Modelling practical certainty and its link with classical propositional logic

Arthur Van Camp and Gert de Cooman

SYSTeMS research group, Ghent University, Belgium

{Arthur.VanCamp,Gert.deCooman}@UGent.be

UNIVERSITEIT GENT

Introduction

Context A subject who is *practically* certain about the occurrence of every event in a collection $\mathscr{T}$.

Accept & reject statements We try to model this certainty in the language of accept and reject statement-based uncertainty models.

Motivation This language is rich enough to encompass the different approaches of Walley and de Finetti. In order to obtain more insight in these approaches, we study different types of assessments.

(Sets of) events The random variable X about which the subject expresses practical certainty takes values in $\mathscr{X}$. All events are collected in the *power set* $\mathscr{P} := \{A : A \subseteq \mathscr{X}\}$. $\emptyset \neq \mathscr{C} \subseteq \mathscr{P}$ is a *filter base* if it is closed under finite intersections (closed under conjunction): if $A, B \in \mathscr{C}$, then also $A \cap B \in \mathscr{C}$. $\mathscr{C}$ is called *proper* if in addition $\emptyset \notin \mathscr{C}$. $\emptyset \neq \mathscr{F} \subseteq \mathscr{P}$ is called a *filter* if: (i) $\mathscr{F}$ is closed under conjunction, and (ii) $\mathscr{F}$ is increasing (closed under modus ponens): if $A \in \mathscr{F}$ and $B \supseteq A$, then also $B \in \mathscr{F}$. $\mathscr{F}$ is called *proper* if in addition $\emptyset \notin \mathscr{F}$. We denote the set of all proper filters by $\mathbb{F}$.

(Sets of) gambles A gamble f is a bounded real-valued function on $\mathscr{X}$. The set of all gambles is $\mathscr{L}$. If $f(x) \geq 0$ for all $x \in \mathscr{X}$, we write $f \geq 0$, and the set of all such gambles is $\mathscr{L}_{\geq 0}$. We write $f > 0$ if $f \geq 0$ and $f \neq 0$. The set of all such gambles is $\mathscr{L}_{>0}$. If $f(x) > 0$ for all $x \in \mathscr{X}$, we write $f \rhd 0$, and the set of all such gambles is $\mathscr{L}_{\rhd 0}$.

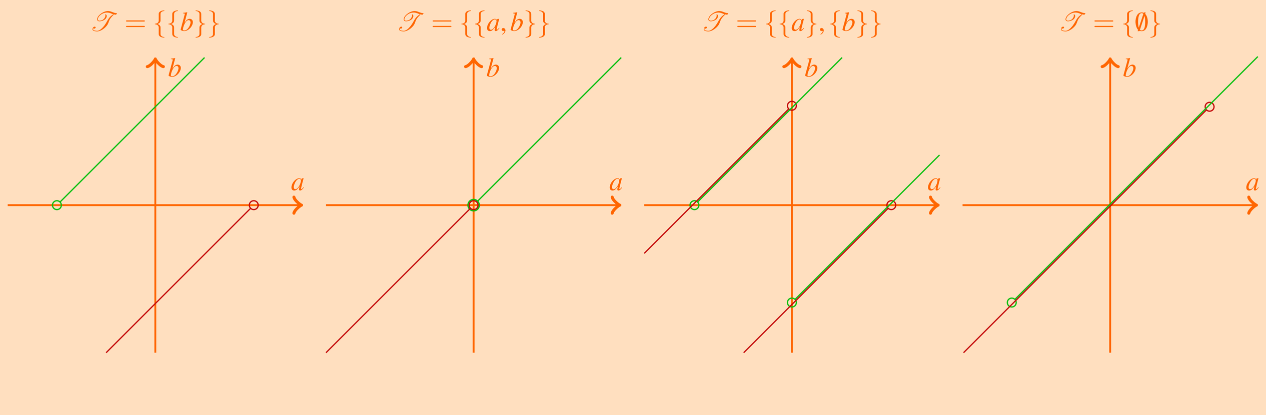
First type: favourability assessment

Favourability A gamble f is *favourable* if $f \in \mathscr{A}_{\rhd} := \mathscr{A}_{\succeq} \cap -\mathscr{A}_{\prec}$: f is accepted and $-f$ is rejected.

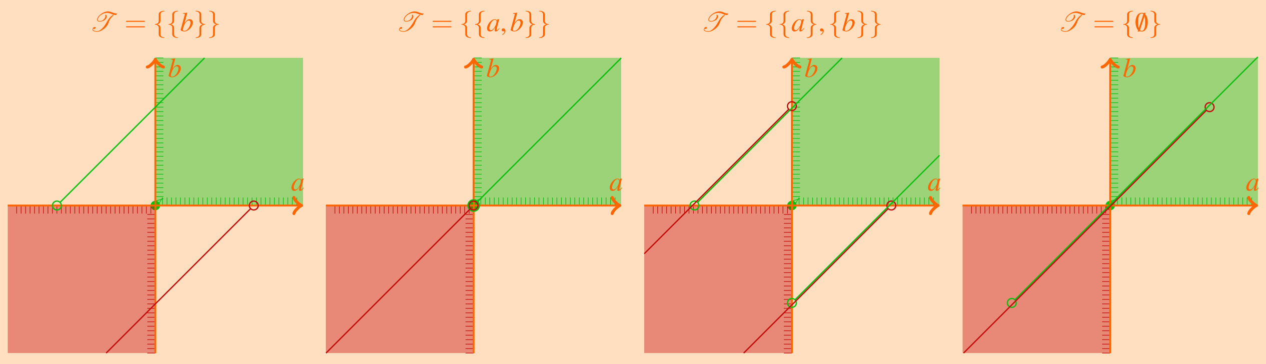
Assessment about one event A If a subject is practically certain that an event A occurs, we will first take this to mean that he finds any gamble in $\mathscr{A}^{A}_{\rhd} := \{-\mathbb{I}_{A^c} + \varepsilon : \varepsilon \in \mathbb{R}_{>0}\}$ favourable: he accepts to bet on A at odds $\varepsilon/_{1-\varepsilon}$ and refuses to bet against A at odds $^{1-\varepsilon}/_{\varepsilon}$.

Assessment about more events $\mathscr{T}$ If he is practically certain that each event in $\mathscr{T} \subseteq \mathscr{P}$ occurs, then his assessment is

$$\mathscr{A} = \langle \mathscr{A}_{\rhd}; -\mathscr{A}_{\rhd} \rangle \text{ with } \mathscr{A}_{\rhd} = \{-\mathbb{I}_{A^c} + \varepsilon : A \in \mathscr{T}, \varepsilon \in \mathbb{R}_{>0}\}.$$



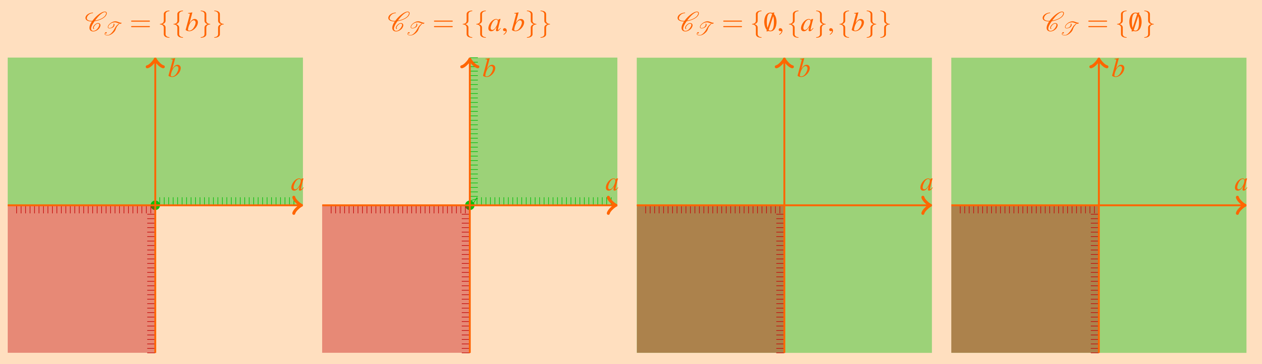
Background model We assume the background model $\mathscr{S} = \langle \mathscr{L}_{\geq 0}; \mathscr{L}_{<0} \rangle$, which yields the smallest assessment $\mathscr{B} = \mathscr{A} \cup \mathscr{S}$ that includes both $\mathscr{A}$ and $\mathscr{S}$.



Deductive closure The deductively closed assessment $\mathscr{D} = \mathrm{ext}_{\mathbf{D}}\mathscr{B} = \langle \mathrm{posi}\,\mathscr{B}_{\succeq}; \mathscr{B}_{\prec} \rangle$ is determined by

$$\mathrm{posi}\,\mathscr{B}_{\succeq} = \{f \in \mathscr{L} : (\exists B \in \mathscr{C}_{\mathscr{T}}) \inf(f|B) > 0\} \cup \mathscr{L}_{\geq 0} =: \mathscr{L}^{>}_{\mathscr{T}} \cup \mathscr{L}_{\geq 0},$$

where $\mathscr{C}_{\mathscr{T}} := \{\bigcap_{k=1}^{n} A_k : n \in \mathbb{N}, A_k \in \mathscr{T}\} \subseteq \mathscr{P}$, the filter base generated by $\mathscr{T}$.

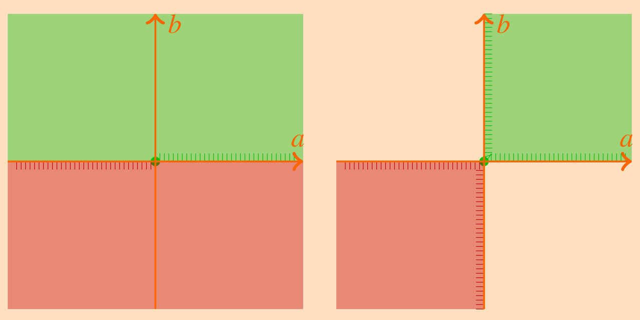


No Confusion is equivalent with the *finite intersection property*:

$$\bigcap_{k=1}^{n} A_k \neq \emptyset \text{ for all } n \in \mathbb{N} \text{ and } A_k \in \mathscr{T}, \text{ or equivalently, } \emptyset \notin \mathscr{C}_{\mathscr{T}}.$$

No Limbo The reckoning extension $\mathscr{M} = \mathrm{ext}_{\mathbf{M}}\mathscr{D}$ is given by

$$\mathscr{M} = \langle \mathscr{L}^{>}_{\mathscr{T}} \cup \mathscr{L}_{\geq 0}; \mathscr{L}^{<}_{\mathscr{T}} \cup \mathscr{L}_{<0} \rangle,$$



so $\mathscr{M}_{\rhd} = \mathscr{L}^{>}_{\mathscr{T}} \cup \mathscr{L}_{\rhd 0} = \mathscr{M}_{\succeq} \setminus \{0\}$. The last two examples have Confusion, what means that the expressed practical certainty is not rational. Therefore, these examples are not continued.

All practical certain events Does the inference procedure described above, which allows us to infer from the set of favourable gambles $\mathscr{A}_{\rhd}$ the larger set of favourable gambles $\mathscr{M}_{\rhd}$, bear any relationship to inference in classical propositional logic? For which events $A \in \mathscr{P}$ is $\mathscr{A}^{A}_{\rhd} \subseteq \mathscr{M}_{\rhd}$?

$$\mathscr{A}^{A}_{\rhd} \subseteq \mathscr{M}_{\rhd} \Leftrightarrow (\exists B \in \mathscr{C}_{\mathscr{T}}) B \subseteq A \Leftrightarrow A \in \mathscr{F}_{\mathscr{T}},$$

where $\mathscr{F}_{\mathscr{T}} := \{B \in \mathscr{P} : (\exists C \in \mathscr{C}_{\mathscr{T}})(C \subseteq B)\}$ is the filter generated by $\mathscr{T}$.

$\mathscr{X}$	$\mathscr{T}$	$\mathscr{C}_{\mathscr{T}}$	$\mathscr{F}_{\mathscr{T}}$
$\{a,b,c\}$	$\{\{a,b\},\{a,c\}\}$	$\{\{a\},\{a,b\},\{a,c\}\}$	$\{\{a\},\{a,b\},\{a,c\},\{a,b,c\}\}$

conjunction — modus ponens

This tells us that on our specific interpretation of it, the logic of practical certainty has the same basic machinery as classical propositional logic.

Accept & reject statements

Accepting & rejecting The subject gives his assessment $\mathscr{A}$ by making accept and reject statements about gambles $f \in \mathscr{L}$.

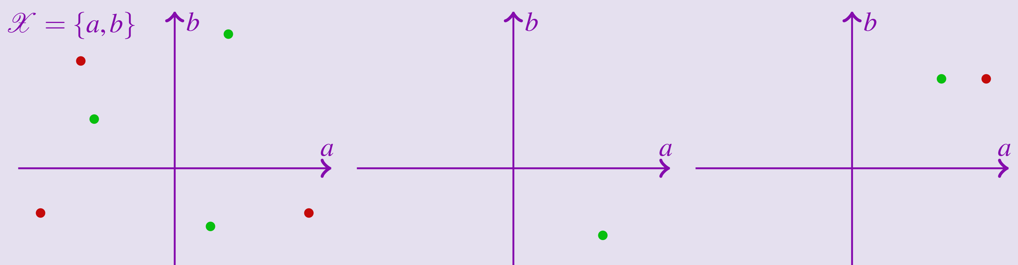
Accepting f implies a commitment for the subject to engage in the following transaction:

(i) the actual value x of X is determined

(ii) he gets the—possible negative—payoff $f(x)$.

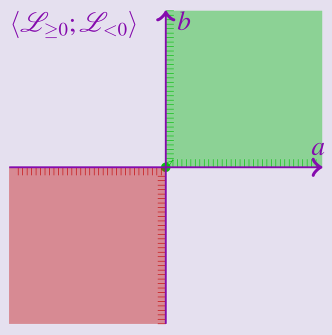
Rejecting f means that the subject excludes f from being accepted.

Assessment $\mathscr{A}$ is a pair of accepted ($\mathscr{A}_{\succeq}$) and rejected ($\mathscr{A}_{\prec}$) gambles: $\mathscr{A} = \langle \mathscr{A}_{\succeq}; \mathscr{A}_{\prec} \rangle$.



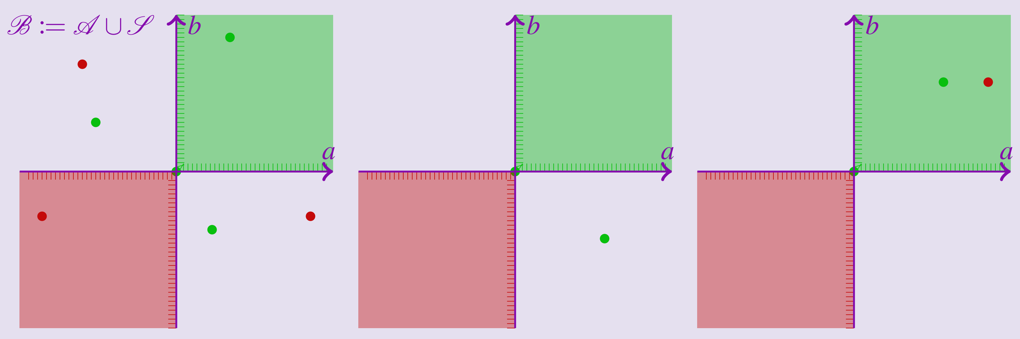
There are *four rationality requirements*.

Background model Before an assessment is given, some gambles can be presumed to be accepted and others to be rejected. Such a priori assumptions can be captured by positing a background model $\mathscr{S}$.



First rationality requirement:

Indifference to status quo: $\langle \{0\}; \emptyset \rangle \subseteq \mathscr{S}$



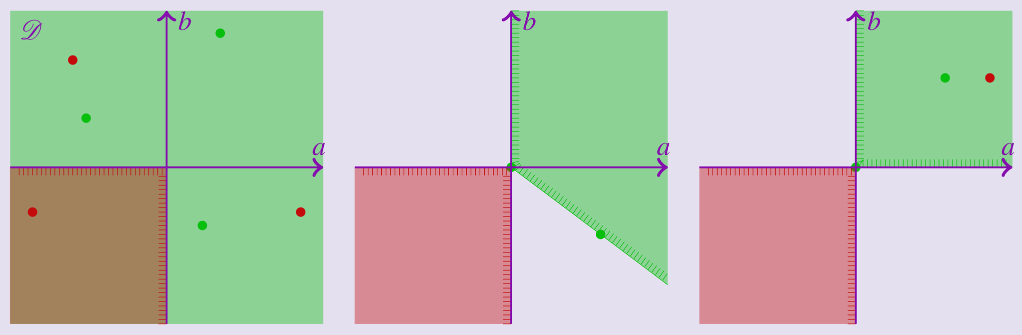
Deductive closure If f and g are acceptable, then so should be $f+g$ and λf, with $\lambda \in \mathbb{R}_{>0}$. These two observations are summarised in the *deductive extension*

$$\mathscr{D} = \mathrm{ext}_{\mathbf{D}}\mathscr{B} := \langle \mathrm{posi}\,\mathscr{B}_{\succeq}; \mathscr{B}_{\prec} \rangle,$$

where $\mathrm{posi}\,\mathscr{B}_{\succeq} := \{\sum_{k=1}^{n} \lambda_k f_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, f_k \in \mathscr{B}_{\succeq}\}$, the positive linear hull of $\mathscr{B}_{\succeq}$.

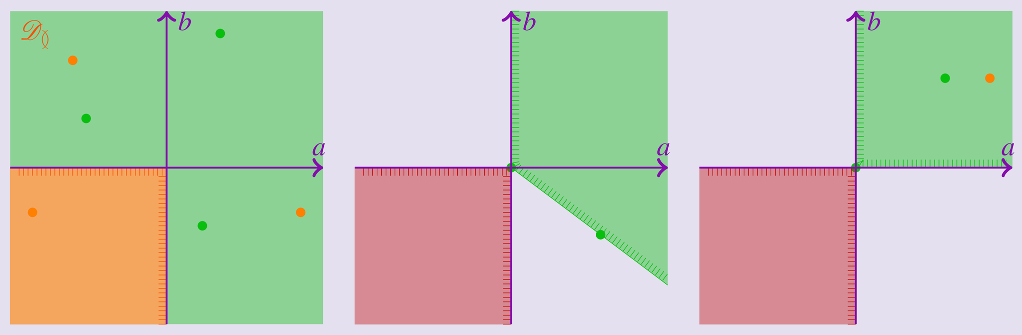
Second rationality requirement:

$\mathscr{A}$ should be deductive closed: $\mathrm{ext}_{\mathbf{D}}\mathscr{A} = \mathscr{A}$



No Confusion Given the interpretation attached to an accept and to a reject statement, we have as a *Third rationality requirement*:

$\mathscr{D}$ should have No Confusion: $\mathscr{D}_{\between} := \mathscr{D}_{\succeq} \cap \mathscr{D}_{\prec} = \emptyset$.

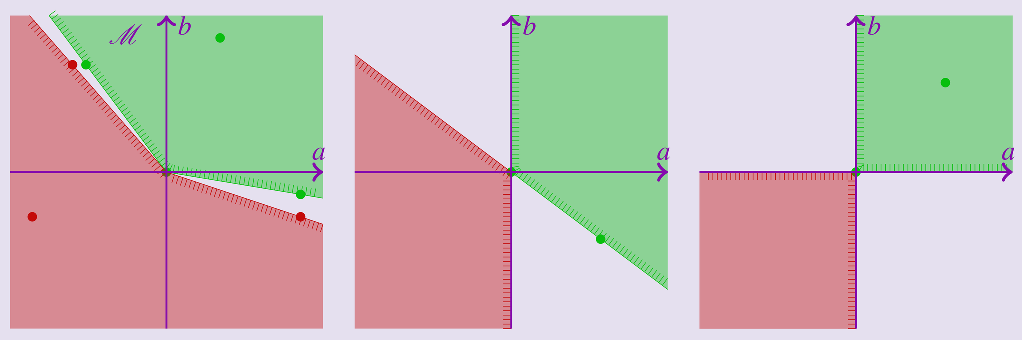


No Limbo One can still accept or reject each unresolved gamble in $\mathscr{D}_{\smile} := (\mathscr{D}_{\succeq} \cup \mathscr{D}_{\prec})^c$. Gambles in Limbo $(\overline{\mathscr{D}_{\prec}} - \mathscr{D}_{\succeq}) \setminus \mathscr{D}_{\prec} \subseteq \mathscr{D}_{\smile}$ can only be rejected, if No Confusion is to be avoided. Here, $\overline{\mathscr{D}_{\prec}} := \{\lambda_k f_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, f_k \in \mathscr{D}_{\prec}\}$ is the positive scalar hull of $\mathscr{D}_{\prec}$. This observation is summarised in the *reckoning extension*

$$\mathscr{M} = \mathrm{ext}_{\mathbf{M}}\mathscr{D} := \langle \mathscr{D}_{\succeq}; \overline{\mathscr{D}_{\prec}} \cup (\overline{\mathscr{D}_{\prec}} - \mathscr{D}_{\succeq}) \rangle.$$

Fourth rationality requirement:

$\mathscr{D}$ should have No Limbo: $\mathrm{ext}_{\mathbf{M}}\mathscr{D} = \mathscr{D}$



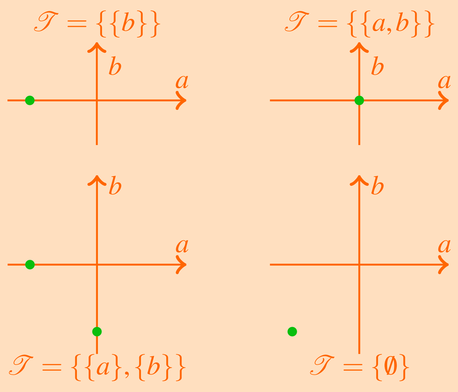
Second type: indifference assessment

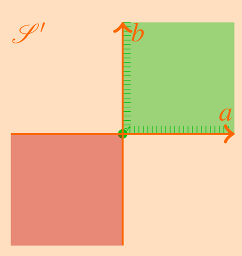
Indifference A gamble f is *indifferent* if $f \in \mathscr{A}_{\simeq} := \mathscr{A}_{\succeq} \cap -\mathscr{A}_{\succeq}$: both f and its negation $-f$ are accepted.

Assessment about one event If a subject is practically certain that an event A occurs, we will now take this to mean that he is indifferent between $\mathbb{I}_A$ and 1, or equivalently, between $-\mathbb{I}_{A^c}$ and 0.

Assessment about more events $\mathscr{T}$ If he is practically certain that each event $\mathscr{T} \subseteq \mathscr{P}$ occurs, then his assessment is

$$\mathscr{A}' = \langle \{-\mathbb{I}_{A^c} : A \in \mathscr{T}\}; \emptyset \rangle.$$





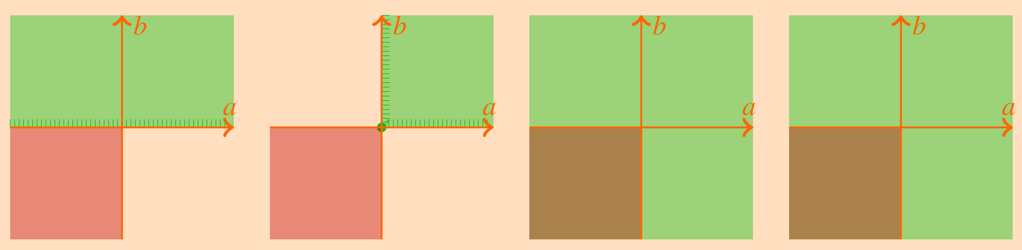
Background model Because $-\mathbb{I}_A \in \mathscr{L}_{<0}$, the nature of this assessment forces us to assume a slightly different background model: $\mathscr{S}' := \langle \mathscr{L}_{\geq 0}; \mathscr{L}_{\lhd 0} \rangle$. This yields the smallest assessment $\mathscr{B}' = \mathscr{S}' \cup \mathscr{A}'$ that includes $\mathscr{S}'$ and $\mathscr{A}'$.

Deductive closure The deductively closed assessment $\mathscr{D}' = \mathrm{ext}_{\mathbf{D}}\mathscr{B}' = \langle \mathrm{posi}\,\mathscr{B}'_{\succeq}; \mathscr{B}'_{\prec} \rangle$ is determined by

$$\mathrm{posi}\,\mathscr{B}'_{\succeq} = \{f \in \mathscr{L} : (\exists B \in \mathscr{C}_{\mathscr{T}}) \mathbb{I}_B f \geq 0\} =: \mathscr{L}^{\geq}_{\mathscr{T}},$$

so the indifferent gambles are

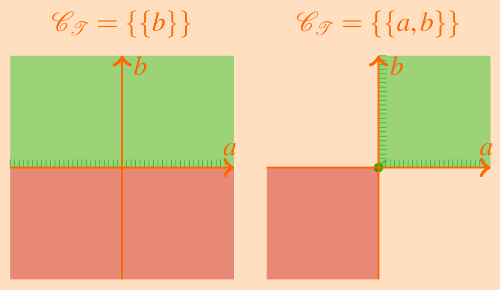
$$\mathscr{D}'_{\simeq} = \{f \in \mathscr{L} : (\exists B \in \mathscr{C}_{\mathscr{T}}) \mathbb{I}_B f = 0\}.$$



No Confusion if and only if $\emptyset \notin \mathscr{C}_{\mathscr{T}}$.

No Limbo Let $\mathscr{L}^{<}_{\mathscr{T}} := \{f \in \mathscr{L} : (\exists B \in \mathscr{C}_{\mathscr{T}})(\forall x \in B) f(x) < 0\}$. The reckoning extension $\mathscr{M}' = \mathrm{ext}_{\mathbf{D}}\mathscr{D}'$ is given by

$$\mathscr{M}' = \langle \mathscr{L}^{\geq}_{\mathscr{T}}; \mathscr{L}^{<}_{\mathscr{T}} \rangle,$$



so $\mathscr{M}'_{\rhd} = \mathscr{L}^{>}_{\mathscr{T}}$ and $\mathscr{M}'_{\simeq} = \mathscr{D}'_{\simeq}$. The last two examples have Confusion. Therefore, these examples are not continued.

All practical certain events To find all such events, we look at the events $A \in \mathscr{P}$ for which we have that $-\mathbb{I}_{A^c} \in \mathscr{M}'_{\simeq}$. We find

$$-\mathbb{I}_{A^c} \in \mathscr{M}'_{\simeq} \Leftrightarrow (\exists B \in \mathscr{C}_{\mathscr{T}}) B \subseteq A \Leftrightarrow A \in \mathscr{F}_{\mathscr{T}},$$

which leads to the same conclusions as for the first type of assessments.

Embedding classical propositional logic

Indifference assessments

Denote the collection of all assessments by $\mathbf{A}$. Consider the family of models for practical certainty following from indifference assessments:

$$\mathbb{C}' := \{\langle \mathscr{L}^{\geq}_{\mathscr{F}}; \mathscr{L}^{<}_{\mathscr{F}} \rangle : \mathscr{F} \in \mathbb{F}\} \subseteq \mathbf{A}.$$

Then we have

$(\mathbf{A}, \mathbb{C}', \subseteq)$ is a strong belief structure,

meaning that (i) $(\mathbf{A}, \subseteq)$ is a complete lattice where $\bigcap$ plays the role of infimum, (ii) $(\mathbb{C}', \subseteq)$ is an intersection structure: for any $\emptyset \neq \mathbf{B} \in \mathbb{C}'$, $\inf \mathbf{B} \in \mathbb{C}'$, (iii) $(\mathbb{C}', \subseteq)$ has no top, and (iv) $(\mathbb{C}', \subseteq)$ is dually atomic: $\hat{\mathbb{C}}' \neq \emptyset$ and $\mathscr{D} = \inf\{\mathscr{D}' \in \hat{\mathbb{C}}' : \mathscr{D} \subseteq \mathscr{D}'\}$ if $\mathscr{D} \in \mathbb{C}'$. We have also

$(\mathbb{C}', \subseteq)$ and $(\mathbb{F}, \subseteq)$ are order isomorphic,

which means that $\mathbb{F}$ and $\mathbb{C}'$ are essentially the same.

Favourability assessments

Consider the family of models for practical certainty following from favourability assessments:

$$\mathbb{C} := \{\langle \mathscr{L}^{>}_{\mathscr{F}} \cup \mathscr{L}_{\geq 0}; \mathscr{L}^{<}_{\mathscr{F}} \cup \mathscr{L}_{<0} \rangle : \mathscr{F} \in \mathbb{F}\} \subseteq \mathbf{A}.$$

Unfortunately, $(\mathbf{A}, \mathbb{C}, \subseteq)$ is no strong belief structure:

$(\mathbf{A}, \mathbb{C}, \subseteq)$ is no intersection structure.

Luckily, we can still find an embedding of $\mathbb{F}$ into $\mathbb{C}$. Consider a coherent set of favourable gambles $\mathscr{D}_{\rhd}$ derived from an assessment that includes $\mathscr{S}$ and take any $\mathscr{A} \subseteq \mathscr{P}$ such that $\mathscr{L}^{>}_{\mathscr{A}} \cup \mathscr{L}_{\geq 0} \subseteq \mathscr{D}_{\rhd}$. Let $\mathscr{F} := \{B \in \mathscr{P} : (\forall \varepsilon \in \mathbb{R}_{>0}) - \mathbb{I}_{B^c} + \varepsilon \in \mathscr{D}_{\rhd}\}$. Then

(i) $\mathscr{F} \in \mathbb{F}$; (ii) $\mathscr{L}^{>}_{\mathscr{F}} \cup \mathscr{L}_{\geq 0} \subseteq \mathscr{D}_{\rhd}$; (iii) $\mathscr{A} \subseteq \mathscr{F}$.