{D}}_D$ of all maximal coherent set

[4] See Ref. [21] for an axiomatisation of imprecise preferences in the context of binary comparisons of horse lotteries.

of desirable options that include a coherent set of desirable options D. In order not to burden the notation, we let

$$C_D^M := C_{\hat{\mathcal{D}}_D}^M = \inf\{C_{\hat{D}} : \hat{D} \in \hat{\mathcal{D}} \text{ and } D \subseteq \hat{D}\}. \qquad (7)$$

Proposition 17. *Consider any coherent set of desirable options $D' \in \bar{\mathcal{D}}$. Then $D' = D_{C_{D'}^M}$ and $C_{D'} \sqsubseteq C_{D'}^M$.*

The inequality in Proposition 17 can be strict—meaning that $C_{D'} \sqsubset C_{D'}^M$ for some coherent set of desirable options D'—as is shown in Example 3.

As another special case, we consider choice functions associated with Levi's [15, Chapter 5] notion of E-admissibility, as suggested by Seidenfeld *et al.* [23], and Troffaes [26]. They are based on a non-empty *set of mass functions*. Consider a *finite* possibility space $\mathcal{X}$, and maps from $\mathcal{X}$ to $\mathbb{R}$ (also called *gambles*), forming the vector space $\mathcal{V} = \mathbb{R}^{\mathcal{X}}$ of *finite* dimension $|\mathcal{X}|$. The vector ordering $\preceq$ we associate with this vector space is the pointwise ordering of real numbers: $u \preceq v \Leftrightarrow (\forall x \in \mathcal{X}) u_x \leq v_x$, where, for instance, u_x is the x-component of the option u. We call any map $p : \mathcal{V} \to \mathbb{R}$ with $(\forall x \in \mathcal{X}) p(x) \geq 0$ and $\sum_{x \in \mathcal{X}} p(x) = 1$ a (probability) *mass function*, and we associate an *expectation* E_p with p by letting $\mathrm{E}_p(u) := \sum_{x \in \mathcal{X}} p(x) u_x$ for all u in $\mathcal{V}$.

With a mass function p, we associate a set of desirable options

$$D_p := \mathcal{V}_{\succ 0} \cup \{u \in \mathcal{V} : \mathrm{E}_p(u) > 0\} \qquad (8)$$

and a choice function C_p defined for all O in $\mathcal{Q}$ by

$$C_p(O) := \{u \in O : (\forall v \in O)(\mathrm{E}_p(u) \geq \mathrm{E}_p(v) \text{ and } u \nprec v)\}. \qquad (9)$$

Proposition 18. *The set of desirable options D_p and the choice function C_p are coherent and compatible, and moreover $C_p = C_{D_p}$.*

This result allows us to introduce the following, second special case of 'infimum of maximality' choice functions.

Definition 9. With any non-empty set of mass functions K,[5] we associate the corresponding *E-admissible* choice function $C_K^E := \inf\{C_p : p \in K\} = C_{\{D_p : p \in K\}}$.

Proposition 19. *Given any non-empty set of mass functions K, we have for all O in $\mathcal{Q}$ that*

$$C_K^E(O) = \{u \in O : (\exists p \in K) u \in \operatorname*{arg\,max}_{v \in O} \mathrm{E}_p(v)\} \cap C_{\mathrm{v}}(O).$$

The following proposition establishes a connection between M-admissible and E-admissible choice functions.

Proposition 20. *For any non-empty set of mass functions K, $C_K^E \sqsubseteq C_{\hat{\mathcal{D}}_K}^M$, where $\hat{\mathcal{D}}_K := \bigcup_{p \in K} \hat{\mathcal{D}}_{D_p} \subseteq \hat{\mathcal{D}}$.*

[5] Although Levi's notion of E-admissibility was originally [15, Chapter 5] concerned with *convex closed* sets of mass functions, we impose no such requirement here on the set K.

The following examples show why choice functions are more powerful than sets of desirable options as uncertainty representations, and elucidates the difference between E-admissible and M-admissible choice functions.

Example 2. Consider the situation where you have a coin with two *identical* sides of unknown type: either both sides are heads (H), or both sides are tails (T). The random variable that represents the outcome of a coin flip assumes a value in the finite possibility space $\mathcal{X} := \{\mathrm{H}, \mathrm{T}\}$. The options we consider are gambles: real-valued functions on $\mathcal{X}$, which constitute the two-dimensional vector space $\mathbb{R}^{\mathcal{X}}$, ordered by the pointwise order. We model this situation using (a) coherent sets of desirable options, (b) M-admissible choice functions, and (c) E-admissible choice functions. In all three cases we start from two simple models: one that describes practical certainty of H and another that describes practical certainty of T, and we take their infimum—the most informative model that is still less informative than both—as a candidate model for the coin problem.

For (a), we use two coherent sets of desirable options D_{H} and D_{T}, expressing practical certainty of H and T, respectively, given by the *maximal* sets of desirable options $D_{\mathrm{H}} := \mathcal{V}_{\succ 0} \cup \{u \in \mathcal{V} : u_{\mathrm{H}} > 0\}$ and $D_{\mathrm{T}} := \mathcal{V}_{\succ 0} \cup \{u \in \mathcal{V} : u_{\mathrm{T}} > 0\}$, where u_{H} and u_{T} denote the values of the gamble u in H and T, respectively. The model for the coin with two identical sides is then $D_{\mathrm{H}} \cap D_{\mathrm{T}} = \mathcal{V}_{\succ 0}$. This vacuous model D_{v} is incapable of distinguishing between this situation and the one where we are completely ignorant about the coin.

For an approach (b) that distinguishes between these two situations, we draw inspiration from Proposition 13: instead of working with the sets of desirable options themselves, we move to the corresponding choice functions $C_{\mathrm{H}} := C_{D_{\mathrm{H}}}$ and $C_{\mathrm{T}} := C_{D_{\mathrm{T}}}$, where

$$\begin{aligned} C_{\mathrm{H}}(O) &= \{u \in O : (\forall v \in O) v - u \notin D_{\mathrm{H}}\} \\ &= \arg\max\{u_{\mathrm{H}} : u \in O\} \cap C_{\mathrm{v}}(O) \text{ for all } O \text{ in } \mathcal{Q} \\ C_{\mathrm{T}}(O) &= \arg\max\{u_{\mathrm{T}} : u \in O\} \cap C_{\mathrm{v}}(O) \text{ for all } O \text{ in } \mathcal{Q}. \end{aligned}$$

We infer that $|C_{\mathrm{H}}(O)| = |C_{\mathrm{T}}(O)| = 1$ for every O in $\mathcal{Q}$. The M-admissible choice function we are looking for is $C_{\{D_{\mathrm{H}}, D_{\mathrm{T}}\}}^M = \inf\{C_{\mathrm{H}}, C_{\mathrm{T}}\}$, which selects at most two options from each option set. It is given by

$$\begin{aligned} &C_{\{D_{\mathrm{H}}, D_{\mathrm{T}}\}}^M(O) \\ &= (\arg\max\{u_{\mathrm{H}} : u \in O\} \cup \arg\max\{u_{\mathrm{T}} : u \in O\}) \cap C_{\mathrm{v}}(O) \end{aligned}$$

for all O in $\mathcal{Q}$, and differs from the vacuous choice function C_{v}. Indeed, consider the particular option set $O = \{u, v, w\}$, where $u = (1, 0)$, $v = (0, 1)$ and $w = (1/2, 1/2)$. Then $C_{\{D_{\mathrm{H}}, D_{\mathrm{T}}\}}^M(O) = \{u, v\} \neq O = C_{\mathrm{v}}(O)$.

For (c), the set of mass functions K consists of the two degenerate mass functions: $K = \{p_{\mathrm{H}}, p_{\mathrm{T}}\}$, where $p_{\mathrm{H}} = (1, 0)$ and $p_{\mathrm{T}} = (0, 1)$. The corresponding expectations $\mathrm{E}_{\mathrm{H}} := \mathrm{E}_{p_{\mathrm{H}}}$ and $\mathrm{E}_{\mathrm{T}} := \mathrm{E}_{p_{\mathrm{T}}}$ satisfy $\mathrm{E}_{\mathrm{H}}(u) = u_{\mathrm{H}}$ and $\mathrm{E}_{\mathrm{T}}(u) = u_{\mathrm{T}}$ for all u

in $\mathcal{V}$. So we see that $C_{p_{\mathrm{H}}} = C_{\mathrm{H}}$ and $C_{p_{\mathrm{T}}} = C_{\mathrm{T}}$, and therefore this approach leads to the same choice function as the previous one: $C^E_{\{p_{\mathrm{H}},p_{\mathrm{T}}\}} = C^M_{\{D_{\mathrm{H}},D_{\mathrm{T}}\}} = \inf\{C_{\mathrm{H}}, C_{\mathrm{T}}\}$. □

Example 3. We consider the same finite possibility space $\mathcal{X} := \{\mathrm{H},\mathrm{T}\}$ as in Example 2, with the same option space and vector ordering. Also consider the vacuous set of desirable options D_{v} and the option set $O := \{0, u, v\}$, where $u = (1, -1/4)$ and $v = (-1/4, 1)$. Because all options in O are pointwise undominated in O, we find that $C_{D_{\mathrm{v}}}(O) = O$. On the other hand, it follows from the definition in Eq. (7) that

$$0 \in C^M_{D_{\mathrm{v}}}(O) \Leftrightarrow (\exists \hat{D} \in \hat{\mathcal{D}}_{D_{\mathrm{v}}})(u \notin \hat{D} \text{ and } v \notin \hat{D}),$$

also taking into account Axiom D_1. But $u \notin \hat{D}$ and $v \notin \hat{D}$ implies that $-u \in \hat{D}$ and $-v \in \hat{D}$ by Proposition 7, and therefore also $-u - v \in \hat{D}$ by Axiom D_4. But $-u - v = (-3/4, -3/4) < 0$, contradicting the coherence [Axiom D_1] of $\hat{D}$. This means that $0 \notin C^M_{D_{\mathrm{v}}}(O)$, so $C_{D'} \sqsubset C^M_{D'}$.

This same example shows that $C_{\mathrm{v}} = C_{\bar{\mathcal{D}}} \sqsubset C_{\hat{\mathcal{D}}} = C^M_{D_{\mathrm{v}}}$. □

To conclude this section, we want to mention that there are other popular choice rules besides maximality and E-admissibility, such as, amongst others, Γ-*maximin*, Γ-*maximax* and *interval dominance* [26]. However, they are not coherent: none of them satisfies Axiom C_4b.

5 Indifference

5.1 Indifference and desirability

For sets of desirable options, there is a systematic way of modelling indifference [8, 7, 17]. Let us recall what it means to express an assessment of indifference there.

In addition to a subject's set of desirable options D—the options he strictly prefers to the zero option—we can also consider the options that he considers to be *equivalent* to the zero option. We call these options *indifferent*. A set of indifferent options I is simply a subset of $\mathcal{V}$, but as before with desirable options, we pay special attention to *coherent* sets of indifferent options.

Definition 10. A set of indifferent options I is called coherent if for all u, v in $\mathcal{V}$ and λ in $\mathbb{R}$:

I_1. $0 \in I$;
I_2. if $u \in \mathcal{V}_{>0} \cup \mathcal{V}_{<0}$ then $u \notin I$;
I_3. if $u \in I$ then $\lambda u \in I$;
I_4. if $u, v \in I$ then $u + v \in I$.

Taken together, Axioms I_3 and I_4 are equivalent to imposing that $\mathrm{span}(I) = I$, and due to Axiom I_1, I is non-empty and therefore a linear subspace of $\mathcal{V}$.

The interaction between indifferent and desirable options is subject to rationality criteria as well: they should be compatible with one another.

Definition 11. Given a set of desirable options D and a coherent set of indifferent options I, we call D *compatible* with I if $D + I \subseteq D$.

The idea behind Definition 11 is that adding an indifferent option to a desirable option does not make it non-desirable.

Since $D \subseteq D + I$ due to Axiom I_1, compatibility of D and I is equivalent to $D + I = D$. An immediate consequence of compatibility between a coherent set of desirable options D and a coherent set of indifferent options I is that $D \cap I = \varnothing$, meaning that no option can be assessed as desirable—strictly preferred to the zero option—and indifferent—equivalent to the zero option—at the same time.

5.2 Indifference and quotient spaces

In order to introduce indifference for choice functions, we shall build on a coherent set of indifferent options I, as defined in Definition 10. Two options u and v are considered to be indifferent, to a subject, whenever $v - u$ is indifferent to the zero option, or in other words $v - u \in I$. The idea behind indifference for choice functions will be that we identify indifferent options, and choose between equivalence classes of indifferent options, rather than between single options. We begin by formalising this idea.

We can collect all options that are indifferent to an option $u \in \mathcal{V}$ into the *equivalence class*

$$[u] := \{v \in \mathcal{V} : v - u \in I\} = \{u\} + I.$$

Of course, $[0] = \{0\} + I = I$ is a linear subspace, and the $[u] = \{u\} + I$ affine subspaces of $\mathcal{V}$. The set of all these equivalence classes is the *quotient space*

$$\mathcal{V}/I := \{[u] : u \in \mathcal{V}\} = \{\{u\} + I : u \in \mathcal{V}\}.$$

This quotient space is a vector space under the vector addition, given by

$$[u] + [v] = \{u\} + I + \{v\} + I = \{u + v\} + I = [u + v] \text{ for } u, v \in \mathcal{V},$$

and the scalar multiplication, given by

$$\lambda[u] = \lambda(\{u\} + I) = \{\lambda u\} + I = [\lambda u],$$

for $u \in \mathcal{V}$ and $\lambda \in \mathbb{R}$. $[0] = I$ is the additive identity of $\mathcal{V}/I$.

That we identify indifferent options, and therefore express preferences between equivalence classes of indifferent options, essentially means that we define choice functions on $\mathcal{Q}(\mathcal{V}/I)$. But in order to characterise coherence for such choice functions, we need to introduce a convenient vector ordering on $\mathcal{V}/I$, that is appropriately related to the vector ordering on $\mathcal{V}$; see Section 2.1. For two elements $[u]$ and $[v]$ of $\mathcal{V}/I$, we define

$$[u] \preceq [v] \Leftrightarrow (\exists w \in I) u \preceq v + w, \tag{10}$$

and as usual, the strict variant of the vector ordering on $\mathcal{V}/I$ is characterised by

$$[u] \prec [v] \Leftrightarrow ([u] \preceq [v] \text{ and } [u] \neq [v]).$$

Proposition 21. *The ordering $\preceq$ on $\mathcal{V}/I$ is a vector ordering, and $[u] \prec [v] \Leftrightarrow (\exists w \in I) u \prec v + w$ for any u, v in $\mathcal{V}$.*

We use the notation $O/I := \{[u] : u \in O\}$ for the option set of equivalence classes $[u]$ associated with the options u in an option set O in $\mathcal{Q}(\mathcal{V})$. $\cdot/I$ is an onto map from $\mathcal{Q}(\mathcal{V})$ to $\mathcal{Q}(\mathcal{V}/I)$ that preserves set inclusion.

Proposition 22. *Given any two option sets O_1 and O_2 in $\mathcal{Q}(\mathcal{V})$ such that $O_1 \subseteq O_2$, then $O_1/I \subseteq O_2/I$.*

5.3 Quotient spaces and sets of desirable options

We use this quotient space to prove interesting characterisations of indifference for sets of desirable options.

Proposition 23. *A set of desirable options $D \subseteq \mathcal{V}$ is compatible with a coherent set of indifferent options I if and only if there is some (*representing*) set of desirable options $D' \subseteq \mathcal{V}/I$ such that $D = \{u : [u] \in D'\} = \bigcup D'$. Moreover, the representing set of desirable options is unique and given by $D' = D/I := \{[u] : u \in D\}$.*

This, together with the definition of compatibility, shows that the correspondence between sets of desirable options on $\mathcal{V}$ and (their representing) sets of desirable options on $\mathcal{V}/I$ is one-to-one and onto. It also preserves coherence.

Proposition 24. *Consider any set of desirable options $D \subseteq \mathcal{V}$ that is compatible with a coherent set of indifferent options I, and its representing set of desirable options $D/I \subseteq \mathcal{V}/I$. Then D is coherent if and only if D/I is.*

5.4 Quotient spaces and choice functions

The discussion above inspires us to combine indifference with choice functions in the following manner: a choice function expresses indifference if its behaviour is completely determined by a choice function on the equivalence classes of indifferent options.

Definition 12. We call a choice function C on $\mathcal{Q}(\mathcal{V})$ *compatible* with a coherent set of indifferent options I if there is some *representing* choice function C' on $\mathcal{Q}(\mathcal{V}/I)$ such that $C(O) = \{u \in O : [u] \in C'(O/I)\}$ for all O in $\mathcal{Q}(\mathcal{V})$.

This definition allows for characterisations that are similar to the ones for desirability in Propositions 23 and 24. If a choice function on $\mathcal{Q}(\mathcal{V})$ is compatible with I then the representing choice function on $\mathcal{Q}(\mathcal{V}/I)$ is necessarily unique, and we denote it by C/I:

Proposition 25. *For any choice function C on $\mathcal{Q}(\mathcal{V})$ that is compatible with some coherent set of indifferent options I, the unique representing choice function C/I on $\mathcal{Q}(\mathcal{V}/I)$ is given by $C/I(O/I) := C(O)/I$ for all O in $\mathcal{Q}(\mathcal{V})$. Hence also*

$$C(O) = O \cap \left(\bigcup C/I(O/I)\right) \text{ for all } O \text{ in } \mathcal{Q}(\mathcal{V}).$$

This, together with the definition of compatibility, shows that the correspondence between choice functions on $\mathcal{Q}(\mathcal{V})$ and (their representing) choice functions on $\mathcal{Q}(\mathcal{V}/I)$ is one-to-one and onto. It also preserves coherence.

Proposition 26. *Consider any choice function C on $\mathcal{Q}(\mathcal{V})$ that is compatible with a coherent set of indifferent options I, and its representing choice function C/I on $\mathcal{Q}(\mathcal{V}/I)$. Then C is coherent if and only if C/I is.*

To conclude this general discussion of indifference for choice functions, we mention that it is closed under arbitrary infima, which enables conservative inference under indifference: we can consider the least informative choice function that is compatible with some assessments and is still compatible with a coherent set of indifferent options.

Proposition 27. *Consider any coherent set of indifferent options I, and any non-empty collection of coherent choice functions $\{C_i : i \in \mathcal{I}\}$ that are compatible with I, then its coherent infimum $\inf\{C_i : i \in \mathcal{I}\}$ is compatible with I as well, and $C/I = \inf\{C_i/I : i \in \mathcal{I}\}$.*

5.5 Relation with desirability

First, we consider a coherent choice function C compatible with some coherent set of indifferent options I, and check whether the corresponding coherent set of desirable options D_C is also compatible with I.

Proposition 28. *Consider any coherent set of indifferent options I, and any compatible coherent choice function C, then the corresponding coherent set of desirable options D_C is also compatible with I, and $D_C/I = D_{C/I}$.*

Next, and conversely, we consider a coherent set of desirable options D compatible with I, and check whether the corresponding coherent choice functions C_D is also compatible with I.

Proposition 29. *Consider any coherent set of indifferent options I, and any compatible coherent set of desirable options D, then the corresponding coherent choice function C_D is also compatible with I, and $C_D/I = C_{D/I}$.*

5.6 Example

To exhibit the power and simplicity of our definition of indifference, we reconsider the finite possibility space $\mathcal{X} := \{\mathrm{H}, \mathrm{T}\}$ of Example 2, where the vector space $\mathcal{V}$ is again the two-dimensional vector space $\mathbb{R}^{\mathcal{X}}$ of real-valued functions on $\mathcal{X}$, or gambles, and the vector ordering $\preceq$ is the usual pointwise ordering of gambles.

We want to express indifference between heads and tails, or in other words between $\mathbb{I}_{\mathrm{H}}$ and $\mathbb{I}_{\mathrm{T}}$, where $\mathbb{I}_{\mathrm{H}} := (1,0)$ and $\mathbb{I}_{\mathrm{T}} := (0,1)$. This means that $\mathbb{I}_{\mathrm{H}} - \mathbb{I}_{\mathrm{T}}$ is considered equivalent to the zero gamble, so the linear space of all gambles that are equivalent to zero—or in other words, the set of indifferent gambles (or options)—is then given by

$$I = \{\lambda(\mathbb{I}_{\mathrm{H}} - \mathbb{I}_{\mathrm{T}}) : \lambda \in \mathbb{R}\} = \{u \in \mathbb{R}^{\mathcal{X}} : \mathrm{E}_p(u) = 0\},$$

where E_p is the expectation associated with the uniform mass function $p = (1/2, 1/2)$ on $\{\mathrm{H},\mathrm{T}\}$, associated with a fair coin: $\mathrm{E}_p(u) := \frac{1}{2}[u_{\mathrm{H}} + u_{\mathrm{T}}]$. So, for any option u in $\mathbb{R}^{\mathcal{X}}$—any real-valued function on $\mathcal{X}$:

$$[u] = \{u\} + I = \{v \in \mathbb{R}^{\mathcal{X}} : \mathrm{E}_p(v) = \mathrm{E}_p(u)\},$$

which tells us that the equivalence class $[u]$ can be characterised by the common uniform expectation $\mathrm{E}_p(u)$ of its elements. Therefore, $\mathbb{R}^{\mathcal{X}}/I$ has unit dimension, and we can identify it with the real line $\mathbb{R}$. The vector ordering between equivalence classes is given by, using Eq. (10):

$$\begin{aligned}
[u] \preceq [v] &\Leftrightarrow (\exists \lambda \in \mathbb{R}) u \leq v + \lambda(\mathbb{I}_{\mathrm{H}} - \mathbb{I}_{\mathrm{T}}) \\
&\Leftrightarrow (\exists \lambda \in \mathbb{R})(u_{\mathrm{H}} \leq v_{\mathrm{H}} + \lambda \text{ and } u_{\mathrm{T}} \leq v_{\mathrm{T}} - \lambda) \\
&\Leftrightarrow (\exists \lambda \in \mathbb{R}) u_{\mathrm{H}} - v_{\mathrm{H}} \leq \lambda \leq -u_{\mathrm{T}} + v_{\mathrm{T}} \\
&\Leftrightarrow u_{\mathrm{H}} - v_{\mathrm{H}} \leq -u_{\mathrm{T}} + v_{\mathrm{T}} \Leftrightarrow \mathrm{E}_p(u) \leq \mathrm{E}_p(v),
\end{aligned}$$

and similarly $[u] \prec [v] \Leftrightarrow \mathrm{E}_p(u) < \mathrm{E}_p(v)$ for all u, v in $\mathbb{R}^{\mathcal{X}}$. Hence, the strict vector ordering $\prec$ on $\mathbb{R}^{\mathcal{X}}/I$ is total, so we infer from the argumentation in Example 1 that there is only one representing choice function, namely the vacuous one. Therefore, there is only one choice function C on $\mathcal{Q}(\mathbb{R}^{\mathcal{X}})$ that is compatible with I, namely, the one that has the vacuous choice function C_{v} on $\mathcal{Q}(\mathbb{R}^{\mathcal{X}}/I)$ as its representation C/I. Recall that for any O in $\mathcal{Q}(\mathbb{R}^{\mathcal{X}})$:

$$\begin{aligned}
C_{\mathrm{v}}(O/I) &= \{[u] : (\forall [v] \in O/I)[u] \nprec [v]\} \\
&= \{[u] : (\forall [v] \in O/I)[v] \preceq [u]\} \\
&= \{[u] : (\forall [v] \in O/I)\mathrm{E}_p(v) \leq \mathrm{E}_p(u)\},
\end{aligned}$$

and therefore

$$C(O) := \{u \in O : (\forall v \in O)\mathrm{E}_p(v) \leq \mathrm{E}_p(u)\} = C^{E}_{\{p\}}(O).$$

The indifference assessment between heads and tails leaves us no choice but to use an E-admissible model for a probability mass function, associated with a fair coin.

The choice function C is therefore based on E-admissibility, but is not compatible with M-admissibility. To see this, consider the set of options $O := \{w, 0, -w\}$ with $w := (1,-1)$, so $w_{\mathrm{H}} + w_{\mathrm{T}} = 0$. Hence $C(O) = O$.

But no M-admissible choice function will select 0 in O: observe that $0 \notin C_{\hat{D}}(O)$ for all $\hat{D} \in \mathcal{D}'$, because $0 \in C_{\hat{D}}(O)$ would imply that $\{w, -w\} \cap \hat{D} = \emptyset$, contradicting that $\hat{D}$ is a maximal set of desirable options by Proposition 7.

6 Conclusion

We have developed a theory of conservative reasoning with choice functions, and related coherent choice functions to coherent sets of desirable options, showing that choice functions are indeed more informative than sets of desirable options as a tool for conservative reasoning. We have also provided an intuitive definition for indifference that subsumes the usual definition for sets of desirable options.

We still intend to address conditioning for choice functions, and look for an elegant conditioning rule that subsumes the one for sets of desirable options—and therefore also Bayes's rule. Another problem to tackle is related to indifference: Seidenfeld [20] (see also [3]) has given another elegant definition for indifference for choice functions, which he has also linked to sequential coherence. We know that our definition implies his, but the question whether the two approaches are equivalent is still open. The connection with sequential coherence is also an open issue, and we expect Axiom C_3 will play an important role in resolving it.

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