A preliminary study of

notions of independence for choice functions

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What about choice functions?

for pmf: p on $\mathscr{X} \times \mathscr{Y}$ is *independent* if $p = p_X p_Y$
(if $p > 0$) $p\,|_X = p_Y$ and $p\,|_Y = p_X$

A preliminary study of notions of independence

for choice functions

very general desirability model

lower previsions, sets of desirable gambles are its special cases

Types of independence

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Types of independence

epistemic irrelevance (from X to Y)	*learning which $x \in \mathscr{X}$ obtains does not change my beliefs about Y*
representational irrelevance (from X to Y)	*the joint model has a representation such that all of its elements satisfy epistemic irrelevance (from X to Y)*
S-irrelevance (from X to Y)	*it is unreasonable to pay to find out which $x \in \mathscr{X}$ obtains to decide about Y*

Types of independence

epistemic **independence**	*learning which $x \in \mathscr{X}$ obtains does not change my beliefs about Y* ***and learning which $y \in \mathscr{Y}$ obtains does not change my beliefs about X***
representational **independence**	*the joint model has a representation such that all of its elements satisfy epistemic* ***independence***
S-**independence**	*it is unreasonable to pay to find out which $x \in \mathscr{X}$ obtains to decide about Y* ***and vice versa***

Choice functions

X uncertain variable

$\mathscr{X}$ finite possibility space

$\mathscr{X} = \{$  ,  $\}$ *Should I take an umbrella today?*

Choice functions

X	uncertain variable
$\mathscr{X}$	finite possibility space
$f : \mathscr{X} \to \mathbb{R}$	gamble
$\mathscr{L}(\mathscr{X})$	all gambles on $\mathscr{X}$

$\mathscr{X} = \{\text{☀}, \text{🌧}\}$ *Should I take an umbrella today?*

$$f_{\text{☂}}(\text{☀}) = -3 \qquad f_{\text{☂}}(\text{🌧}) = 5$$

$$f_{\cancel{\text{☂}}}(\text{☀}) = 1 \qquad f_{\cancel{\text{☂}}}(\text{🌧}) = -10$$

Choosing between $f_{\text{☂}}$ *and* $f_{\cancel{\text{☂}}}$ *depends on my beliefs about the weather that day.*

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F	gamble set
$\mathscr{Q}(\mathscr{X})$	all gamble sets

$\mathscr{X} = \{$  ,  $\}$ *Consider a gamble of taking a raincoat:*

f $($  $) = -1$ f $($  $) = 3.$

Form a gamble set $F = \{f$$, f$$, f$$\}$.

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I believe it is likely to rain today. My choice is

$C(F) = \{f$  $, f$  $\} \subseteq F$

since I reject $R(F) = F \setminus C(F) = \{f$  $\}$.

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R	rejection function
K	set of desirable gamble sets

$\mathscr{X} = \{\text{☀}, \text{🌧}\}$ *Consider a gamble of taking a raincoat:*

$$f_{\text{🧥}}(\text{☀}) = -1 \qquad f_{\text{🧥}}(\text{🌧}) = 3.$$

Form a gamble set $F = \{f_{\text{☂}}, f_{\cancel{\text{☂}}}, f_{\text{🧥}}\}$.

I believe it is likely to rain today. My choice is

$$C(F) = \{f_{\text{☂}}, f_{\text{🧥}}\} \subseteq F$$

since I reject $R(F) = F \setminus C(F) = \{f_{\cancel{\text{☂}}}\}$.

If 0 *is rejected for* $F \cup \{0\}$, *we have*

$$K := \{F \in \mathscr{Q} : 0 \in R(F \cup \{0\})\},$$

called a set of desirable gamble sets.

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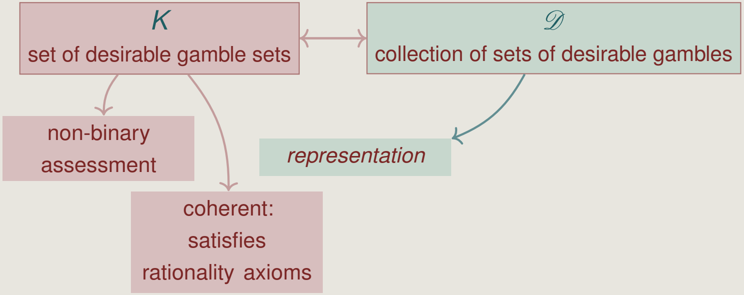
$\mathscr{D}$	representation



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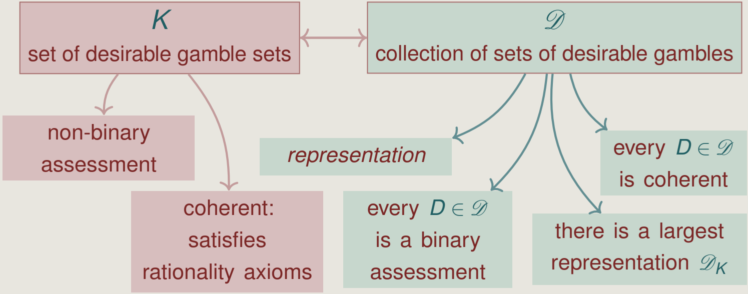
$\mathscr{D}$	representation
$\mathscr{D}_K$	largest representation



Epistemic independence

K_X is epistemically irrelevant to K_Y if learning that any event E_X obtains does not change my assessment in K_Y.

$\mathscr{X}, \mathscr{Y}$	possibility spaces
$E_X \subseteq \mathscr{X}$	non-empty event on $\mathscr{X}$
$E_Y \subseteq \mathscr{Y}$	non-empty event on $\mathscr{Y}$
K_X	set of des. gamble sets on $\mathscr{X}$
K_Y	set of des. gamble sets on $\mathscr{Y}$

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K_X and K_Y are epistemically independent if they are epistemically irrelevant to each other:

$$\mathrm{Marg}_Y(K\rfloor E_X) = \mathrm{Marg}_Y(K),$$
$$\mathrm{Marg}_X(K\rfloor E_Y) = \mathrm{Marg}_X(K).$$

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$\rfloor$	conditioning

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$$\boxed{K = K_X \otimes K_Y}$$

epistemically independent product

marginals

epistemic product

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$\otimes$	epistemic product

Representational independence

K is representationally independent if every D in its representation $\mathscr{D}$ satisfies epistemic independence.

$\mathscr{X}$, $\mathscr{Y}$ possibility spaces

K set of des. gamble sets on $\mathscr{X} \times \mathscr{Y}$

$\mathscr{D}$ representation of K

K_X, K_Y sets of des. gamble sets on $\mathscr{X}$, $\mathscr{Y}$

$\mathscr{D}_{K_X}$, $\mathscr{D}_{K_Y}$ largest representations of K_X, K_Y

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$D = D_X \otimes D_Y$
epistemic product of
$D_X \in \mathscr{D}_{K_X}$, $D_Y \in \mathscr{D}_{K_Y}$

$D = D_X \boxtimes D_Y$
strong product of
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For both, we find their representationally independent natural extensions: $K_X \circledcirc K_Y$ and $K_X \boxdot K_Y$.

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For both, we find their representationally independent natural extensions: $K_X \boxcircle K_Y$ and $K_X \boxbox K_Y$.

We also show $K_X \otimes K_Y \subseteq K_X \boxcircle K_Y \subseteq K_X \boxbox K_Y$.

$\mathscr{X}$, $\mathscr{Y}$ possibility spaces

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$\mathscr{D}$ representation of K

K_X, K_Y
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$\mathscr{D}_{K_X}$, $\mathscr{D}_{K_Y}$
largest representations of K_X, K_Y

$\otimes$ epistemic product

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S-independence

choose partition $\mathscr{P}$ of $\mathscr{X}$

observe which event E occurs

for each E, choose f_E on $\mathscr{Y}$

construct $\sum_{E \in \mathscr{P}} \mathbb{I}_E(X) f_E(Y)$

$\mathscr{X}, \mathscr{Y}$	possibility spaces
X, Y	uncertain variables
$\mathscr{P}$	partition of $\mathscr{X}$
$E \in \mathscr{P}$	event on $\mathscr{X}$
$\mathbb{I}_E(X)$	indicator gamble

S-independence

If X is S-irrelevant to Y, it is unreasonable to pay to observe X to make decision about Y:
reject $\sum_{E\in\mathscr{P}} \mathbb{I}_E(X) f_E(Y) - \varepsilon$ from the gamble set $\{f_E : E \in \mathscr{P}\} \cup \{\sum_{E\in\mathscr{P}} \mathbb{I}_E(X) f_E(Y) - \varepsilon\}$.

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K is S-irrelevant from X to Y if

$$\{\textstyle\sum_{G\in\mathscr{P}\setminus\{E\}} \mathbb{I}_G[f_E - f_G] + \varepsilon : E \in \mathscr{P}\} \in K,$$

for all partitions $\mathscr{P}$ of $\mathscr{X}$, all $\varepsilon > 0$ and all $f_E \in \mathscr{L}(\mathscr{Y})$.

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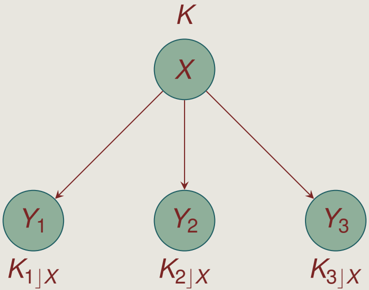
S-independence

- *K is S-independent if and only if K_D is S-independent for all $D \in \mathscr{D}$.*
- *S-independent natural extension $K_X \circledast K_Y$ for mixing and Archimedean K.*
- *For mixing K, representational independence implies S-independence.*
- *For mixing and Archimedean K, representational independence and S-independence coincide.*

mixing K $\mathscr{D}$ = lexicographic Ds

mixing and Archimedean K
$\mathscr{D}$ = linear previsions

Conclusions



How to combine the local models for the children into a joint model?

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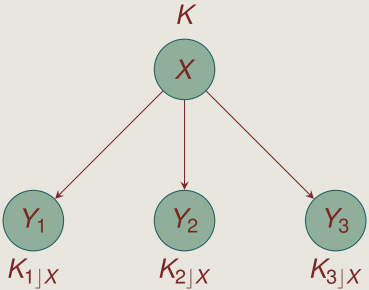


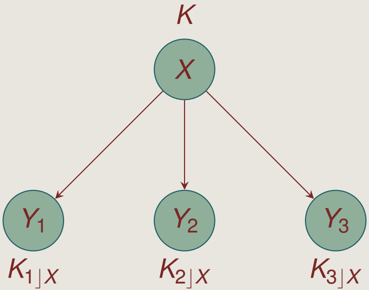
How to combine the local models for the children into a joint model?

Let $\diamond$ be an independent product of choice, then

$$K_{\rfloor X} := \diamond_{\ell=1}^{3} K_{\ell\rfloor X}$$

*is the joint model, provided that $\diamond$ **is associative**.*

Conclusions



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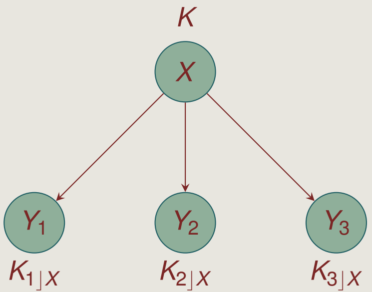

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What is the resulting representation of the joint model?

Depending on $\diamond$, does K have the same properties as its independent marginals?