Modelling practical certainty and its link with classical propositional logic

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Abstract

We model practical certainty in the language of accept & reject statement-based uncertainty models. We present three different ways, each time using a different nature of assessment: we study coherent models following from (i) favourability assessments, (ii) acceptability assessments, and (iii) indifference assessments. We argue that a statement of favourability, when used with an appropriate background model, essentially boils down to stating a belief of practical certainty using acceptability assessments. We show that the corresponding models do not form an intersection structure, in contradistinction with the coherent models following from an indifferenc assessment. We construct embeddings of classical propositional logic into each of our models for practical certainty.

Keywords. Imprecise probabilities, accept & reject statement-based uncertainty models, classical propositional logic, strong belief structure.

1 Introduction

In classical propositional logic, a subject who is certain of the truth of some propositions, or equivalently, of the occurrence of the corresponding events, models this by giving his set of certain events—or true propositions. In this paper, we investigate to what extent classical propositional logic can be embedded in accept and reject statement-based uncertainty models [9]. The embedding is not perfect, therefore we speak of *practical uncertainty*. The language of the uncertainty models used is rich enough to encompass the different approaches of Walley and de Finetti. In order to obtain more insight in these approaches, we study three different types of assessments.

The first type of assessments fits well into Walley's approach to defining lower previsions, and focusses on strict preference. The second type appears to be weaker, as it focusses on weak preference, but we show that the difference essentially does not matter: the derived coherent models from both types of assessments are the same. The third and last type of assessments is more in line with de Finetti's approach to defining previsions, and focusses on indifference.

Because strong belief structures [2] have nice properties, we investigate whether the derived coherent models constitute such structures. It turns out that only the coherent models derived from the third type of assessments do.

The basic concepts are introduced in Section 2. In the subsequent three sections, we study three different ways of modelling practical certainty. We start with favourability assessments in Section 3, and study the consequences of the rationality requirements of No Confusion, Deductive Closure and No Limbo. We proceed with acceptability assessments in Section 4, where we also investigate the connection with the models of the previous section. The last type of assessments—those based on indifference—are discussed in Section 5. In Section 6, we find the corresponding coherent lower prevision models, and we investigate when they are coherent. We make the link with (strong) belief structures in Section 7. Finally, in Section 8, we embeds classical propositional logic into the models introduced in this paper.

2 Notations and concepts

We consider a subject who is uncertain about the value of a variable X that takes values in the—not necessarily finite but non-empty—possibility space $\mathscr{X}$. We want to model this subject's beliefs about the value that X assumes.

2.1 Events and sets of events

An *event* is a subset of $\mathscr{X}$, or equivalently, an element of the *power set* $\mathscr{P} := \{A \colon A \subseteq \mathscr{X}\}$: the collection of all events. A non-empty subset $\mathscr{C}$ of $\mathscr{P}$ is called a *filter base* if it is *closed under finite intersections* (closed under *conjunction*): if both A and B are elements of $\mathscr{C}$, then also $A \cap B \in \mathscr{C}$. A filter base $\mathscr{C}$ is called *proper* if additionally $\emptyset \notin \mathscr{C}$. A non-empty subset $\mathscr{F}$ of $\mathscr{P}$ is called a *filter* if: (i) $\mathscr{F}$ is closed under finite intersections, and (ii) $\mathscr{F}$

is *increasing* (closed under *modus ponens*): if $A \in \mathscr{F}$ and $A \subseteq B$, then also $B \in \mathscr{F}$. A filter $\mathscr{F}$ is called *proper* if additionally $\emptyset \notin \mathscr{F}$, or equivalently, $\mathscr{F} \neq \mathscr{P}$. We denote the set of all proper filters by $\mathbb{F}$. A proper filter $\mathscr{U}$ is called an *ultrafilter* if either $A \in \mathscr{U}$ or $A^c \in \mathscr{U}$ for all events A. We denote the set of all ultrafilters by $\mathbb{U}$.

As an example, consider a filter base $\mathscr{C}$, then the set $\{A \in \mathscr{P} \colon (\exists B \in \mathscr{C}) B \subseteq A\}$ is a filter—the filter generated by the filter base $\mathscr{C}$. It is proper if and only if the filter base $\mathscr{C}$ is.

2.2 Gambles and sets of gambles

A *gamble* f is a bounded real-valued function on the possibility space $\mathscr{X}$. It is interpreted as an uncertain reward $f(X)$. If the value of the variable X turns out to be x, it results in a—positive or negative—payoff $f(x)$, expressed in units of some predetermined linear utility scale. The set of all gambles on $\mathscr{X}$ is denoted by $\mathscr{L}$.

We can compare two gambles f and g in $\mathscr{L}$. We write $f \geq g$ if $f(x) \geq g(x)$ for all x in $\mathscr{X}$. For example, $f \geq 0$ if f is nowhere negative, and we then say that f is *non-negative*. The subset $\mathscr{L}_{\geq 0}$ of $\mathscr{L}$ is the set of all non-negative gambles. We write $f > g$ if $f \geq g$ and $f \neq g$. For example, $f < 0$ if f is nowhere positive—so $f \leq 0$—and $f(x) < 0$ for at least one x in $\mathscr{X}$, and we then say that f is *negative*. The subset $\mathscr{L}_{<0}$ of $\mathscr{L}$ is the set of all negative gambles. We write $f \gtrdot g$ if $\inf(f-g) > 0$. For example, $f \lessdot 0$ if $\sup f < 0$, meaning that the gamble f is negative and bounded away from zero. The subset $\mathscr{L}_{\lessdot 0}$ of $\mathscr{L}$ is the set of all such gambles. We write $f \rhd g$ if $f(x) > g(x)$ for all x in $\mathscr{X}$. For example, $f \rhd 0$ if f is everywhere (strictly) positive, and we then say that f is *point-wise positive*. The subset $\mathscr{L}_{\rhd 0}$ of $\mathscr{L}$ is the set of all such gambles.

We also introduce a number of operations on sets of gambles $\mathscr{K}, \mathscr{K}' \subseteq \mathscr{L}$. The first is the *Minkowski sum* $\mathscr{K} + \mathscr{K}' := \{f + g \colon f \in \mathscr{K}, g \in \mathscr{K}'\}$. The *positive scalar hull* $\overline{\mathscr{K}} := \{\lambda f \colon \lambda \in \mathbb{R}_{>0}, f \in \mathscr{K}\}$ is the collection of all positive multiples of gambles in $\mathscr{K}$, where we use the notation $\mathbb{R}_{>a}$ for the set of real numbers (strictly) greater than the real number a. The *positive linear hull* $\operatorname{posi}\mathscr{K}$ is the collection of all positive linear combinations of gambles in $\mathscr{K}$:

$$\operatorname{posi}\mathscr{K} := \left\{ \sum_{k=1}^{n} \lambda_k f_k \colon n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, f_k \in \mathscr{K} \right\},$$

where we use the notation $\mathbb{N}$ for the set of natural numbers (positive integers). Observe that

$$\operatorname{posi}(\mathscr{K} \cup \mathscr{K}') = \operatorname{posi}\mathscr{K} \cup \operatorname{posi}\mathscr{K}' \cup (\operatorname{posi}\mathscr{K} + \operatorname{posi}\mathscr{K}'). \quad (1)$$

We call a set $\mathscr{K}$ a *cone* if $\operatorname{posi}\mathscr{K} = \mathscr{K}$.

2.3 Accept & reject statement-based uncertainty models

In order to have greater flexibility in expressing beliefs, we use the framework and language of *accept & reject statement-based uncertainty models*, as introduced and described in detail by Quaeghebeur et al. [9]. In contrast with the slightly older and more common framework of *sets of desirable gambles* [1, 4, 13], where the subject gives only one set of gambles, in this framework a subject is supposed to give two sets: a set of *acceptable* gambles $\mathscr{A}_{\succeq} \subseteq \mathscr{L}$, and a set of *rejected* gambles $\mathscr{A}_{\prec} \subseteq \mathscr{L}$. An assessment is then represented by $\mathscr{A} = \langle \mathscr{A}_{\succeq}; \mathscr{A}_{\prec} \rangle$. Following the discussion in Ref. [9], a subject's accepting a gamble f implies a commitment for him to engage in the following transaction: (i) the actual value x of the variable X is determined, and (ii) the subject gets the—possibly negative—payoff $f(x)$. On the other hand, the subject's rejecting a gamble implies that he excludes it from being accepted.

From an assessment, one can derive other types of statements. For any gamble $f \in \mathscr{A}_{\simeq} := \mathscr{A}_{\succeq} \cap -\mathscr{A}_{\succeq}$, the subject accepts both f and its negation $-f$. We say that he is *indifferent* about f, and $\mathscr{A}_{\simeq}$ is his set of indifferent gambles. For any $f \in \mathscr{A}_{\rhd} := \mathscr{A}_{\succeq} \cap -\mathscr{A}_{\prec}$, the subject accepts f and rejects its negation $-f$. We say that he finds f *favourable*, and $\mathscr{A}_{\rhd}$ is his set of favourable gambles. These are the gambles that the subject strictly prefers to 0, which is the interpretation that is usually given to desirable gambles [1, 4]. Finally, gambles in the set $\mathscr{A}_{\smile} := \mathscr{L} \setminus (\mathscr{A}_{\succeq} \cup \mathscr{A}_{\prec})$ are called *unresolved*. For unresolved gambles no accept or reject statement has been made.

3 Modelling practical certainty using favourability

3.1 Assessment

If a subject is *practically* certain that a proposition is true, or that the corresponding event A occurs, we will first take this to mean that he finds any gamble of the form $\mathbb{I}_A - 1 + \varepsilon$, with $\varepsilon \in \mathbb{R}_{>0}$, favourable.[1] Here $\mathbb{I}_A$ is the *indicator* of the event A, which assumes the value 1 on A (if the proposition is true) and 0 elsewhere. Finding $\mathbb{I}_A - 1 + \varepsilon$ favourable means: (i) the transaction that yields ε if A occurs and $\varepsilon - 1$ otherwise, is accepted, and (ii) the transaction that yields $-\varepsilon$ if A occurs, and $1 - \varepsilon$ otherwise, is rejected (excluded from being accepted). The first assessment means that the subject accepts to bet on A at odds $\varepsilon/(1-\varepsilon)$, and the second that he excludes accepting a bet against A at odds $(1-\varepsilon)/\varepsilon$. So our subject accepts to bet on A at all odds, and rejects betting against A at any odds.

A subject can be practically certain about a number of

[1] Actually, it is enough to look at $\varepsilon \in (0,1)$, because for $\varepsilon \geq 1$, $\mathbb{I}_A - 1 + \varepsilon \in \mathscr{L}_{\geq 0}$ already belongs to the background model; see further on.

events. We collect the events he is practically certain about in the set $\mathscr{T} \subseteq \mathscr{P}$. So his initial assessment is:

$$\mathscr{A} = \langle \mathscr{A}_{\triangleright}; -\mathscr{A}_{\triangleright} \rangle \\ \text{with } \mathscr{A}_{\triangleright} = \{\mathbb{I}_A - 1 + \varepsilon \colon A \in \mathscr{T}, \varepsilon \in \mathbb{R}_{>0}\}. \quad (2)$$

Even before an assessment is given, some gambles can be presumed to be accepted and others to be rejected. Such *a priori* assumptions can be captured by positing a *background model* $\mathscr{S}$. In the context of favourability assessments, it follows from the discussion in Ref. [9, Section 5] that it is convenient to use the following background model:

$$\mathscr{S} = \langle \mathscr{L}_{\geq 0}; \mathscr{L}_{<0} \rangle,$$

so we take for granted that all non-negative gambles should be accepted, and all negative gambles should be rejected—be excluded from being accepted. The background model $\mathscr{S}$ is an instance of a *favour-indifference model* [9, Section 4.3], meaning that it fulfils the two conditions $-\mathscr{S}_{\prec} \subseteq \mathscr{S}_{\succeq}$ and $\mathscr{S}_{\succeq} = \mathscr{S}_{\triangleright} \cup \mathscr{S}_{\simeq}$.

We use $\mathscr{B} := \mathscr{A} \cup \mathscr{S} = \langle \mathscr{A}_{\triangleright} \cup \mathscr{L}_{\geq 0}; -\mathscr{A}_{\triangleright} \cup \mathscr{L}_{<0} \rangle$ to denote the smallest assessment that includes both the subject's assessment $\mathscr{A}$ and the background model $\mathscr{S}$.

Clearly, we will have to impose conditions on the set $\mathscr{T}$ of events that the subject is practically certain to occur. To give just one example, suppose $\mathscr{T} = \mathscr{P}$, then the subject is practically certain about the occurrence of every event and of its complement, which—as we shall see—is not a rational belief. The conditions we impose on the set $\mathscr{T}$ follow from three rationality criteria, described in full detail in Ref. [9]. In the next three sections, we discuss these rationality criteria and the resulting requirements on $\mathscr{T}$.

3.2 Deductive closure

That we are working with a linear utility scale for rewards has certain consequences. If the gambles f and g are acceptable, then so should be $f + g$, and λf, with $\lambda \in \mathbb{R}_{>0}$. These two observations are summarised in the *deductive extension* $\text{ext}_{\mathbf{D}}$:

$$\text{ext}_{\mathbf{D}} \mathscr{B} := \langle \text{posi}\, \mathscr{B}_{\succeq}; \mathscr{B}_{\prec} \rangle,$$

and we call an assessment $\mathscr{D}$ deductively closed if $\text{ext}_{\mathbf{D}} \mathscr{D} = \mathscr{D}$. This leads us to the first rationality criterion: assessments should be deductively closed.

Proposition 1. *The positive linear hull of $\mathscr{B}_{\succeq}$ is given by* $\text{posi}\, \mathscr{B}_{\succeq} = \mathscr{L}_{\geq 0} \cup \mathscr{L}^{\gtrdot}_{\mathscr{T}}$*, where we use the notations* $\mathscr{L}^{\gtrdot}_{\mathscr{T}} := \{f \in \mathscr{L} \colon (\exists B \in \mathscr{C}_{\mathscr{T}}) \inf(f|B) > 0\}$,[2] $\inf(f|B) := \inf\{f(x) \colon x \in B\}$ *and*

$$\mathscr{C}_{\mathscr{T}} := \left\{ \bigcap_{k=1}^{n} A_k \colon n \in \mathbb{N}, A_k \in \mathscr{T} \right\} \quad (3)$$

is the collection of all finite intersections of elements of $\mathscr{T}$. Note that $\text{posi}\, \mathscr{B}_{\succeq} \neq \mathscr{L}$ *if and only if $\emptyset \notin \mathscr{C}_{\mathscr{T}}$, meaning that $\mathscr{T}$ has the intuitively appealing* finite intersection property.

Proof. We infer from Eq. (1) that $\text{posi}(\mathscr{A}_{\triangleright} \cup \mathscr{L}_{\geq 0}) = \text{posi}\, \mathscr{A}_{\triangleright} \cup \mathscr{L}_{\geq 0} \cup (\text{posi}\, \mathscr{A}_{\triangleright} + \mathscr{L}_{\geq 0})$. Since $0 \in \mathscr{L}_{\geq 0}$, we see that $\text{posi}\, \mathscr{A}_{\triangleright} + \mathscr{L}_{\geq 0} \supseteq \text{posi}\, \mathscr{A}_{\triangleright}$, and therefore $\text{posi}(\mathscr{A}_{\triangleright} \cup \mathscr{L}_{\geq 0}) = \mathscr{L}_{\geq 0} \cup (\text{posi}\, \mathscr{A}_{\triangleright} + \mathscr{L}_{\geq 0})$. A gamble f belongs to $\text{posi}\, \mathscr{A}_{\triangleright} + \mathscr{L}_{\geq 0}$ if and only if there are $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{>0}$, $A_1, \dots, A_n \in \mathscr{T}$ and $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{R}_{>0}$ such that

$$f \geq \sum_{k=1}^{n} \lambda_k (\varepsilon_k - \mathbb{I}_{A_k^c}).$$

By an appropriate choice of the $\lambda_k > 0$ and the $\varepsilon_k \in (0,1)$, the lower bound in the inequality above can be made arbitrarily low (negative) provided that $\bigcap_{k=1}^{n} A_k = \emptyset$, and only then. This shows that $\emptyset \in \mathscr{C}_{\mathscr{T}} \Leftrightarrow \text{posi}\, \mathscr{A}_{\triangleright} + \mathscr{L}_{\geq 0} = \mathscr{L}$. So let us assume that $\emptyset \notin \mathscr{C}_{\mathscr{T}}$.

Consider any gamble f in $\text{posi}\, \mathscr{A}_{\triangleright} + \mathscr{L}_{\geq 0}$, then there are $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{>0}$, $A_1, \dots, A_n \in \mathscr{T}$ and $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{R}_{>0}$ such that $f \geq \sum_{k=1}^{n} \lambda_k (\varepsilon_k - \mathbb{I}_{A_k^c})$, and therefore $\inf(f | \bigcap_{k=1}^{n} A_k) \geq \sum_{k=1}^{n} \lambda_k \varepsilon_k > 0$.

Conversely, if $\inf(f | \bigcap_{k=1}^{n} A_k) =: \delta > 0$ for some $n \in \mathbb{N}$ and $A_1, \dots, A_n \in \mathscr{T}$, then let all $\lambda_k > \lambda := \delta - \inf f \geq 0$ and all $\varepsilon_k := \frac{\delta}{n \lambda_k} > 0$, so

$$\begin{aligned} \sum_{k=1}^{n} \lambda_k (\varepsilon_k - \mathbb{I}_{A_k^c}) &\leq \mathbb{I}_{\bigcap_{k=1}^{n} A_k} \delta + \mathbb{I}_{\bigcup_{k=1}^{n} A_k^c} (\delta - \lambda) \\ &= \mathbb{I}_{\bigcap_{k=1}^{n} A_k} \delta + \mathbb{I}_{\bigcup_{k=1}^{n} A_k^c} \inf f \leq f, \end{aligned}$$

meaning that $f \in \text{posi}\, \mathscr{A}_{\triangleright} + \mathscr{L}_{\geq 0}$. □

For notational convenience, we define $\mathscr{L}^{\lessdot}_{\mathscr{T}} := -\mathscr{L}^{\gtrdot}_{\mathscr{T}}$.

The set $\mathscr{C}_{\mathscr{T}}$, as defined in Eq. (3), satisfies all the requirements for a filter base. It is called the *filter base generated by* the set $\mathscr{T}$.

The deductively closed $\text{ext}_{\mathbf{D}} \mathscr{B}$ is not yet "perfect enough": for it to be a so-called *model*, we need to further impose the criteria of No Confusion and No Limbo.

3.3 No Confusion

Given the interpretation attached to an accept and to a reject statement, there should be no gambles in the set $(\text{ext}_{\mathbf{D}} \mathscr{B})_{\between} := (\text{ext}_{\mathbf{D}} \mathscr{B})_{\succeq} \cap (\text{ext}_{\mathbf{D}} \mathscr{B})_{\prec}$: a gamble cannot be accepted and rejected at the same time. This observation leads us to the second rationality criterion: deductively closed assessments need to have

$$\text{No Confusion:} \quad (\text{ext}_{\mathbf{D}} \mathscr{B})_{\between} = \emptyset.$$

The following proposition gives the conditions to be imposed on $\mathscr{T}$ in order to have No Confusion.

[2] We let $\inf(f|\emptyset)$ be $+\infty$ everywhere.

Proposition 2. *The deductively closed assessment* $\operatorname{ext}_{\mathbf{D}} \mathscr{B}$ *has No Confusion if and only if* $\mathscr{T}$ *satisfies the* finite intersection property*:* $\bigcap_{k=1}^{n} A_k \neq \emptyset$ *for all* $n \in \mathbb{N}$ *and all* $A_1, \dots, A_n \in \mathscr{T}$*, or equivalently,* $\emptyset \notin \mathscr{C}_{\mathscr{T}}$*.*

Proof. $\operatorname{ext}_{\mathbf{D}} \mathscr{B}$ has No Confusion if and only if $\mathscr{L}_{\mathscr{T}}^{\gtrdot} \cap -\mathscr{A}_{\rhd} = \emptyset$, $\mathscr{L}_{\geq 0} \cap -\mathscr{A}_{\rhd} = \emptyset$, $\mathscr{L}_{\mathscr{T}}^{\gtrdot} \cap \mathscr{L}_{<0} = \emptyset$ and $\mathscr{L}_{\geq 0} \cap \mathscr{L}_{<0} = \emptyset$. The last intersection is obviously empty, and the condition for the third one to be empty is clearly that $\emptyset \notin \mathscr{C}_{\mathscr{T}}$, taking into account Prop. 1.

The second intersection $\mathscr{L}_{\geq 0} \cap -\mathscr{A}_{\rhd}$ is empty if and only if $\mathbb{I}_{A^c} - \varepsilon \not\geq 0$ for all events A in $\mathscr{T}$ and all $\varepsilon \in \mathbb{R}_{>0}$, which is equivalent with $\emptyset \notin \mathscr{T}$.

The first intersection is non-empty if and only if there are $A \in \mathscr{T}$ and $B \in \mathscr{C}_{\mathscr{T}}$ such that $\inf(\mathbb{I}_{A^c} - \varepsilon | B) > 0$ for some $\varepsilon \in \mathbb{R}_{>0}$, or equivalently, such that $B \cap A = \emptyset$. This tells us that the first intersection is empty if and only if

$$(\forall A \in \mathscr{T})(\forall B \in \mathscr{C}_{\mathscr{T}}) B \cap A \neq \emptyset,$$

which is equivalent with $\emptyset \notin \mathscr{C}_{\mathscr{T}}$. □

Because of its form, $\mathscr{C}_{\mathscr{T}}$ is a filter base. Moreover, No Confusion is equivalent to $\mathscr{C}_{\mathscr{T}}$ being a *proper* filter base: in addition to $\mathscr{C}_{\mathscr{T}}$ being closed under finite intersections, it cannot contain the empty set. From now on, we consider only proper filter bases $\mathscr{C}_{\mathscr{T}}$, or equivalently, sets $\mathscr{T}$ that satisfy the finite intersection property.

3.4 No Limbo

For $\operatorname{ext}_{\mathbf{D}} \mathscr{B}$ to be a *model*, besides being deductively closed and having No Confusion, it also needs to satisfy a third and last rationality criterion: it must have No Limbo.

To see what this means, consider any deductively closed assessment $\mathscr{D} = \langle \mathscr{D}_{\succeq}; \mathscr{D}_{\prec} \rangle$ with No Confusion. At this point, all the gambles in $\mathscr{D}_{\smile} = \mathscr{L} \setminus (\mathscr{D}_{\succeq} \cup \mathscr{D}_{\prec})$ are unresolved, and can therefore in principle still be accepted or rejected. But it is proved in Ref. [9, Corollary 6] that the gambles in the so-called *limbo*

$$\left(\overline{\mathscr{D}_{\prec}} - (\mathscr{D}_{\succeq} \cup \{0\})\right) \setminus \mathscr{D}_{\prec}, \tag{4}$$

which is a subset of $\mathscr{D}_{\smile}$, cannot be made acceptable without creating Confusion. In other words, these are the unresolved gambles that have exactly the same effect as gambles in $\mathscr{D}_{\prec}$: if we considered them as acceptable too, the resulting assessment would have Confusion. So they are still in an unresolved state, but if we want to avoid Confusion, there is nothing for it: we must also reject them.

Starting from the deductively closed assessment $\mathscr{D}$ with No Confusion, additionally rejecting the gambles that are in its limbo results in its *reckoning extension* $\operatorname{ext}_{\mathbf{M}}$:

$$\operatorname{ext}_{\mathbf{M}} \mathscr{D} := \left\langle \mathscr{D}_{\succeq}; \overline{\mathscr{D}_{\prec}} \cup \left(\overline{\mathscr{D}_{\prec}} - \mathscr{D}_{\succeq}\right) \right\rangle, \tag{5}$$

and we say that a deductively closed assessment $\mathscr{D}$ without Confusion has No Limbo if and only if $\operatorname{ext}_{\mathbf{M}} \mathscr{D} = \mathscr{D}$, or equivalently, if and only if the set in Eq. (4) is empty.

We end up with $\mathscr{M} := \operatorname{ext}_{\mathbf{M}} \operatorname{ext}_{\mathbf{D}} \mathscr{B}$, a model that is deductively closed and has No Limbo and No Confusion; see Ref. [9, Prop. 7] for details. We call it a *coherent model*. The next proposition characterises $\mathscr{M}$, where the notation emphasises the set of favourable gambles.

Proposition 3. *The coherent model* $\mathscr{M} = \operatorname{ext}_{\mathbf{M}} \operatorname{ext}_{\mathbf{D}} \mathscr{B}$ *is given by* $\mathscr{M} = \langle \mathscr{M}_{\rhd} \cup \{0\}; -\mathscr{M}_{\rhd} \rangle$*, with*

$$\mathscr{M}_{\rhd} := \mathscr{L}_{\mathscr{T}}^{\gtrdot} \cup \mathscr{L}_{>0}.$$

Proof. The proof for the set of acceptable gambles $\mathscr{M}_{\succeq}$ follows from Prop. 1, $(\operatorname{ext}_{\mathbf{M}} \mathscr{D})_{\succeq} = \mathscr{D}_{\succeq}$ and $\mathscr{L}_{\geq 0} = \mathscr{L}_{>0} \cup \{0\}$. Taking into account Eq. (5) and Prop. 1, the set of rejected gambles is given by $\mathscr{M}_{\prec} = \overline{\mathscr{B}_{\prec}} \cup \left(\overline{\mathscr{B}_{\prec}} - (\mathscr{L}_{\geq 0} \cup \mathscr{L}_{\mathscr{T}}^{\gtrdot})\right) = \overline{\mathscr{B}_{\prec}} - (\mathscr{L}_{\geq 0} \cup \mathscr{L}_{\mathscr{T}}^{\gtrdot})$, where we used the fact that $0 \in \mathscr{L}_{\geq 0}$. Because $\overline{\mathscr{A}_{\prec} \cup \mathscr{L}_{<0}} = \overline{\mathscr{A}_{\prec}} \cup \mathscr{L}_{<0}$, it follows that

$$\begin{aligned} \mathscr{M}_{\prec} &= (\overline{\mathscr{A}_{\prec}} \cup \mathscr{L}_{<0}) - (\mathscr{L}_{\geq 0} \cup \mathscr{L}_{\mathscr{T}}^{\gtrdot}) \\ &= \left(\overline{\mathscr{A}_{\prec}} - \mathscr{L}_{\geq 0}\right) \cup \left(\overline{\mathscr{A}_{\prec}} - \mathscr{L}_{\mathscr{T}}^{\gtrdot}\right) \\ &\quad \cup \left(\mathscr{L}_{<0} - \mathscr{L}_{\geq 0}\right) \cup \left(\mathscr{L}_{<0} - \mathscr{L}_{\mathscr{T}}^{\gtrdot}\right). \end{aligned} \tag{6}$$

We first prove that $\mathscr{L}_{<0} \cup \mathscr{L}_{\mathscr{T}}^{\lessdot} \subseteq \mathscr{M}_{\prec}$. Observe that $\mathscr{L}_{<0} \subseteq \overline{\mathscr{A}_{\prec}} \cup \mathscr{L}_{<0} \subseteq (\overline{\mathscr{A}_{\prec}} \cup \mathscr{L}_{<0}) - (\mathscr{L}_{\geq 0} \cup \mathscr{L}_{\mathscr{T}}^{\gtrdot}) = \mathscr{M}_{\prec}$, where the last inclusion holds because $0 \in \mathscr{L}_{\geq 0}$. To show that also $-\mathscr{L}_{\mathscr{T}}^{\gtrdot} \subseteq \mathscr{M}_{\prec}$, use the next Lem. 1 to see that $-\mathscr{L}_{\mathscr{T}}^{\gtrdot} = (-\mathscr{L}_{\mathscr{T}}^{\gtrdot}) + \mathscr{L}_{<0}$, and by Eq. (6), this is a subset of $\mathscr{M}_{\prec}$.

Next, we prove that $\mathscr{L}_{<0} \cup \mathscr{L}_{\mathscr{T}}^{\lessdot} \supseteq \mathscr{M}_{\prec}$. We prove that each of the four terms of the union of Eq. (6) is a subset of $\mathscr{L}_{<0} \cup \mathscr{L}_{\mathscr{T}}^{\lessdot}$. To do so, it is useful to remark that

$$\mathscr{L}_{\mathscr{T}}^{\lessdot} = \operatorname{posi}(\mathscr{A}_{\prec} + \mathscr{L}_{\leq 0}) \supseteq \operatorname{posi} \mathscr{A}_{\prec} \supseteq \overline{\mathscr{A}_{\prec}}. \tag{7}$$

For $\overline{\mathscr{A}_{\prec}} - \mathscr{L}_{\geq 0}$, use Eq. (7) to infer that $\overline{\mathscr{A}_{\prec}} \subseteq -\mathscr{L}_{\mathscr{T}}^{\gtrdot}$, so $\overline{\mathscr{A}_{\prec}} - \mathscr{L}_{\geq 0} \subseteq -\mathscr{L}_{\mathscr{T}}^{\gtrdot} - \mathscr{L}_{\geq 0} = -\mathscr{L}_{\mathscr{T}}^{\gtrdot}$, where the equality follows from Lem. 1. For $\overline{\mathscr{A}_{\prec}} - \mathscr{L}_{\mathscr{T}}^{\gtrdot}$, use Eq. (7) to obtain $\overline{\mathscr{A}_{\prec}} - \mathscr{L}_{\mathscr{T}}^{\gtrdot} \subseteq -\mathscr{L}_{\mathscr{T}}^{\gtrdot} - \mathscr{L}_{\mathscr{T}}^{\gtrdot} = -\mathscr{L}_{\mathscr{T}}^{\gtrdot}$, where the equality follows from the fact that $-\mathscr{L}_{\mathscr{T}}^{\gtrdot}$ is a cone. Since $\mathscr{L}_{<0} - \mathscr{L}_{\geq 0} = \mathscr{L}_{<0}$, it only remains to consider the last term: use Lem. 1 to find that $\mathscr{L}_{<0} - \mathscr{L}_{\mathscr{T}}^{\gtrdot} = -\mathscr{L}_{\mathscr{T}}^{\gtrdot}$. □

Lemma 1. *For any collection of events* $\mathscr{T} \subseteq \mathscr{P}$ *that satisfies the finite intersection property,* $\mathscr{L}_{\mathscr{T}}^{\gtrdot} = \mathscr{L}_{\mathscr{T}}^{\gtrdot} + \mathscr{L}_{>0} = \mathscr{L}_{\mathscr{T}}^{\gtrdot} + \mathscr{L}_{\geq 0}$*.*

Proof. Since $0 \in \mathscr{L}_{\geq 0}$, we have $\mathscr{L}_{\mathscr{T}}^{\gtrdot} \subseteq \mathscr{L}_{\mathscr{T}}^{\gtrdot} + \mathscr{L}_{\geq 0}$, and since $\mathscr{L}_{>0} \subseteq \mathscr{L}_{\geq 0}$, also $\mathscr{L}_{\mathscr{T}}^{\gtrdot} + \mathscr{L}_{>0} \subseteq \mathscr{L}_{\mathscr{T}}^{\gtrdot} + \mathscr{L}_{\geq 0}$. The proof is complete if we can prove that $\mathscr{L}_{\mathscr{T}}^{\gtrdot} + \mathscr{L}_{\geq 0}$ is also included in both $\mathscr{L}_{\mathscr{T}}^{\gtrdot}$ and $\mathscr{L}_{\mathscr{T}}^{\gtrdot} + \mathscr{L}_{>0}$. Consider any gamble $f \in \mathscr{L}_{\mathscr{T}}^{\gtrdot} + \mathscr{L}_{\geq 0}$, so there are $g \in \mathscr{L}$ and $B \in \mathscr{C}_{\mathscr{T}}$ such that $\delta := \inf(g|B) > 0$ and $f \geq g$. This means that also $\inf(f|B) > 0$, and therefore $f \in \mathscr{L}_{\mathscr{T}}^{\gtrdot}$. Also, consider the gamble $h := \delta/2 \mathbb{I}_B + g \mathbb{I}_{B^c} < f$. Because $\inf(h|B) = \delta/2 > 0$, it follows that $h \in \mathscr{L}_{\mathscr{T}}^{\gtrdot}$ and therefore $f = h + (f - h) \in \mathscr{L}_{\mathscr{T}}^{\gtrdot} + \mathscr{L}_{>0}$. □

To summarise, we started out with the assessment $\mathscr{A}$ of a subject who is practically certain of the occurrence of all events in $\mathscr{T}$, and added the background model $\mathscr{S}$, leading to a larger assessment $\mathscr{B} = \mathscr{A} \cup \mathscr{S}$. Using deductive and reckoning extension, and by imposing restrictions on $\mathscr{T}$, namely that $\mathscr{T}$ has the finite intersection property, we added acceptable as well as rejected gambles to end up with the coherent model $\mathscr{M} = \mathrm{ext}_{\mathbf{M}}\,\mathrm{ext}_{\mathbf{D}}\,\mathscr{B}$. Prop. 3 guarantees that, as was the case for the initial assessment of Eq. (2), the coherent model $\mathscr{M}$ is fully determined by the set of favourable gambles

$$\mathscr{M}_{\triangleright} = \{f \in \mathscr{L} \colon (\exists B \in \mathscr{C}_{\mathscr{T}}) \inf(f|B) > 0\} \cup \mathscr{L}_{>0},$$

leaving aside the always indifferent zero gamble. Because $\mathscr{M}$ is a coherent model, we call this set $\mathscr{M}_{\triangleright}$ a *coherent set of favourable gambles*. In this model $\mathscr{M}$, 0 is the only indifferent gamble: $\mathscr{M}_{\simeq} = \mathscr{M}_{\succeq} \cap -\mathscr{M}_{\succeq} = \{0\}$, and $\mathscr{M}$ is an instance of a favour-indifference model, because $-\mathscr{M}_{\prec} \subseteq \mathscr{M}_{\succeq}$ and $\mathscr{M}_{\succeq} = \mathscr{M}_{\triangleright} \cup \mathscr{M}_{\simeq}$.

3.5 Finding all practically certain events

We now ask ourselves whether the inference procedure described above, which allowed us to infer from the set of favourable gambles $\mathscr{A}_{\triangleright}$ the larger set of favourable gambles $\mathscr{M}_{\triangleright}$, bears any relationship to inference in classical propositional logic? Which are the other events, besides the ones in $\mathscr{T}$, that the inference procedure tells us our subject, if he is rational, should also be practically certain of?

As we have suggested above, a subject who is certain about an event A expresses this as finding favourable the gambles of the form $-\mathbb{I}_{A^c} + \varepsilon$, with $\varepsilon \in \mathbb{R}_{>0}$. We denote the corresponding set of favourable gambles by $\mathscr{A}^A_{\triangleright} := \{-\mathbb{I}_{A^c} + \varepsilon \colon \varepsilon \in \mathbb{R}_{>0}\}$. The question therefore becomes: *for which events A is the set $\mathscr{A}^A_{\triangleright}$ a subset of $\mathscr{M}_{\triangleright}$?* As the gambles in $\mathscr{A}^A_{\triangleright}$ are, for ε small enough, positive only on A, the answer to this question is immediate:

$$\mathscr{A}^A_{\triangleright} \subseteq \mathscr{M}_{\triangleright} \Leftrightarrow (\exists B \in \mathscr{C}_{\mathscr{T}}) B \subseteq A.$$

This tells us that the subject should be practically certain of all events in the filter generated by $\mathscr{T}$:

$$\mathscr{F}_{\mathscr{T}} := \{A \in \mathscr{P} \colon (\exists B \in \mathscr{C}_{\mathscr{T}})(B \subseteq A)\}.$$

This is a proper filter provided that $\emptyset \notin \mathscr{C}_{\mathscr{T}}$. Also observe that $\mathscr{L}^{>}_{\mathscr{T}} = \{f \in \mathscr{L} \colon (\exists B \in \mathscr{F}_{\mathscr{T}}) \inf(f|B) > 0\}$.

Any filter is a set-theoretic counterpart of a collection of propositions that is deductively closed (closed under conjunction and modus ponens), and the generated filter corresponds to the deductive closure of a set of propositions, in classical propositional logic. We see that on our specific interpretation of it—or semantics for it—the logic of practical certainty has the same basic machinery as classical propositional logic. In simple terms: if someone is practically certain that both the events A and B occur, it is reasonable to be practically certain of $A \cap B$; and if someone is practically certain that the event A occurs, then it is reasonable to be practically certain of every event $B \supseteq A$.

Our argument goes further than that, because it also allows us to infer which gambles a subject should find favourable if he is practically certain that all events in $\mathscr{T}$ occur: all gambles in $\mathscr{M}_{\triangleright}$, which are the gambles that are strictly positive, or that have a strictly positive return, bounded away from zero, on some practically certain event.

4 Modelling practical certainty using acceptability

When a subject is practically certain that an event A occurs, we have taken this to mean, in Section 3, that he finds favourable every gamble of the form $-\mathbb{I}_{A^c} + \varepsilon$, with $\varepsilon \in \mathbb{R}_{>0}$. Here, we repeat the same reasoning with a weaker assessment of acceptability, rather than favourability: if a subject is practically certain that an event A occurs, we now take this to mean that he finds every gamble of the form $-\mathbb{I}_{A^c} + \varepsilon$, with $\varepsilon \in \mathbb{R}_{>0}$, acceptable. With $\mathscr{T}$ the collection of events he is practically certain of, his assessment is therefore:

$$\mathscr{A}^- := \langle \{-\mathbb{I}_{A^c} + \varepsilon : A \in \mathscr{T}, \varepsilon \in \mathbb{R}_{>0}\} ; \emptyset \rangle.$$

We make the same a priori assumptions summarised in the background model $\mathscr{S} = \langle \mathscr{L}_{\geq 0}; \mathscr{L}_{<0} \rangle$, leading to the smallest background-including assessment $\mathscr{B}^- = \mathscr{A}^- \cup \mathscr{S}$.

In the next proposition, we determine the relationship between $\mathscr{M}^- = \mathrm{ext}_{\mathbf{M}}\,\mathrm{ext}_{\mathbf{D}}\,\mathscr{B}^-$ and $\mathscr{M}$, and show that the (apparently) weaker acceptability assessment leads to the same conclusions.[3]

Proposition 4. *Using deductive extension we obtain* $\mathrm{ext}_{\mathbf{D}}\,\mathscr{B}^- = \left\langle \mathscr{L}_{\geq 0} \cup \mathscr{L}^{>}_{\mathscr{T}}; \mathscr{L}_{<0} \right\rangle$. *This deductively closed assessment has No Confusion if and only if $\mathscr{T}$ satisfies the finite intersection property. The corresponding coherent model is* $\mathscr{M}^- = \mathrm{ext}_{\mathbf{M}}\,\mathrm{ext}_{\mathbf{D}}\,\mathscr{B}^- = \mathscr{M}$.

Proof. The argument is analogous to, but less involved than, that in the proofs of Props. 1–3. □

5 An alternative way of modelling practical certainty using indifference

Williams [15] and Walley [12] define a lower prevision $\underline{p}$ for a gamble f as a supremum acceptable buying price:

[3] The equivalence between the implications of favourability and acceptability assessments does not hold in more general cases. As an example, consider the background model $\langle \mathscr{L}_{\geq 0}; \emptyset \rangle$. Then the conclusions from every non-empty favourability assessment differs from the corresponding acceptability assessment.

the highest price $\underline{p}$ such that $f-\underline{p}+\varepsilon$ is acceptable—or equivalently as it turns out, favourable—for all $\varepsilon>0$. In the previous sections, we have used an approach with a very similar flavour to account for practical certainty: the supremum acceptable buying price for (the indicator of) a practically certain event is 1.

The approach that Bruno de Finetti [5, 7] follows in defining the (precise) prevision p for a gamble f, is rather different:[4] it is the unique number p such that the subject is indifferent between the uncertain f and the fixed p, or equivalently, between $f-p$ and 0.

We therefore also propose an alternative way of modelling practical certainty, more along the lines of de Finetti's approach to previsions: we model a subject's practical certainty of the occurrence of an event A by an assessment of indifference between $\mathbb{I}_A$ and 1, or equivalently, between $\mathbb{I}_{A^c}$ and 0. This amounts to a statement of acceptability for both $\mathbb{I}_{A^c}$ and its negation $-\mathbb{I}_{A^c}$. But, since $\mathbb{I}_{A^c}\geq 0$, and since we will use $\mathscr{L}_{\geq 0}$ as a background model for acceptability, meaning that all non-negative gambles are a priori assumed to be acceptable (see further on), we need only explicitly state the acceptability of $-\mathbb{I}_{A^c}$. This assessment is stronger than the corresponding one in the previous sections: here the subject actually accepts the gamble $-\mathbb{I}_{A^c}$, whereas before he only accepted gambles of the form $-\mathbb{I}_{A^c}+\varepsilon$, with $\varepsilon\in\mathbb{R}_{>0}$.

If our subject is practically certain of every event in the collection $\mathscr{T}\subseteq\mathscr{P}$, this leads to the (indifference) assessment:

$$\mathscr{A}':=\langle\{-\mathbb{I}_{A^c}\colon A\in\mathscr{T}\};\emptyset\rangle.$$

Before, we used the background model $\mathscr{S}=\langle\mathscr{L}_{\geq 0};\mathscr{L}_{<0}\rangle$. The nature of an indifference assessment no longer allows us to use $\mathscr{S}$ as background model, as this would lead to difficulties: since $-\mathbb{I}_{A^c}\in\mathscr{L}_{<0}$ if $A^c\neq\emptyset$, in order to avoid No Confusion, the set $\mathscr{T}$ can only contain the trivial certain event $\mathscr{X}$.[5] For this reason, we propose a slightly more conservative background model:

$$\mathscr{S}'=\langle\mathscr{L}_{\geq 0};\mathscr{L}_{\lhd 0}\rangle,$$

where we take for granted that all non-negative gambles should be accepted, and all gambles that are point-wise (strictly) negative should be rejected:

$$\mathscr{L}_{\lhd 0}:=\{f\in\mathscr{L}\colon(\forall x\in\mathscr{X})f(x)<0\}.$$

We use $\mathscr{B}':=\mathscr{A}'\cup\mathscr{S}'=\left\langle\mathscr{A}'_{\succeq}\cup\mathscr{L}_{\geq 0};\mathscr{L}_{\lhd 0}\right\rangle$ to denote the smallest assessment that includes both the subject's indifference assessment $\mathscr{A}'$ and the background model $\mathscr{S}'$.

In this section, due to page limitations, and because the reasoning uses similar arguments to the ones in Section 3, we will omit the proofs.

[4] For an extensive discussion of the difference between the two approaches, we refer to Refs. [9] and [11].

[5] See also the discussion in Ref. [9, Section 5].

As before, in order to obtain a coherent model, we have to impose rationality conditions on the set $\mathscr{T}$ of practically certain events, which we explore next.

5.1 Deductive closure

The first rationality criterion states that we have to accept every gamble that can be deduced from $\mathscr{B}'_{\succeq}$: the deductive closure is $\operatorname{ext}_{\mathbf{D}}\mathscr{B}'=\left\langle\operatorname{posi}\mathscr{B}'_{\succeq};\mathscr{B}'_{\prec}\right\rangle$.

Proposition 5. *The positive linear hull of $\mathscr{B}'_{\succeq}$ is*

$$\operatorname{posi}\mathscr{B}'_{\succeq}=\mathscr{L}^{\geq}_{\mathscr{T}}:=\{f\in\mathscr{L}\colon(\exists B\in\mathscr{C}_{\mathscr{T}})\mathbb{I}_B f\geq 0\},$$

with $\mathscr{C}_{\mathscr{T}}$ defined as in Eq. (3)*. Note that* $\operatorname{posi}\mathscr{B}'_{\succeq}\neq\mathscr{L}$ *if and only if $\emptyset\notin\mathscr{C}_{\mathscr{T}}$.*

Compared with $\operatorname{posi}\mathscr{B}_{\succeq}$, $\operatorname{posi}\mathscr{B}'_{\succeq}$ contains more gambles: those gambles f that are non-negative on an event B in $\mathscr{C}_{\mathscr{T}}$, but for which $\inf(f|B)$ is zero.

5.2 No Confusion

The second rationality criterion requires that the deductively closed assessment $\operatorname{ext}_{\mathbf{D}}\mathscr{B}'$ should have No Confusion. This leads to the same condition on $\mathscr{T}$ as before in Section 3:

Proposition 6. *The deductively closed assessment* $\operatorname{ext}_{\mathbf{D}}\mathscr{B}'$ *has No Confusion if and only if $\mathscr{T}$ satisfies the finite intersection property, or equivalently, if $\emptyset\notin\mathscr{C}_{\mathscr{T}}$.*

As in Section 3, the second rationality criterion turns $\mathscr{C}_{\mathscr{T}}$ into a proper filter base. From now on, we will assume $\mathscr{C}_{\mathscr{T}}$ to be proper.

5.3 No Limbo

The final rationality criterion of No Limbo leads us to apply the reckoning extension $\operatorname{ext}_{\mathbf{M}}$ to the deductive extension $\operatorname{ext}_{\mathbf{D}}\mathscr{B}'$ with No Confusion, leading to a coherent model $\mathscr{M}':=\operatorname{ext}_{\mathbf{M}}\operatorname{ext}_{\mathbf{D}}\mathscr{B}'$.

Proposition 7. *The coherent model $\mathscr{M}'$ is given by $\mathscr{M}'=\left\langle\mathscr{M}'_{\succeq};\mathscr{M}'_{\prec}\right\rangle$, with $\mathscr{M}'_{\succeq}=\mathscr{L}^{\geq}_{\mathscr{T}}$ and $\mathscr{M}'_{\prec}=-\mathscr{L}^{\rhd}_{\mathscr{T}}=\mathscr{L}^{\lhd}_{\mathscr{T}}:=\{f\in\mathscr{L}\colon(\exists B\in\mathscr{C}_{\mathscr{T}})(\forall x\in B)f(x)<0\}$.*

The corresponding set of favourable gambles $\mathscr{M}'_{\rhd}$ is:

$$\begin{aligned}\mathscr{M}'_{\rhd}&=\mathscr{M}'_{\succeq}\cap-\mathscr{M}'_{\prec}=\mathscr{L}^{\geq}_{\mathscr{T}}\cap\mathscr{L}^{\rhd}_{\mathscr{T}}=\mathscr{L}^{\rhd}_{\mathscr{T}}\\&=\{f\in\mathscr{L}\colon(\exists B\in\mathscr{C}_{\mathscr{T}})(\forall x\in B)f(x)>0\}.\end{aligned}$$

5.4 Finding all practically certain events

As in Section 3.5, we ask ourselves whether, in addition to the events in $\mathscr{T}$, the criteria of rationality allow the subject to infer the practical certainty of more events. Since, here, we are modelling practical certainty via indifference, we look at the indifferent gambles $\mathscr{M}'_{\simeq}$ in the coherent model $\mathscr{M}'$, and our subject is practically certain about an event A precisely when he is indifferent about $\mathbb{I}_{A^c}$, meaning that $-\mathbb{I}_{A^c}$ (in addition to $\mathbb{I}_{A_c}$) belongs to his inferred set of indifferent gambles $\mathscr{M}'_{\simeq} = \mathscr{M}'_{\succeq} \cap -\mathscr{M}'_{\succeq}$.

So let us look for an expression for $\mathscr{M}'_{\simeq}$. This set contains the gambles for which there are B and B' in $\mathscr{C}_{\mathscr{T}}$ such that both $\mathbb{I}_B f \geq 0$ and $\mathbb{I}_{B'} f \leq 0$. Since $\mathscr{C}_{\mathscr{T}}$ is closed under finite intersections, we find that

$$\begin{aligned}\mathscr{M}'_{\simeq} &= \{f \in \mathscr{L} \colon (\exists B \in \mathscr{C}_{\mathscr{T}})\mathbb{I}_B f = 0\}\\ &= \{f \in \mathscr{L} \colon (\exists B \in \mathscr{F}_{\mathscr{T}})\mathbb{I}_B f = 0\},\end{aligned}$$

and therefore also

$$-\mathbb{I}_{A^c} \in \mathscr{M}'_{\simeq} \Leftrightarrow (\exists B \in \mathscr{C}_{\mathscr{T}}) A^c \cap B = \emptyset \Leftrightarrow A \in \mathscr{F}_{\mathscr{T}}.$$

This tells us that the subject can be practically certain of all events A in the proper filter $\mathscr{F}_{\mathscr{T}}$ generated by $\mathscr{T}$, as in Section 3.5. Here too, our approach allows us to say even more: the subject should regard as favourable all gambles that are (strictly) positive on some practically certain event, and be indifferent about any gamble that is zero on some practically certain event.

6 Coherent lower prevision and coherent lower probability

6.1 Coherent lower prevision

With every set of favourable gambles we can associate a *lower prevision* $\underline{P}$ and an *upper prevision* $\overline{P}$. Lower previsions (or lower expectation functionals) $\underline{P}$ as wel as upper previsions (or upper expectation functionals) $\overline{P}$ are real-valued functionals defined on $\mathscr{L}$. Given any set of favourable gambles $\mathscr{D}$, then the corresponding lower prevision $\underline{P}$ and upper prevision $\overline{P}$ are defined by:

$$\begin{aligned}&\underline{P}(f) := \sup\{\mu \in \mathbb{R} \colon f - \mu \in \mathscr{D}\} \text{ and}\\ &\overline{P}(f) := \inf\{\mu \in \mathbb{R} \colon \mu - f \in \mathscr{D}\} \text{ for every } f \text{ in } \mathscr{L}.\end{aligned}$$

If the defining set of favourable gambles is coherent, then we call $\underline{P}$ and $\overline{P}$ coherent. Since $\underline{P}(f) = -\overline{P}(-f)$ for every $f \in \mathscr{L}$, lower and upper previsions contain the same information, and we focus on lower previsions.

Let us calculate the coherent lower prevision corresponding with $\mathscr{M}_{\triangleright}$. For any gamble f, $\underline{P}(f)$ is the supremum μ such that $f - \mu$ is an element of $\mathscr{M}_{\triangleright}$, or equivalently, it is the supremum μ such that

$$\mu < f \text{ or } (\exists B \in \mathscr{C}_{\mathscr{T}})\mu < \inf(f|B).$$

This tells us that $\underline{P}(f)$ is the maximum of $\inf f$ and $\sup_{B \in \mathscr{C}_{\mathscr{T}}} \inf(f|B)$. Since the latter number is never smaller than the former, we conclude:[6]

$$\underline{P}(f) = \sup_{B \in \mathscr{C}_{\mathscr{T}}} \inf(f|B) = \sup_{B \in \mathscr{F}_{\mathscr{T}}} \inf(f|B).$$

To make explicit the proper filter of events $\mathscr{C}_{\mathscr{T}}$ we are using, we denote this lower prevision also as $\underline{P}_{\mathscr{F}_{\mathscr{T}}}$. Observe that $\underline{P}_{\mathscr{F}_{\mathscr{T}}}$ is coherent if and only if $\mathscr{C}_{\mathscr{T}}$ is a proper filter base.

Using a similar argument as above, it follows that the lower prevision $\underline{P}'$ corresponding with the set of favourable gambles $\mathscr{M}'_{\triangleright}$ is the supremum μ such that $(\exists B \in \mathscr{C}_{\mathscr{T}}) \inf(f|B) > \mu$, whence $\underline{P}'(f) = \sup_{B \in \mathscr{C}_{\mathscr{T}}} \inf(f|B) = \underline{P}(f)$ for every gamble $f \in \mathscr{L}$. This tells us that, regardless of whether we formulate practical certainty using favourability or indifference assessments, we end up with the same corresponding coherent lower prevision.

6.2 Coherent lower probability

With every lower prevision $\underline{P}$, we can associate a lower probability $\underline{Q}$. A lower probability $\underline{Q}$ is a real-valued set function defined on $\mathscr{P}$. Given a lower prevision $\underline{P}$, then the corresponding lower probability $\underline{Q}$ is defined by:

$$\underline{Q}(A) := \underline{P}(\mathbb{I}_A) \text{ for each event } A \text{ in } \mathscr{P}.$$

If the defining lower prevision is coherent, then the corresponding lower probability is called coherent as well.

We look at the lower probability $\underline{R}_{\mathscr{F}_{\mathscr{T}}}$ corresponding with the lower prevision $\underline{P}_{\mathscr{F}_{\mathscr{T}}}$. For any event A, the lower probability $\underline{R}_{\mathscr{F}_{\mathscr{T}}}(A)$ equals $\sup_{B \in \mathscr{C}_{\mathscr{T}}} \inf(\mathbb{I}_A|B)$. Since $\inf(\mathbb{I}_A|B)$ is 1 if $B \subseteq A$ and 0 otherwise, we have:

$$\underline{R}_{\mathscr{F}_{\mathscr{T}}}(A) = \begin{cases} 1 & \text{if } (\exists B \in \mathscr{C}_{\mathscr{T}}) B \subseteq A \\ 0 & \text{otherwise.}\end{cases} = \begin{cases} 1 & \text{if } A \in \mathscr{F}_{\mathscr{T}} \\ 0 & \text{otherwise.}\end{cases}$$

This lower probability is coherent because $\underline{P}_{\mathscr{F}_{\mathscr{T}}}$ is. This tells us that the subject is willing to bet at all odds on the occurrence of every event $A \in \mathscr{F}_{\mathscr{T}}$. For all other events, he has no commitment whatsoever: he is only willing to bet on these other evens at zero odds. Compare this with the discussion in Sections 3.5 and 5.4.

Conversely, an event A for which the upper probability is zero—which means that the subject is willing to bet at all odds against the occurrence of A—reflects practical certainty that A does not occur. For an event A, the upper probability $\overline{R}_{\mathscr{F}_{\mathscr{T}}}(A)$ equals $\inf_{B \in \mathscr{C}_{\mathscr{T}}} \sup(\mathbb{I}_A|B)$, which is zero iff $A \in \mathscr{I} := \{B^c \colon B \in \mathscr{T}\}$. This ideal[7] of subsets $\mathscr{I}$ is the set of events the agent is practically certain will not occur.

[6] This lower prevision constitute a particular case of the so-called filter maps, see [3].

[7] The notion of an ideal is the dual notion of a filter: an ideal $\mathscr{I}$ is a subset of $\mathscr{P}$ that is *closed under finite unions* ($A \cup B \in \mathscr{I}$ when $A, B \in \mathscr{I}$) and *decreasing* (if $A \in \mathscr{I}$ and $B \subseteq A$, then also $B \in \mathscr{I}$).

7 Connection with strong belief structures

7.1 Strong belief structures

For this section, we will need some extra notation. We call $\mathbf{A}$ the collection of all the assessments—with or without Confusion: $\mathbf{A} = \{\langle \mathscr{D}_{\succeq}; \mathscr{D}_{\prec} \rangle : \mathscr{D}_{\succeq}, \mathscr{D}_{\prec} \subseteq \mathscr{L}\}$. Assessments in $\mathbf{A}$ can be partially ordered by set inclusion $\subseteq$: with two assessments $\mathscr{D}$ and $\mathscr{D}'$ in $\mathbf{A}$, we write $\mathscr{D} \subseteq \mathscr{D}'$ if and only if $\mathscr{D}_{\succeq} \subseteq \mathscr{D}'_{\succeq}$ and $\mathscr{D}_{\prec} \subseteq \mathscr{D}'_{\prec}$. The corresponding partially ordered set is denoted by $(\mathbf{A}, \subseteq)$.

Not all assessments in $\mathbf{A}$ are of interest; we can restrict our attention to some generic subclass of models $\mathbb{M} \subseteq \mathbf{A}$. This $\mathbb{M}$ inherits the partial order $\subseteq$ from $\mathbf{A}$. We call $\hat{\mathbb{M}}$ the set of *maximal*, or undominated, models in $\mathbb{M}$: $\hat{\mathbb{M}} := \{\mathscr{D} \in \mathbb{M} \colon (\forall \mathscr{D}' \in \mathbb{M})(\mathscr{D} \subseteq \mathscr{D}' \Rightarrow \mathscr{D} = \mathscr{D}')\}$. In contradistinction with $\mathbf{A}$, where $\hat{\mathbf{A}} = \{\langle \mathscr{L}; \mathscr{L} \rangle\}$ is its top (and unique maximal element), the family of models $\mathbb{M}$ may have no, one or multiple maximal elements.

We are interested in whether the structure $(\mathbf{A}, \mathbb{M}, \subseteq)$ is a *strong belief structure* [2], meaning that it satisfies the following four criteria:

S1. $(\mathbf{A}, \subseteq)$ is a complete lattice: for any subset $\mathbf{B}$ of $\mathbf{A}$, its supremum $\sup \mathbf{B}$ and its infimum $\inf \mathbf{B}$ with respect to the order $\subseteq$ exist. Here the component-wise union operator $\bigcup$ plays the role of supremum and the component-wise intersection operator $\bigcap$ that of infimum.

S2. $(\mathbb{M}, \subseteq)$ is a (component-wise) *intersection structure*, meaning that $\mathbb{M}$ is closed under arbitrary non-empty infima: for any non-empty subset $\mathbf{B}$ of $\mathbb{M}$, $\inf \mathbf{B} \in \mathbb{M}$.

S3. The partially ordered set $(\mathbb{M}, \subseteq)$ has no top.

S4. The partially ordered set $(\mathbb{M}, \subseteq)$ is *dually atomic*: $\hat{\mathbb{M}} \neq \emptyset$ and $\mathscr{D} = \inf\{\mathscr{D}' \in \hat{\mathbb{M}} \colon \mathscr{D} \subseteq \mathscr{D}'\}$ if $\mathscr{D} \in \mathbb{M}$.

A structure $(\mathbf{A}, \mathbb{M}, \subseteq)$ that satisfies requirements S1–S3 is called a *belief structure*. The relevance of the additional requirement S4 is that the maximal coherent models can be used to construct any coherent model. We want to investigate whether the coherent models encountered in Sections 3 and 5 constitute strong belief structures.

7.2 Favourability of acceptability assessments

We consider the family of models for practical certainty following from favourability or acceptability assessments, as described in Sections 3 and 4:

$$\mathbb{C} := \left\{\left\langle \mathscr{L}^{>}_{\mathscr{F}} \cup \mathscr{L}_{\geq 0}; \mathscr{L}^{<}_{\mathscr{F}} \cup \mathscr{L}_{<0} \right\rangle : \mathscr{F} \in \mathbb{F}\right\}.$$

For this family $\mathbb{C}$, it is not difficult to show by means of a counterexample that $(\mathbf{A}, \mathbb{C}, \subseteq)$ does not constitute a strong belief structure: it is not even a belief structure as it violates requirement S2.

7.3 Indifference assessments

We consider the family of models for practical certainty following from indifference assessments, as described in Section 5:

$$\mathbb{C}' := \left\{\left\langle \mathscr{L}^{\geq}_{\mathscr{F}}; \mathscr{L}^{\lhd}_{\mathscr{F}} \right\rangle : \mathscr{F} \in \mathbb{F}\right\}.$$

The elements of $\mathbb{C}'$ are the coherent models identified in Prop. 7, and to make explicit which filter we are using, we denote them by $\mathscr{M}'(\mathscr{F}) = \left\langle \mathscr{M}'_{\succeq}(\mathscr{F}); \mathscr{M}'_{\prec}(\mathscr{F}) \right\rangle := \left\langle \mathscr{L}^{\geq}_{\mathscr{F}}; \mathscr{L}^{\lhd}_{\mathscr{F}} \right\rangle$. In contrast with the structure considered in Section 7.2, $(\mathbf{A}, \mathbb{C}', \subseteq)$ is a strong belief structure.

Proposition 8. *$(\mathbf{A}, \mathbb{C}', \subseteq)$ is a strong belief structure.*

Proof. We have to prove that $(\mathbf{A}, \mathbb{C}', \subseteq)$ fulfils the requirements S1–S4. S1 is fulfilled thanks to [9, Section 2.6]. S2 is fulfilled thanks to the next Lem. 2. For S3, consider Lem. 4 and take into account that the set of maximal elements of $\mathbb{F}$ is the set of ultrafilters $\mathbb{U}$, so $\mathbb{C}'$ has no top. S4 is fulfilled thanks to Lem. 4 and the *Ultrafilter Theorem* [10]. □

Lemma 2. *$(\mathbb{C}', \subseteq)$ is an intersection structure.*

Proof. Consider an arbitrary non-empty subset $\mathbf{B} \subseteq \mathbb{C}'$. We can describe $\mathbf{B}$ using a family of filters $\mathscr{F}_i$, $i \in I$ with a non-empty index set $I \neq \emptyset$: $\mathbf{B} = \{\mathscr{M}'(\mathscr{F}_i) : i \in I\}$. We now have to prove that $\inf \mathbf{B} \in \mathbb{C}'$, or equivalently, that $\bigcap_{i \in I} \mathscr{M}'(\mathscr{F}_i) \in \mathbb{C}'$, since taking infima corresponds to taking component-wise intersections. Consider any gamble f, then:

$$\begin{aligned} f \in \bigcap_{i \in I} \mathscr{M}'_{\succeq}(\mathscr{F}_i) &\Leftrightarrow (\forall i \in I)(\exists B_i \in \mathscr{F}_i)(\forall x \in B_i) f(x) \geq 0 \\ &\Leftrightarrow (\exists B \in \bigcap_{i \in I} \mathscr{F}_i)(\forall x \in B) f(x) \geq 0. \end{aligned}$$

For the second equivalence the converse implication is trivial. The direct implication holds because it follows that $(\forall x \in \bigcup_{i \in I} B_i) f(x) \geq 0$ and $\bigcup_{i \in I} B_i$ belongs to all $\mathscr{F}_j$, $j \in I$. By the next Lem. 3, $\bigcap_{i \in I} \mathscr{F}_i$ is a proper filter. Using a completely similar argument leads to a similar conclusion for the rejected gambles $\bigcap_{i \in I} \mathscr{M}'_{\prec}(\mathscr{F}_i)$. □

The proof of Lem. 2 tells us more than that $\mathbb{C}'$ is closed under arbitrary non-empty intersections; it also tells us how to find the filter that is associated with this intersection:

$$\bigcap_{i \in I} \mathscr{M}'(\mathscr{F}_i) = \mathscr{M}'\left(\bigcap_{i \in I} \mathscr{F}_i\right). \tag{8}$$

Lemma 3. *Given a non-empty family of proper filters $\mathscr{F}_i$, $i \in I$, $\mathscr{F} := \bigcap_{i \in I} \mathscr{F}_i \in \mathbb{F}$.*

Proof. Since $\emptyset \notin \mathscr{F}_i$, also $\emptyset \notin \mathscr{F}$. Because $\mathscr{X} \in \mathscr{F}_i$ for every $i \in I$, also $\mathscr{F} \neq \emptyset$. Furthermore, let $A, B \in \mathscr{F}$, meaning that $A, B \in \mathscr{F}_i$ for every $i \in I$. Then also also $A \cap B \in \mathscr{F}_i$ for every $i \in I$, what tells us that $A \cap B \in \mathscr{F}$, meaning that $\mathscr{F}$ is closed under conjunction. Finally, let $A \in \mathscr{F}$ and $B \supseteq A$. Then $B \in \mathscr{F}_i$ for every $i \in I$, whence $B \in \mathscr{F}$, meaning that $\mathscr{F}$ is increasing. □

Lemma 4. *The partially ordered sets* $(\mathbb{C}', \subseteq)$ *and* $(\mathbb{F}, \subseteq)$ *are* order isomorphic*, meaning that there is a bijection* ϕ *from* $\mathbb{C}'$ *to* $\mathbb{F}$ *such that* $\mathscr{M}'(\mathscr{F}_k) \subseteq \mathscr{M}'(\mathscr{F}_\ell)$ *if and only if* $\phi(\mathscr{M}'(\mathscr{F}_k)) \subseteq \phi(\mathscr{M}'(\mathscr{F}_\ell))$ *for all* $\mathscr{M}'(\mathscr{F}_k), \mathscr{M}'(\mathscr{F}_\ell) \in \mathbb{C}'$.

Proof. Consider the map ϕ from $\mathbb{C}'$ to $\mathbb{F}$ defined by $\phi(\mathscr{M}'(\mathscr{F})) := \mathscr{F}$. ϕ is clearly injective and surjective, and therefore a bijection. We then have to prove for all $\mathscr{F}_k, \mathscr{F}_\ell \in \mathbb{F}$ that $\mathscr{M}'(\mathscr{F}_k) \subseteq \mathscr{M}'(\mathscr{F}_\ell)$ if and only if $\mathscr{F}_k \subseteq \mathscr{F}_\ell$. The 'if' is immediate from the definition of the $\mathscr{F}_i$. For the 'only if', start with $\mathscr{M}'(\mathscr{F}_k) \subseteq \mathscr{M}'(\mathscr{F}_\ell)$, and focuss on the accepted gambles. It follows that $\mathscr{M}'_{\succeq}(\mathscr{F}_k) \subseteq \mathscr{M}'_{\succeq}(\mathscr{F}_\ell)$. This is equivalent with $(\forall B_k \in \mathscr{F}_k)(\exists B_\ell \in \mathscr{F}_\ell) B_\ell \subseteq B_k$. Since $\mathscr{F}_\ell$ is increasing, it follows that $\mathscr{F}_k \subseteq \mathscr{F}_\ell$. □

8 Embedding classical propositional logic into models for practical certainty

We want to formally embed classical propositional logic into our framework. Since, in contradistinction with the models following from favourability assessments, the models that follow from indifference assessments constitute an intersection structure, this embedding is easier for the latter models.

8.1 Indifference assessments

Eq. (8) and Lem. 4 tell us that that language of proper filters is interchangeable with the language of models following from indifference assessments as far as modelling practical certainty is concerned.

8.2 Favourability assessments

Since the partially ordered set $(\mathbb{C}, \subseteq)$ is no intersection structure, there is no counterpart to Eq. (8):

$$\bigcap_{i \in I} \mathscr{M}(\mathscr{F}_i) \supseteq \mathscr{M}\left(\bigcap_{i \in I} \mathscr{F}_i\right),$$

where $\mathscr{M}(\mathscr{F}) := \left\langle \mathscr{L}^{\gtrdot}_{\mathscr{F}} \cup \mathscr{L}_{\geq 0}; \mathscr{L}^{\lessdot}_{\mathscr{F}} \cup \mathscr{L}_{<0} \right\rangle \in \mathbb{C}$, but the converse inclusion does not generally hold. Despite of this observation, Prop. 9 guarantees that we can still find an embedding of the set of filters $\mathbb{F}$ into $\mathbb{C}$.

Proposition 9. *Consider a coherent set of favourable gambles* $\mathscr{D}_\triangleright$ *derived from a coherent model that includes the background model* $\mathscr{S}$ *and take any collection of events* $\mathscr{A} \subseteq \mathscr{P}$ *such that* $\mathscr{M}_\triangleright(\mathscr{A}) \subseteq \mathscr{D}_\triangleright$. *Let* $\mathscr{F} := \{B \in \mathscr{P} \colon (\forall \varepsilon \in \mathbb{R}_{>0}) - \mathbb{I}_{B^c} + \varepsilon \in \mathscr{D}_\triangleright\}$*, then*

(i) $\mathscr{F} \in \mathbb{F}$; *(ii)* $\mathscr{M}_\triangleright(\mathscr{F}) \subseteq \mathscr{D}_\triangleright$; *(iii)* $\mathscr{A} \subseteq \mathscr{F}$.

Proof. Due to [9, Prop. 8 (iii)], $\mathscr{D}_\triangleright$ is a cone, and $\mathscr{S}_\triangleright \subseteq \mathscr{D}_\triangleright$. This guarantees, by the way, that we can always find such $\mathscr{A}$: if $\mathscr{D}_\triangleright = \mathscr{S}_\triangleright$, use $\mathscr{A} = \{\mathscr{X}\}$. No Confusion guarantees that $\mathscr{L}_{\leq 0}$ and $\mathscr{D}_\triangleright$ are disjoint, ensuring that $\emptyset \notin \mathscr{F}$. Since $\varepsilon \in \mathscr{D}_\triangleright$ for all $\varepsilon \in \mathbb{R}_{>0}$, we see that $\mathscr{X} \in \mathscr{F}$, ensuring that $\mathscr{F} \neq \emptyset$. Consider two events $A, B \in \mathscr{F}$, then both $-\mathbb{I}_{A^c} + \varepsilon_1$ and $-\mathbb{I}_{B^c} + \varepsilon_2 \in \mathscr{D}_\triangleright$ for all $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0}$, so also $\varepsilon_1 + \varepsilon_2 - \mathbb{I}_{A^c} - \mathbb{I}_{B^c} \in \mathscr{D}_\triangleright$. From this, we infer $\varepsilon_1 + \varepsilon_2 - \mathbb{I}_{A^c} - \mathbb{I}_{B^c} \leq \varepsilon_1 + \varepsilon_2 - \mathbb{I}_{(A \cap B)^c} \in \mathscr{D}_\triangleright$ for all $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0}$, so $A \cap B \in \mathscr{F}$, meaning that $\mathscr{F}$ is closed under finite intersections. Consider an event $A \in \mathscr{F}$ and $B \supseteq A$, then $-\mathbb{I}_{A^c} + \varepsilon \in \mathscr{D}_\triangleright$ for all $\varepsilon \in \mathbb{R}_{>0}$. Because $-\mathbb{I}_{A^c} \leq -\mathbb{I}_{B^c}$, also $-\mathbb{I}_{B^c} + \varepsilon \in \mathscr{D}_\triangleright$, so $B \in \mathscr{F}$, meaning that $\mathscr{F}$ is increasing. This proves (i).

For (ii), consider any gamble $f \in \mathscr{M}_\triangleright(\mathscr{F})$. Then there is some $B \in \mathscr{F}$ such that $\inf(f|B) =: \delta > 0$, and it follows that $f \geq \mathbb{I}_{B^c} \inf f + \mathbb{I}_B \delta = \mathbb{I}_{B^c} \gamma + \delta$, where $\gamma := \inf(f) - \delta \leq 0$. Because of the definition of $\mathscr{F}$ and taking into account that $\mathscr{D}_\triangleright$ is a cone that includes $\mathbb{R}_{>0}$, $\{-\lambda \mathbb{I}_{B^c} + \varepsilon \colon \lambda \in \mathbb{R}_{\geq 0}, \varepsilon \in \mathbb{R}_{>0}\} \subseteq \mathscr{D}_\triangleright$, hence $f \in \mathscr{D}_\triangleright$.

For (iii), consider any event $B \in \mathscr{A}$. Then $-\mathbb{I}_{B^c} + \varepsilon \in \mathscr{M}_\triangleright(\mathscr{A})$ because $\inf(-\mathbb{I}_{B^c} + \varepsilon | B) > 0$. Since $\mathscr{M}_\triangleright(\mathscr{A}) \subseteq \mathscr{D}_\triangleright$ by assumption, then also $-\mathbb{I}_{B^c} + \varepsilon \in \mathscr{D}_\triangleright$, which tells us that $B \in \mathscr{F}$. □

9 Conclusions

We have shown that the language of accept & reject statement-based uncertainty models is well-suited for describing practical certainty about the validity of some propositions, or the occurrence of the corresponding events. We have studied three different ways of translating such beliefs of practical certainty into this language, each time modelled by a different type of assessment. All three types formulations lead to the same logical inferences: a collection of events the subject is practically certain of must be closed under conjunction and modus ponens. This conclusion can be drawn as well by calculating the corresponding coherent lower probability: it is formulated in terms of a filter. We concluded with the result that the collection of coherent models following from the latter type of assessments constitute a strong belief structure, and we found a belief embedding of classical propositional logic into all our models for practical certainty.

Future goals include deriving belief expansion and belief revision operators in the language of sets of favourable gambles, inspired by the ideas in [2].

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