A preliminary study of notions of independence for choice functions

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Abstract. We study independence concepts for imprecise-probabilistic choice functions. Leveraging the idea that a choice function is represented by a collection of coherent sets of desirable gambles, we introduce representational independence as the requirement of having an independent representation. We derive the representational product, which is the smallest coherent choice function that marginalises to given choice functions and satisfies representational independence. By applying similar proof techniques, we also obtain the independent product for the existing S-independence. We compare these products, together with the existing epistemic product, and derive some of their properties. Finally, we discuss open questions concerning the independence notions.

Keywords: Independence · Sets of desirable gamble sets · Choice functions · Natural extension.

1 Introduction

If an agent's joint beliefs about two uncertain variables (X, Y) are described by a probability mass function (pmf) p on their joint finite possibility space,[1] then X and Y are (stochastically) independent if conditioning p on any value of one variable does not change the agent's beliefs about the other. Given marginal mass functions p_X and p_Y, the (unique) independent joint distribution with these marginals is their product $p_X p_Y$. This construction is a form of *independent natural extension*, which we study here in an imprecise-probabilistic setting.

In this paper we investigate independence for choice functions, one of the most expressive belief models in the imprecise-probabilistic framework. Their added flexibility allows expressing alternative notions of independence, such as requiring independence in every element of a representation, or judging it unreasonable to use observations of X to make decisions about Y. We introduce two new notions of independence and relate them to two existing ones.

The paper is organised as follows. In Section 2, we introduce the relevant results about choice functions, or their equivalent model of sets of desirable gamble sets. Section 3 contains the main results: after reviewing marginalisation

[1] For convenience, we assume all probabilities are positive.

and conditioning, we introduce the independence notions. Section 4 concludes with a brief discussion and directions for future research. In order to adhere to the page limit, we have omitted the proofs.

2 Choice functions

Consider an uncertain variable X taking values in a possibility space $\mathcal{X}$, which we assume to be *finite* throughout this paper. A gamble f is a real-valued function on $\mathcal{X}$ that maps each possibility x in $\mathcal{X}$ to a reward $f(x)$, expressed in unites of a predetermined utility scale. If the agent accepts a gamble f, she accepts to receive the – possibly negative – reward $f(x)$ if the possibility x is realised.

Because X is an *uncertain* variable, a gamble f on $\mathcal{X}$ involves risk so it is a risky transaction. An agent's preferences between gambles therefore reveal her beliefs about X. A *choice function* C is a function that describes the agent's preference between gambles in a very general way, without assuming that they are determined by a binary preference order [7,12]. In this paper, we study choice functions as a belief model, but for mathematical convenience, we shall focus on the equivalent model of *sets of desirable gamble sets* [4,5].

To introduce choice functions, we first need to introduce some relevant notation. We denote all gambles on $\mathcal{X}$ as $\mathcal{L}(\mathcal{X})$ – or simply $\mathcal{L}$ when it is clear from the context what the domain is. The set of all gambles $\mathcal{L}(\mathcal{X})$ is an $|\mathcal{X}|$-dimensional linear space under the pointwise addition and scalar multiplication of gambles. For two gambles f and g, we use $f \leq g$ to denote $f(x) \leq g(x)$ for all x in $\mathcal{X}$, and we use $f < g$ for '$f \leq g$ and $f(x) < g(x)$ for some x in $\mathcal{X}$'. A constant gamble is a gamble that returns the same value α everywhere, and we identify such gamble with a real number α. We collect the gambles that result in a partial gain in the *set of positive gambles* $\mathcal{L}_{>0}(\mathcal{X}) \coloneqq \{f \in \mathcal{L}(\mathcal{X}) : f > 0\}$, and the gambles that cannot make you win utility in the *set of non-positive gambles* $\mathcal{L}_{\leq 0}(\mathcal{X}) \coloneqq \{f \in \mathcal{L}(\mathcal{X}) : f \leq 0\}$ – or simply $\mathcal{L}_{>0}$ and $\mathcal{L}_{\leq 0}$ respectively, when the domain is clear from the context

Choice functions take a *finite* set of gambles, called *gamble set*, as an argument. We denote the set of all gamble sets on $\mathcal{X}$ as $\mathcal{Q}(\mathcal{X}) \coloneqq \{F \subseteq \mathcal{L}(\mathcal{X}) : |F| \in \mathbb{N}\}$, or simply $\mathcal{Q}$. A choice function C on $\mathcal{X}$ is a function $\mathcal{Q}(\mathcal{X}) \to \mathcal{Q}(\mathcal{X})$ such that a set of admissible gambles $C(F)$ is a subset of F for every F in $\mathcal{Q}(\mathcal{X})$. The corresponding set of rejected gambles $R(F) \coloneqq F \setminus C(F)$ is an equivalent representation of this assessment.

Imprecise-probabilistic choice functions have first been introduced by Seidenfeld et al. [7,12], where they have provided an axiomatisation for coherent choice between *horse lotteries* rather than gambles. Since then, Refs. [13,4,5,17,14,20], among others, have proposed an axiomatisation based on it, but specified to gambles. Due to page limitations, we will not give an overview of the rationality axioms for choice functions here, but rather introduce them directly in Section 2.1 in their equivalent and more convenient form – sets of desirable gamble sets – established in Refs. [4,5]. We stress that the rationality axioms allow for a gamble f to be admissible from both $\{f, g\}$ and $\{f, h\}$, while being inadmissible from $\{f, g, h\}$, which proves that choice functions can express more than pairwise preferences.

2.1 Sets of desirable gamble sets

In this section, we discuss relevant results about coherent sets of desirable gamble sets, which were introduced in Refs. [4,5] as an equivalent representation of coherent choice functions. Given the equivalence, everything we establish for sets of desirable gamble sets, will immediately hold for choice functions too, *mutatis mutandis*.

Definition 1 (Sets of desirable gamble sets). *A set of desirable gamble sets K on $\mathcal{X}$ is a subset of $\mathcal{Q}(\mathcal{X})$.*

The idea is that a set of desirable gamble sets K contains all the gamble sets F that have at least one gamble that the agent prefers to the constant gamble 0, called the *status quo*. Such a gamble set F is then called a *desirable gamble set*.

Given a rejection function R, if $0 \in R(\{0\} \cup F)$ then we interpret this as meaning that the agent finds at least one gamble f in F desirable, or in other words, prefers f to 0, and thus F should be a desirable gamble set. Then $K \coloneqq \{F \in \mathcal{Q} : 0 \in R(\{0\} \cup F)\}$ is the set of desirable gamble sets corresponding to R. Conversely, given a set of desirable gamble sets K, the corresponding rejection function R is given by $R(F) \coloneqq \{f \in F : F - \{f\} \in K\}$, where $F - \{f\} \coloneqq \{g - f : g \in F\}$ for every F in $\mathcal{Q}$. Under conditions implied by coherence, both operations commute; see Ref. [4].

Definition 2 (Coherent sets of desirable gamble sets [4, Definition 4]). *A set of desirable gamble sets $K \subseteq \mathcal{Q}$ is called* coherent *if:*

K_0. *$\emptyset \notin K$;*

K_1. *$F \in K \Rightarrow F \setminus \{0\} \in K$;*

K_2. *$\{f\} \in K$ for all f in $\mathcal{L}_{>0}$;*

K_3. *if $F, G \in K$ and $(\lambda^{f,g}, \mu^{f,g}) > 0$ for every pair (f, g) in $F \times G$, then $\{\lambda^{f,g} f + \mu^{f,g} g : f \in F, g \in G\} \in K$;*[2]

K_4. *if $F \in K$ and $F \subseteq G$, then $G \in K$.*

We collect all the coherent sets of desirable gamble sets in the collection $\overline{\mathcal{K}}(\mathcal{X})$, or simply $\overline{\mathcal{K}}$ when the possibility space is clear from the context.

We refer to Refs. [4,5] for a justification of the coherence axioms.

A set of desirable gamble sets K_1 is called *at least as informative as* a set of desirable gamble sets K_2 when $K_2 \subseteq K_1$, simply because an agent with K_1 finds more gamble sets desirable than an agent with K_2. The resulting partially ordered set $(\overline{\mathcal{K}}, \subseteq)$ is a complete meet-semilattice [4, Theorem 8]. This guarantees that if a partially specified set of desirable gamble sets $\mathcal{A} \subseteq \mathcal{Q}$ has a coherent superset, in which case $\mathcal{A}$ is called *consistent*, then it has a unique smallest (or

[2] We use $(\lambda^{f,g}, \mu^{f,g}) > 0$ to mean '$\lambda^{f,g} \geq 0$ and $\mu^{f,g} \geq 0$ but not $\lambda^{f,g} = \mu^{f,g} = 0$.'

least informative) coherent superset $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A}) \coloneqq \bigcap\{K \in \overline{\mathcal{K}} : \mathcal{A} \subseteq K\}$, called the *natural extension* of $\mathcal{A}$ [4, Theorem 9].

The natural extension $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A})$ of a consistent assessment $\mathcal{A}$ obtains a closed-form expression; however, we focus on its more convenient formulation in terms of *representations*, which we discuss in Sections 2.2 and 2.3.

2.2 Sets of desirable gambles

Given a coherent rejection function R, the gambles in $D_R \coloneqq \{f \in \mathcal{L} : 0 \in R(\{0, f\})\}$ are preferred in a *pairwise preference* to 0. Similarly, for a coherent set of desirable gamble sets K, the set $D_K \coloneqq \{f \in \mathcal{L} : \{f\} \in K\}$ consists of the gambles that are preferred to 0. If R and K contain the same information, the sets D_R and D_K coincide. We therefore call D_K the *binary part* of K.

The set D_K is more commonly known as a *set of desirable gambles*: it contains the gambles that the agent prefers to 0. Sets of desirable gambles were introduced by Seidenfeld et al. [10], and have been further developed by a range of authors, for instance in Refs. [21,22,19,9,2,3].

Definition 3 (Sets of desirable gambles). *A set of desirable gambles D on $\mathcal{X}$ is a subset of $\mathcal{L}(\mathcal{X})$.*

The idea is that the set of desirable gambles D contains all the gambles f that are preferred to 0. Sets of desirable gambles have the following rationality axioms.

Definition 4 (Coherent sets of desirable gambles). *A set of desirable gambles $D \subseteq \mathcal{L}$ is called* coherent *if:*
D_1. $0 \notin D$;
D_2. $\mathcal{L}_{>0} \subseteq D$;
D_3. *if* $f, g \in D$ *and* $(\lambda, \mu) > 0$, *then* $\lambda f + \mu g \in D$.
We collect all the coherent sets of desirable gambles in $\overline{\mathcal{D}}(\mathcal{X}) \subseteq \mathscr{P}(\mathcal{L}(\mathcal{X}))$,[3] or simply $\overline{\mathcal{D}}$ when the possibility space is clear from the context.

Coherent sets of desirable gambles are convex cones – sets $A \subseteq \mathcal{L}$ for which $\mathrm{posi}(A) \coloneqq \{\sum_{k=1}^{n} \lambda_k f_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, f_k \in A\} \subseteq A$ – that do not contain 0 but contain all the positive gambles. Given a pmf p, we let $D_p \coloneqq \{f \in \mathcal{L} : E_p(f) > 0 \text{ or } f > 0\}$ be the corresponding set of desirable gambles, which collects all the gambles whose p-expectation E_p is positive, and the set $\mathcal{L}_{>0}$. It is a matter of straightforward verification that D_p is a coherent set of desirable gambles. If $p(x) > 0$ for all x in $\mathcal{X}$ then D_p is an open half space.

A set of desirable gambles D_1 is called *at least as informative as* a set of desirable gambles D_2 when $D_2 \subseteq D_1$. The resulting partially ordered set $(\overline{\mathcal{D}}, \subseteq)$ is also a complete meet-semilattice, and if a partially specified set of desirable gambles $A \subseteq \mathcal{L}$ has a coherent superset, in which case A is called *consistent*, then it has a unique smallest (or least informative) coherent superset $\mathrm{cl}_{\overline{\mathcal{D}}}(A) \coloneqq \bigcap\{D \in \overline{\mathcal{D}} : A \subseteq D\}$, called the *natural extension* of A. The natural extension for sets of desirable gambles has a convenient closed-form expression.

[3] In this paper, we denote the power set of a given set A by $\mathscr{P}(A)$, and the set of all non-empty subsets of A by $\mathscr{P}^{+}(A) = \mathscr{P}(A) \setminus \{\emptyset\}$.

Theorem 1 ([19, Theorem 1]). *Consider any assessment $A \subseteq \mathcal{L}$. Then A is consistent iff $0 \notin \operatorname{posi}(\mathcal{L}_{>0} \cup A)$. If this is the case, then $\operatorname{cl}_{\overline{\mathcal{D}}}(A) = \operatorname{posi}(\mathcal{L}_{>0} \cup A)$.*

Maximal sets of desirable gambles. The maximal elements of the partially ordered set $(\overline{\mathcal{D}}(\mathcal{X}), \subseteq)$ are called *maximal sets of desirable gambles*. We collect them in the set $\mathcal{M}(\mathcal{X})$, or simply $\mathcal{M}$ when the possibility space is clear from the context.

Maximal sets of desirable gambles have an easy characterisation: a coherent set of desirable gambles D is maximal if and only if $f \in D$ or $-f \in D$ for all gambles $f \neq 0$ [19, Proposition 2]. This implies [3, Lemma 17] that every element of $\mathcal{M}(\mathcal{X})$ is a superset of D_p, where p is a pmf. As such, elements M of $\mathcal{M}(\mathcal{X})$ can be seen as more informative than probabilities: if a non-zero gamble f's expectation $E_p(f)$ is zero, then M still contains either f or $-f$.

For any set of gambles $A \subseteq \mathcal{L}$, denote the set of its dominating maximal sets of desirable gambles by $m(A) \coloneqq \{M \in \mathcal{M} : A \subseteq M\}$. Given a coherent set of desirable gambles D, the set $m(D)$ can be viewed as a representation of D.

Theorem 2 ([19, Corollary 4]). *A set of desirable gambles A is consistent iff $m(A) \neq \emptyset$. Moreover, its natural extension can be obtained by $\operatorname{cl}_{\overline{\mathcal{D}}}(A) = \bigcap m(A)$.*

In particular, Theorem 2 implies that any coherent set of desirable gambles D can be represented by the maximal sets of desirable gambles above it.

2.3 Representation

Given a coherent set of desirable gambles D, there may be multiple coherent sets of desirable gamble sets K such that $D_K = D$, but the smallest – or least informative – of them is $K_D \coloneqq \{F \in \mathcal{Q} : F \cap D \neq \emptyset\}$ [15, Proposition 5]. Because K_D is completely determined by D, and hence by *binary comparisons*, we call any coherent K_D a *binary* set of desirable gamble sets. Binary sets of desirable gambles are useful because they allow for a representation of any coherent set of desirable gamble sets.

Theorem 3 ([5, Theorem 9]). *For any sets of desirable gamble sets K, the following two expressions are equivalent:*
- *(i) K is coherent;*
- *(ii) there is a non-empty set $\mathcal{D} \subseteq \overline{\mathcal{D}}$ of coherent sets of desirable gambles such that $K = \bigcap\{K_D : D \in \mathcal{D}\}$. If this is the case, then K is said to be* represented *by $\mathcal{D}$, and $\mathcal{D}$ is a* representation *of K.*

Moreover, K's (unique) largest representing set is $\mathcal{D}_K \coloneqq \{D \in \overline{\mathcal{D}} : K \subseteq K_D\}$.

Ref. [20, Equation (24)] guarantees that Theorem 3 implies the following convenient expression for the natural extension.

Theorem 4. *An assessment $\mathcal{A} \subseteq \mathcal{Q}$ is consistent iff $\mathcal{D}_{\mathcal{A}} \neq \emptyset$. Moreover, if this is the case, then $\operatorname{cl}_{\overline{\mathcal{K}}}(\mathcal{A}) = \bigcap\{K_D : D \in \mathcal{D}_{\mathcal{A}}\}$.*

3 Three concepts of independence

In this section, we compare three notions of independence for sets of desirable gamble sets: epistemic independence, representational independence and S-independence. Each of them captures a different intuitive concept of independence. The most widely used one, epistemic independence, has been introduced in Ref. [21, Sections 9.1 and 9.2] for lower previsions and further developed for sets of desirable gambles in Ref [18] and sets of desirable gamble sets in Ref. [14]. S-independence has been introduced in Ref. [12] and studied in more detail for choice functions in Ref. [6]. We introduce a new notion in this context – which we call representational independence – based on epistemic and strong products for sets of desirable gambles; see Ref. [18].

To prioritise conceptual clarity and minimise technicalities, we have decided to focus on independence between two uncertain variables X and Y, with finite possibility spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively.

3.1 Marginalisation and conditioning

In this section, we discuss sets of desirable gamble sets that represent beliefs about X and Y *jointly*, necessary to our discussion of independence next. The discussion and results in Section 2 above hold for the possibility space $\mathcal{X} \times \mathcal{Y}$ without restriction.

For any gamble f in the $|\mathcal{X}||\mathcal{Y}|$-dimensional linear space $\mathcal{L}(\mathcal{X} \times \mathcal{Y})$ and any $x \in \mathcal{X}$, the partial map $f(x, \cdot)$ is a gamble on $\mathcal{Y}$; and similarly, for any $y \in \mathcal{Y}$, the partial map $f(\cdot, y)$ is a gamble on $\mathcal{X}$. In this section, the roles of X and Y are symmetric; to avoid repetition we decide to focus on one of them, but *swapping X for Y* would yield analogous statements or results. It will be convenient to relate a gamble f on $\mathcal{X}$ to a gamble on $\mathcal{X} \times \mathcal{Y}$.

Definition 5 (Cylindrical extension). *Given any gamble f on $\mathcal{X}$, we let its* cylindrical extension *f^Y to $\mathcal{X} \times \mathcal{Y}$ be defined by $f^Y(x, y) \coloneqq f(x)$ for all x in $\mathcal{X}$ and y in $\mathcal{Y}$. Similarly, given any set of gambles $F \subseteq \mathcal{L}(\mathcal{X})$, we let its* cylindrical extension *$F^Y \subseteq \mathcal{L}(\mathcal{X} \times \mathcal{Y})$ be defined as $F^Y \coloneqq \{f^Y : f \in F\}$.*

Formally, f is a gamble on $\mathcal{X}$, while f^Y is a gamble on $\mathcal{X} \times \mathcal{Y}$, but since it is constant on $\mathcal{Y}$, its behavioural content is contained in $\mathcal{X}$ – just like for f. We will therefore, from now on, *make no notational distinction between a gamble and its cylindrical extension.* This notational trick allows us to regard f as an element of $\mathcal{L}(\mathcal{X} \times \mathcal{Y})$, and $\mathcal{L}(\mathcal{X})$ as a subset of $\mathcal{L}(\mathcal{X} \times \mathcal{Y})$ – even though their dimensions might differ – and similarly for $\mathcal{Q}(\mathcal{X}) \subseteq \mathcal{Q}(\mathcal{X} \times \mathcal{Y})$.

Marginalisation. Suppose that we have a coherent belief model D or K about (X, Y) jointly and we are interested in marginalising to X, or extracting the information about X solely. For a set of desirable gambles D, we collect the gambles that belong to D but only depend on $\mathcal{X}$ in its X-marginal $\operatorname{marg}_X D$.

Definition 6 ([15]). *For any set of desirable gambles* $D \subseteq \mathcal{L}(\mathcal{X} \times \mathcal{Y})$*, its* X-marginal $\operatorname{marg}_X D \subseteq \mathcal{L}(\mathcal{X})$ *is defined as* $\operatorname{marg}_X D \coloneqq D \cap \mathcal{L}(\mathcal{X})$.

Similarly, for a set of desirable gamble sets K, we collect the gamble sets that belong to K, but only depend on $\mathcal{X}$, in its X-marginal $\operatorname{Marg}_X K$.

Definition 7 ([15]). *For any set of desirable gamble sets* $K \subseteq \mathcal{Q}(\mathcal{X} \times \mathcal{Y})$*, its* X-marginal $\operatorname{Marg}_X K \subseteq \mathcal{Q}(\mathcal{X})$ *is defined as* $\operatorname{Marg}_X K \coloneqq K \cap \mathcal{Q}(\mathcal{X})$.

Definition 7 generalises Definition 6, in the sense that for all K' in $\overline{\mathcal{K}}$ and D' in $\overline{\mathcal{D}}$, we have [15, Proposition 10] $\operatorname{Marg}_X K_{D'} = K_{\operatorname{marg}_X D'}$ and $\operatorname{marg}_X D_{K'} = D_{\operatorname{Marg}_X K'}$. Marginalisation preserves the order for both sets of desirable gambles and set of desirable gamble sets: if $D_1 \subseteq D_2$, then $\operatorname{marg}_X D_1 \subseteq \operatorname{marg}_X D_2$ and if $K_1 \subseteq K_2$, then $\operatorname{Marg}_X K_1 \subseteq \operatorname{Marg}_X K_2$. Likewise, marginalisation preserves coherence: if a set of desirable gambles D is coherent, then so is $\operatorname{marg}_X D$; and if a set of desirable gamble sets K is coherent, then so is $\operatorname{Marg}_X K$. Moreover, using the lifted version $\operatorname{marg}_X \mathcal{D} \coloneqq \{\operatorname{marg}_X D : D \in \mathcal{D}\}$, if a coherent set of desirable gamble sets K is represented by $\mathcal{D}$, then $\operatorname{Marg}_X K$ is represented by $\operatorname{marg}_X \mathcal{D}$; see Ref. [14, Proposition 8].

Conditioning Assume that we have a model D or K representing the agent's beliefs about (X, Y) jointly. Suppose we receive new information that X takes values in some event $E \in \mathscr{P}^+(\mathcal{X})$. We aim to update our model to reflect this information, which is to *condition* on the event E.

Calling off a gamble f on $E \times \mathcal{Y}$ with respect to E yields the gamble $\mathbb{I}_E f \in \mathcal{L}(\mathcal{X} \times \mathcal{Y})$ defined, for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$ by

$$\mathbb{I}_E f(x, y) \coloneqq \begin{cases} f(x, y), & \text{if } x \in E; \\ 0, & \text{otherwise.} \end{cases}$$

This is a *called-off* version of f: the risky transaction occurs only if E happens; if not, the gamble gets called off. In turn, calling off the gamble set F on the event E yields $\mathbb{I}_E F = \{\mathbb{I}_E f : f \in F\}$.

Conditioning D on E yields the conditional set of desirable gambles $D\rfloor E \coloneqq \{f \in \mathcal{L}(E \times \mathcal{Y}) : \mathbb{I}_E f \in D\}$ and if D is coherent, then so is $D\rfloor E$. We refer to [21, Appendix F] and [19, Section 3.2] for more information.

Similarly, conditioning K on E yields the conditional set of desirable gamble sets $K\rfloor E \coloneqq \{F \in \mathcal{Q}(E \times \mathcal{Y}) : \mathbb{I}_E F \in K\}$ and if K is coherent, then so is $K\rfloor E$. Moreover, if $\mathcal{D}$ is some representation of K, then $K\rfloor E$ is represented by $\mathcal{D}\rfloor E = \{D\rfloor E : D \in \mathcal{D}\}$ [14, Proposition 7].

3.2 Independence concepts

In this section we provide an overview of three concepts of independence. In order to streamline the discussion, let K denote a coherent set of desirable gamble sets on the joint (finite) possibility space $\mathcal{X} \times \mathcal{Y}$.

Epistemic independence The idea of epistemic irrelevance from X to Y is:

> Learning that a non-empty event $E_X \subseteq \mathcal{X}$ obtains does not affect the beliefs about Y.

The set K is said in Ref. [14, Section 6] to satisfy *epistemic irrelevance from X to Y* when conditioning on any non-empty event $E_X \subseteq \mathcal{X}$ leaves the marginal beliefs about Y unchanged. Formally, for all E_X in $\mathscr{P}^+(\mathcal{X})$, $\mathrm{Marg}_Y(K\rfloor E_X) = \mathrm{Marg}_Y K$. Epistemic irrelevance is asymmetric, and to obtain its symmetric counterpart – called *epistemic independence* – one must require it in both directions. For K, this requirement is

$$\mathrm{Marg}_Y(K\rfloor E_X) = \mathrm{Marg}_Y K \text{ and } \mathrm{Marg}_X(K\rfloor E_Y) = \mathrm{Marg}_X K \tag{1}$$

for all E_X in $\mathscr{P}^+(\mathcal{X})$ and E_Y in $\mathscr{P}^+(\mathcal{Y})$. If Equation (1) holds for all such events, we say that K satisfies *epistemic independence.*

A K on $\mathcal{X} \times \mathcal{Y}$ is called an *epistemic product* of coherent K_X on $\mathcal{X}$ and K_Y on $\mathcal{Y}$ if it satisfies epistemic independence and has K_X and K_Y as its marginals. It is established in Ref. [14, Theorem 15] that the smallest epistemic product of K_X and K_Y exists, and we denote it by $K_1 \otimes K_2$.

The requirement of epistemic irrelevance can also be applied on sets of desirable gambles. A coherent set of desirable gambles D on $\mathcal{X} \times \mathcal{Y}$ is said in Ref. [18] to satisfy *epistemic irrelevance from X to Y* if $\mathrm{marg}_Y(D\rfloor E_X) = \mathrm{marg}_Y D$ for all E_X in $\mathscr{P}^+(\mathcal{X})$. This notion is again asymmetric, and to obtain a symmetric counterpart – called *epistemic independence* – one must require it in both directions. Similarly as for sets of desirable gamble sets, given two coherent sets of desirable gambles D_X on $\mathcal{X}$ and D_Y on $\mathcal{Y}$, a set D on $\mathcal{X} \times \mathcal{Y}$ is called an *epistemic product* of D_X and D_Y if it satisfies epistemic independence and has D_X and D_Y as its marginals. It is established in Ref. [18, Theorem 19] that the smallest epistemic product of D_X and D_Y exists, and we denote it by $D_X \otimes D_Y$.

Fortunately, for pmfs, the epistemic product for sets of desirable gambles coincides with the standard independence definition, in the following sense.

Proposition 1. *Consider a pmf p on $\mathcal{X}$ and a pmf q on $\mathcal{Y}$ such that $p(x) > 0$ for all x in $\mathcal{X}$ and $q(y) > 0$ for all y in $\mathcal{X}$. Then $D_{pq} = D_p \otimes D_q$.*

Representational independence Consider a *committee of experts* in which each member expresses their beliefs by a coherent set of desirable gambles D. The collective information of the committee is captured by its collection $\mathcal{D}$ of the individual Ds, which, by Theorem 3, represents a coherent set of desirable gamble sets K. The independence requirement for K we introduce here, which we call *representational independence*, is that the each member of the committee must agree about the independence:

> There is an *independent representation*, that is, a representation whose every member satisfies epistemic independence.

Definition 8 (Representational independence). *A coherent set of desirable gamble sets K on $\mathcal{X} \times \mathcal{Y}$ satisfies* representational independence *if it has a representation $\mathcal{D}$ such that every D in $\mathcal{D}$ satisfies epistemic independence.*

Recall from Theorem 3 that a coherent K may obtain multiple representations. Representational independence is the requirement that *at least one such representation* is independent.

Given two coherent sets of desirable gamble sets K_X on $\mathcal{X}$ and K_Y on $\mathcal{Y}$, we call K on $\mathcal{X} \times \mathcal{Y}$ a *representational product* of K_X and K_Y if it satisfies representational independence and has K_X and K_Y as its marginals. We establish that the smallest representational product of K_X and K_Y – which we call the *representationally independent natural extension of* K_X *and* K_Y – exists and has a convenient representation in terms of the epistemically independent natural extension of the elements in the representations of K_X and K_Y.

Theorem 5. *Consider any coherent marginal sets of desirable gamble sets $K_X \in \overline{\mathcal{K}}(\mathcal{X})$ and $K_Y \in \overline{\mathcal{K}}(\mathcal{Y})$. Then the representationally independent natural extension of K_X and K_Y exists and is given by*

$$K_X \circledcirc K_Y := \bigcap\nolimits_{D_X \in \mathcal{D}_{K_X}, D_Y \in \mathcal{D}_{K_Y}} K_{D_X \otimes D_Y}, \tag{2}$$

which has $\{D_X \otimes D_Y : D_X \in \mathcal{D}_{K_X}, D_Y \in \mathcal{D}_{K_Y}\}$ as an independent representation.

In Definition 8 we have used epistemic independence for sets of desirable gambles. Another popular type of independence for sets of desirable gambles is *strong independence*. Given coherent sets of desirable gambles D_X on $\mathcal{X}$ and D_Y on $\mathcal{Y}$, their strong product is defined as $D_X \boxtimes D_Y := \bigcap\{M_X \otimes M_Y : M_X \in m(D_X), M_Y \in m(D_Y)\}$. Similarly to the representationally independent product, the strong product is an intersection of epistemic products of the members of representations $m(D_X)$ and $m(D_Y)$ of D_X and D_Y, respectively. In the context of coherent sets of desirable gambles, [18, Proposition 27] establishes that $D_X \boxtimes D_Y$ is indeed an epistemic product of D_X and D_Y, but not necessarily the smallest one [18, Appendix A.3].

This tells us that there is a choice to be made in Equation (2) about the type of product of the D_X and D_Y. In our Definition 8 of representational independence, we have chosen to require *epistemic* independence, but the results in Ref. [18] allows us to use a *strong* product instead. We thus obtain a different independence notion, which we call *strong representational independence*.

Definition 9 (Strong representational independence). *A coherent set of desirable gamble sets K on $\mathcal{X} \times \mathcal{Y}$ satisfies* strong representational independence *if it has a representation $\mathcal{D}$ such that every D in $\mathcal{D}$ is a strong product.*

Since every strong product is an epistemic product, if K satisfies strong representational independence, then it also satisfies representational independence. Similar to what we have done above, we call K on $\mathcal{X} \times \mathcal{Y}$ a *strong representational product* of the two coherent sets of desirable gamble sets K_X on $\mathcal{X}$ and K_Y on $\mathcal{Y}$, if it satisfies strong representational independence and has K_X and K_Y as its marginals. Here too, it turns out that the smallest strong representational product of K_X and K_Y – which we call the *strong representationally independent natural extension of* K_X *and* K_Y – exists and has a convenient representation in terms of strong products of elements in the representations of K_X and K_Y.

Theorem 6. *Consider any coherent marginal sets of desirable gamble sets* $K_X \in \overline{\mathcal{K}}(\mathcal{X})$ *and* $K_Y \in \overline{\mathcal{K}}(\mathcal{Y})$*. Then the strong representationally independent natural extension of* K_X *and* K_Y *exists and is given by*

$$K_X \boxbox K_Y := \bigcap\nolimits_{K_X \in \mathcal{D}_{K_X}, K_Y \in \mathcal{D}_{K_Y}} K_{D_X \boxtimes D_Y},$$

which has $\{D_X \boxtimes D_Y : D_X \in \mathcal{D}_{K_X}, D_Y \in \mathcal{D}_{K_Y}\}$ *as an independent representation.*

Interestingly, epistemic independence is a weaker requirement than strong representational independence.

Proposition 2. *Consider any* $K \in \overline{\mathcal{K}}(\mathcal{X} \times \mathcal{Y})$*. If* K *satisfies (strong) representational independence, then* K *satisfies epistemic independence.*

As a consequence, the strong representational product of two coherent sets of desirable gamble sets K_X and K_Y is an epistemic product of K_X and K_Y, but it is not necessarily the smallest one.

Corollary 1. *Consider any* $K_X \in \overline{\mathcal{K}}(\mathcal{X})$ *and* $K_Y \in \overline{\mathcal{K}}(\mathcal{Y})$*. Then* $K_X \otimes K_Y \subseteq K_X \boxcircle K_Y \subseteq K_X \boxbox K_Y$.

S-independence The idea of *S-irrelevance* of X to Y is [6, Section 4.1]:

> It is unreasonable to *spend resources* in order to use the observed value of X to choose between gambles that only depend on Y.

When X is S-irrelevant to Y and Y is S-irrelevant to X, the two uncertain variables are called *S-independent*.

Observing the value of X is modelled by selecting a partition $\mathcal{P}$ of the possibility space $\mathcal{X}$ and determining which event $E \in \mathcal{P}$ occurs, so that $X \in E$. For each event E in the partition, the agent chooses the gamble f_E on $\mathcal{Y}$. The action of observing X and then selecting a gamble on Y in this manner corresponds to the composite gamble $\sum_{E \in \mathcal{P}} \mathbb{I}_E(X) f_E(Y)$.

If X is S-irrelevant to Y, the requirement is that the agent choosing between the gambles on Y should reject the option to pay any real $\epsilon > 0$ for the composite gamble: from the gamble set $\{f_E : E \in \mathcal{P}\} \cup \{\sum_{G \in \mathcal{P}} \mathbb{I}_G f_G - \epsilon\}$, the agent should reject the latter gamble. Translated to sets of desirable gamble sets, this requirement becomes as follows [6, Definition 9].

Definition 10 (S-independence). *A coherent set of desirable gamble sets* K *on* $\mathcal{X} \times \mathcal{Y}$ *satisfies* S-irrelevance from X to Y *if* $\{\sum_{G \in \mathcal{P} \setminus \{E\}} \mathbb{I}_G [f_E - f_G] + \epsilon : E \in \mathcal{P}\} \in K$*, for all partitions* $\mathcal{P}$ *of* $\mathcal{X}$*, all* $f_E \in \mathcal{L}(\mathcal{Y})$ *and all* $\epsilon > 0$*.* K *satisfies* S-independent *if it satisfies S-irrelevance both from* X *to* Y *and from* Y *to* X*.*

Using representations of K, we can simplify the requirement in Definition 10, which will be useful later on this paper, thereby generalising [6, Proposition 12].

Proposition 3. *Consider any coherent set of desirable gamble sets K on $\mathcal{X}\times\mathcal{Y}$, and any representation $\mathcal{D}$ of K. Then K satisfies S-irrelevance from X to Y iff for all D in $\mathcal{D}$, the binary K_D satisfies S-irrelevance from X to Y.*

The theory of S-independence was significantly developed in Ref. [6]. To fully appreciate and build upon this work, we must introduce two additional axioms for sets of desirable gamble sets, the first being *mixingness*. A coherent set of desirable gamble sets K satisfies mixingness if

K_5. If $G \in K$ and $F \subseteq G \subseteq \operatorname{posi} F$, then also $F \in K$.

The effect of adding this axiom, is that any mixing set of desirable gamble sets K has a representation $\mathcal{D}$ – see Theorem 3 – that consists of *mixing sets of desirable gambles*, which are coherent sets of desirable gambles D for which the complement is also a convex cone: $\operatorname{posi}(D^c) = D^c$.[4] All maximal sets of desirable gambles, as well as all D_p for pmfs p such that $p(x) > 0$ for every x in $\mathcal{X}$, are mixing.

The second additional axiom is *Archimedeanity*, for which we refer to Ref. [17] for a detailed description. What matters for our analysis is that, once this requirement is imposed on K, it can be represented by $\mathcal{D}$ containing only sets of desirable gambles of the form $D_{\underline{P}} := \{f \in \mathcal{L} : \underline{P}(f) > 0\}$, where $\underline{P}$ is a coherent lower prevision. A coherent lower prevision is a lower expectation functional that satisfies certain rationality axioms; we refer to [21,8] for more information about lower previsions. This means that we can represent an Archimedean set of desirable gamble sets K by a set $\underline{\mathcal{P}}$ of coherent lower previsions, and its largest representation in terms of lower previsions is $\underline{\mathcal{P}}_K := \{\underline{P}$ coherent lower prevision : $K \subseteq K_{D_{\underline{P}}}\}$; see Ref. [17] for more details.

A *linear prevision*, or (precise) expectation operator, is a lower prevision that corresponds to a precise probability. Imposing both Archimedeanity and mixingness on K has the effect that the set $\underline{\mathcal{P}}_K$ consists of linear previsions only; see [6, Theorem 2]. Together with a technical condition called *credible indeterminacy*,[5] the following surprising result is established in Ref [6].

Theorem 7 ([6, Theorem 10]). *Consider any Archimedean set of desirable gamble sets K on $\mathcal{X}\times\mathcal{Y}$ whose X- and Y-marginals are credibly indeterminate. If K satisfies S-independence, then every $\underline{P}$ in $\underline{\mathcal{P}}_K$ is a linear prevision that moreover satisfies independence.*

Theorem 7 and Proposition 1 imply that Archimedean and credibly indeterminate sets of desirable gamble sets that satisfy S-independence, also satisfy representational independence. We call K on $\mathcal{X}\times\mathcal{Y}$ an *S-independent product*

[4] Such sets of desirable gambles have also been called *lexicographic* in Refs. [16,1].

[5] For convenience, we use a stronger version of credible indeterminacy than in Ref. [6], but we give it the same name. This guarantees that under our more restrictive requirements, the results in Ref. [6] remain valid. We call a set of desirable gamble sets K_X on $\mathcal{X}$ *credibly indeterminate* if for every outcome x in $\mathcal{X}$ there is a real $\epsilon > 0$ such that $\{\mathbb{I}_{\{x\}} - \epsilon\} \in K$.

of two coherent sets of desirable gamble sets K_X on $\mathcal{X}$ and K_Y on $\mathcal{Y}$ if it is Archimedean, satisfies S-independence and has K_X and K_Y as its marginals. It turns out that this independence concept, too, has a smallest S-independent product of K_X and K_Y, at least when K_X and K_Y are mixing, Archimedean and credibly indeterminate, which we call the *S-independent natural extension.*

Theorem 8. *Consider Archimedean, mixing and credibly indeterminate sets of desirable gamble sets $K_X \in \overline{\mathcal{K}}(\mathcal{X})$ and $K_Y \in \overline{\mathcal{K}}(\mathcal{Y})$. Then the S-independent natural extension of K_X and K_Y exists and is given by*

$$K_X \circledast K_Y \coloneqq \bigcap\nolimits_{P_X \in \underline{\mathcal{P}}_{K_X}, P_Y \in \underline{\mathcal{P}}_{K_Y}} K_{D_{P_X P_Y}},$$

which has $\{D_{P_X P_Y} : P_X \in \underline{\mathcal{P}}_{K_X}, P_Y \in \underline{\mathcal{P}}_{K_Y}\}$ as an (independent) representation.

It turns out that the new notion of representational independence is expressive enough to connect K to S-independence without the need to impose Archimedeanity, as is required in Theorem 7.

Proposition 4. *Given any mixing set of desirable gamble sets K on $\mathcal{X} \times \mathcal{Y}$, if K satisfies (strong) representational independence, then K satisfies S-independence.*

Together with Theorem 7 and Proposition 1, Proposition 4 implies that for Archimedean, mixing and credibly inderterminate sets of desirable gamble sets S-independence and representational independence are equivalent.

4 Discussion

We introduced a new notion of independence for sets of desirable gamble sets – representational independence – together with its variant, strong representational independence. For these notions, as well as for the existing S-independence, we established independent natural extensions of marginal sets of desirable gamble sets, and compared them with the epistemically independent natural extension.

We verify that epistemic independence, representational independence, and S-independence all coincide with the standard notion of independence for probability measures, in the following sense. Let p and q be pmfs on $\mathcal{X}$ and $\mathcal{Y}$, respectively, such that $p(x) > 0$ and $q(y) > 0$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, and define $K_p \coloneqq K_{D_p}$, and analogously for q and pq. Then the results in this paper guarantee that

$$K_p \otimes K_q = K_p \boxcircle K_q = K_p \circledast K_q = K_{pq}.$$

Indeed, Proposition 1 yields $K_p \otimes K_q = K_{pq}$. Since K_p is Archimedean, mixing, and credibly indeterminate with singleton representation $\underline{\mathcal{P}}_{K_p} = \{E_p\}$ (and similarly for q and pq), Theorem 8 implies $K_p \circledast K_q = K_{pq}$. As the S-independent product is a representational product, it is a superset of the smallest such product $K_p \boxcircle K_q$. Together with Proposition 2, this gives $K_{pq} = K_p \otimes K_q \subseteq K_p \boxcircle K_q \subseteq K_p \circledast K_q = K_{pq}$, and hence all three products coincide in this case.

Several directions for further research remain open. We have shown that strong representational independence for mixing sets of desirable gamble sets implies S-independence, but it is still an open question whether this result can be strengthened to an equivalence, and under which conditions this would be possible. Additionally, the relation with mixingness opens up more questions, such as whether there are products – other than the S-independent product – that preserve mixingness without imposing Archimedeanity. We also wonder to which extent the Archimedeanity condition is necessary in Ref. [6], and whether there are ways to obtain a closed-form expression for the S-independent product without relying on Archimedeanity.

One of the motivations for this work is to obtain a suitable notion of independence for use in Bayesian networks with sets of desirable gambles as local models. For this purpose, the associated independent product should ideally be associative. Whether any of the proposed products satisfies associativity is still unknown, which is why we introduced all notions for two uncertain variables only. Furthermore, for iterative applications of independent products, we must either find a canonical representation of the product, or allow for the representations of the marginals to be arbitrary, neither of which is currently established for any of the products.

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References

1. Benavoli, A., Facchini, A., Zaffalon, M., Vicente-Pérez, J.: A polarity theory for sets of desirable gambles. In: Antonucci, A., Corani, G., Couso, I., Destercke, S. (eds.) Proceedings of the Tenth International Symposium on Imprecise Probability: Theories and Applications. Proceedings of Machine Learning Research, vol. 62, pp. 37–48. PMLR (10–14 Jul 2017).
2. Casanova, A.: Rationality and desirability – a foundational study. Ph.D. thesis, Università della Svizzera Italiana (2023).
3. Couso, I., Moral, S.: Sets of desirable gambles: Conditioning, representation, and precise probabilities. International Journal of Approximate Reasoning **52**(7), 1034–1055 (2011). https://doi.org/https://doi.org/10.1016/j.ijar.2011.04.004
4. De Bock, J., De Cooman, G.: A desirability-based axiomatisation for coherent choice functions. In: Uncertainty Modelling in Data Science (Proceedings of SMPS 2018). pp. 46–53 (2018). https://doi.org/10.1007/978-3-319-97547-4_7
5. De Bock, J., De Cooman, G.: Interpreting, axiomatising and representing coherent choice functions in terms of desirability. Proceedings of Machine Learning Research **103**, 125–134 (2019), ISIPTA 2019 — Proceedings of the Eleventh International Symposium on Imprecise Probability: Theories and Applications
6. De Bock, J., De Cooman, G.: On a notion of independence proposed by Teddy Seidenfeld. In: Augustin, T., Cozman, F.G., Wheeler, G. (eds.) Reflections on the Foundations of Probability and Statistics: Essays in Honor of Teddy Seidenfeld, pp. 243–284. Springer (2023). https://doi.org/10.1007/978-3-031-15436-2_11

7. Kadane, J.B., Schervish, M.J., Seidenfeld, T.: A Rubinesque theory of decision. In: DasGupta, A. (ed.) A Festschrift for Herman Rubin, Institute of Mathematical Statistics Lecture Notes – Monograph Series, vol. 45, pp. 45–55 (2004). https://doi.org/10.1214/lnms/1196285378
8. Miranda, E.: A survey of the theory of coherent lower previsions. International Journal of Approximate Reasoning **48**(2), 628–658 (2008). https://doi.org/10.1016/j.ijar.2007.12.001
9. Quaeghebeur, E.: Introduction to imprecise probabilities. chap. Desirability. John Wiley & Sons (2014). https://doi.org/10.1002/9781118763117.ch1
10. Seidenfeld, T., Schervish, M.J., Kadane, J.B.: Decisions without ordering. In: Sieg, W. (ed.) Acting and reflecting: The Interdisciplinary Turn in Philosophy, Synthese Library, vol. 211, pp. 143–170. Kluwer, Dordrecht (1990). https://doi.org/10.1007/978-94-009-2476-5_11, reprinted in [11], pp. 40–68
11. Seidenfeld, T., Schervish, M.J., Kadane, J.B.: Rethinking the Foundations of Statistics. Cambridge University Press, Cambridge (1999). https://doi.org/10.1017/CBO9781139173230.001
12. Seidenfeld, T., Schervish, M.J., Kadane, J.B.: Coherent choice functions under uncertainty. Synthese **172**(1), 157–176 (2010). https://doi.org/10.1007/s11229-009-9470-7
13. Van Camp, A.: Choice functions as a tool to model uncertainty. Ph.D. thesis, Universiteit Gent (2018), https://hdl.handle.net/1854/LU-8546928
14. Van Camp, A., Blackwell, K., Konek, J.: Independent natural extension for choice functions. International Journal of Approximate Reasoning **152**, 390–413 (2023). https://doi.org/10.1016/j.ijar.2022.11.003
15. Van Camp, A., Miranda, E.: Modelling epistemic irrelevance with choice functions. International Journal of Approximate Reasoning **125**, 49–72 (2020). https://doi.org/10.1016/j.ijar.2020.06.010
16. Van Camp, A., De Cooman, G., Miranda, E.: Lexicographic choice functions. International Journal of Approximate Reasoning **92**, 97–119 (2018). https://doi.org/10.1016/j.ijar.2017.10.011
17. De Cooman, G.: Coherent and Archimedean choice in general Banach spaces. International Journal of Approximate Reasoning **140**, 255–281 (2022). https://doi.org/10.1016/j.ijar.2021.09.005
18. De Cooman, G., Miranda, E.: Irrelevant and independent natural extension for sets of desirable gambles. Journal of Artificial Intelligence Research **45**, 601–640 (2012). https://doi.org/10.1613/jair.3770
19. De Cooman, G., Quaeghebeur, E.: Exchangeability and sets of desirable gambles. International Journal of Approximate Reasoning **53**(3), 363–395 (2012). https://doi.org/10.1016/j.ijar.2010.12.002, special issue in honour of Henry E. Kyburg, Jr.
20. De Cooman, G., Van Camp, A., De Bock, J.: The logic behind desirable sets of things, and its filter representation. International Journal of Approximate Reasoning **172**, 109241 (2024). https://doi.org/10.1016/j.ijar.2024.109241
21. Walley, P.: Statistical Reasoning with Imprecise Probabilities. Chapman and Hall, London (1991)
22. Walley, P.: Towards a unified theory of imprecise probability. International Journal of Approximate Reasoning **24**, 125–148 (2000). https://doi.org/10.1016/S0888-613X(00)00031-1