Natural extension of choice functions

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Overview

Goal of the paper

- Coherent choice functions can be used as a model of the rational behaviour of an individual or a group.
- They were extended by Seidenfeld et al. to allow for incomparability, that arises naturally with imprecise information.
- Previous works assume that the choice function is determined for *all* options, something unreasonable in practice.
- Given a partially specified choice function, can we determine its implications on other option sets, using *only* the axioms of coherence?

Choice functions

We consider a real vector space $\mathcal{V}$ that represents our option space, and let $\mathcal{Q}$ be the set of all non-empty *finite* subsets of $\mathcal{V}$.

A choice function C on $\mathcal{V}$ is a map

$$C\colon \mathcal{Q} \to \mathcal{Q} \cup \{\emptyset\}\colon A \mapsto C(A) \text{ such that } C(A) \subseteq A.$$

Equivalently to a choice function C, we may consider its rejection function R, defined by $R(A) \coloneqq A \setminus C(A)$ for all A in $\mathcal{Q}$.

We will assume that $\mathcal{V}$ is ordered by a vector ordering $\preceq$, and that $\prec$ is its associated strict partial order $\prec$.

Coherent choice functions

We call a rejection function R on $\mathcal{V}$ coherent if for all A, A_1 and A_2 in $\mathcal{Q}$, all u and v in $\mathcal{V}$, and all λ in $\mathbb{R}_{>0}$:

R1. $R(A) \neq A$;

R2. if $u \prec v$ then $u \in R(\{u, v\})$;

R3. a. if $A_1 \subseteq R(A_2)$ and $A_2 \subseteq A$ then $A_1 \subseteq R(A)$;
b. if $A_1 \subseteq R(A_2)$ and $A \subseteq A_1$ then $A_1 \setminus A \subseteq R(A_2 \setminus A)$;

R4. a. if $A_1 \subseteq R(A_2)$ then $\lambda A_1 \subseteq R(\lambda A_2)$;
b. if $A_1 \subseteq R(A_2)$ then $A_1 + \{u\} \subseteq R(A_2 + \{u\})$.

Overview

Natural extension: definition

Let $\mathcal{Q}_0$ denote those option sets that include 0. Given an assessment $\mathcal{B} \subseteq \mathcal{Q}_0$, it has the interpretation that 0 should be rejected from every option set B in $\mathcal{B}$.

The natural extension of $\mathcal{B}$ is the rejection function

$$\begin{aligned}\mathcal{E}(\mathcal{B}) &:= \inf\{R \text{ coherent} : (\forall B \in \mathcal{B}) 0 \in R(B)\} \\ &= \inf\{R \text{ coherent} : R \text{ extends } \mathcal{B}\},\end{aligned}$$

where we let $\inf \emptyset$ be equal to $\mathrm{id}_{\mathcal{Q}}$, the identity rejection function that maps every option set to itself.

An operational definition

For any set of options A in $\mathcal{Q}$, let $R_{\mathcal{B}}(A)$ be given by

$$\Big\{u \in A : (\exists A' \in \mathcal{Q})\Big(A' \supseteq A \text{ and } (\forall v \in \{u\} \cup (A' \setminus A))$$
$$((A' - \{v\}) \cap \mathcal{V}_{\succ 0} \neq \emptyset \text{ or } (\exists B \in \mathcal{B}, \exists \mu \in \mathbb{R}_{>0})\{v\} + \mu B \precsim A')\Big)\Big\},$$

where $\mathcal{V}_{\succ 0} \coloneqq \{u \in \mathcal{V} : 0 \prec u\}$.

- $R_{\mathcal{B}}$ is the least informative rejection function that satisfies Axioms R2–R4 and extends $\mathcal{B}$.
- We say that $\mathcal{B}$ avoids complete rejection when $R_{\mathcal{B}}$ satisfies Axiom R1.

Characterizing the natural extension

For any assessment $\mathcal{B} \subseteq \mathcal{Q}_0$, the following are equivalent:

(i) $\mathcal{B}$ avoids complete rejection;

(ii) $\mathcal{B}$ has a coherent extension;

(iii) $\mathcal{E}(\mathcal{B}) \neq \mathrm{id}_{\mathcal{Q}}$;

(iv) $\mathcal{E}(\mathcal{B})$ is coherent;

(v) $\mathcal{E}(\mathcal{B})$ is the least informative rejection function that is coherent and extends $\mathcal{B}$.

When any of these equivalent statements hold, then $\mathcal{E}(\mathcal{B}) = R_{\mathcal{B}}$.

Overview

Binary comparisons: sets of desirable options

A desirability assessment $B \subseteq \mathcal{V}$ is a set of options that we strictly prefer to the zero option.

We call a set of desirable options $D \subseteq \mathcal{V}$ coherent if for all u and v in $\mathcal{V}$ and λ in $\mathbb{R}_{>0}$:

D1. $0 \notin D$;

D2. if $0 \prec u$ then $u \in D$;

D3. if $u \in D$ then $\lambda u \in D$;

D4. if $u, v \in D$ then $u + v \in D$.

Desirable options and choice functions

There is a one-to-one correspondence between sets of options $B \subseteq \mathcal{V}$ and sets of binary comparisons: $\mathcal{B}_B := \{\{0, u\} : u \in B\}$.

With this correspondence, if D is a coherent set of desirable options D, then the rejection function R_D given by

$$R_D(A) = \{u \in A : (\forall v \in A) v - u \notin D\}$$

for all A in $\mathcal{Q}$ is coherent.

More generally, given $B \subseteq \mathcal{V}$, we say that $D \subseteq \mathcal{V}$ extends B if $B \subseteq D$.

$$D \text{ extends } B \Leftrightarrow R_D \text{ extends } \mathcal{B}_B.$$

Natural extension of sets of desirable options

Consider any desirability assessment $B \subseteq \mathcal{V}$. We say that it avoids non-positivity when it is included in some coherent set of desirable options. In that case, the smallest such set is its natural extension

$$\mathcal{E}^{\mathbf{D}}(B) := \inf\{D \text{ coherent} : B \subseteq D\} = \operatorname{posi}(\mathcal{V}_{\succ 0} \cup B),$$

where we let $\inf \emptyset = \mathcal{V}$.

- B avoids non-positivity $\Leftrightarrow \mathcal{B}_B$ avoids complete rejection.
- In that case, $\mathcal{E}(\mathcal{B}_B) = R_{\mathcal{E}^{\mathbf{D}}(B)}$.

Connection between the natural extensions

$$
\begin{array}{ccc}
B & \xrightarrow{\quad \mathcal{E}^{\mathbf{D}} \quad} & \mathcal{E}^{\mathbf{D}}(B) = D_{\mathcal{E}(\mathcal{B}_B)} \\
\mathcal{B}_\bullet \Big\downarrow & & R_\bullet \Big\downarrow \quad \Big\uparrow D_\bullet \\
\mathcal{B}_B & \xrightarrow{\quad \mathcal{E} \quad} & \mathcal{E}(\mathcal{B}_B) = R_{\mathcal{E}^{\mathbf{D}}(B)}
\end{array}
$$

Implication: not all coherent choice functions are determined by binary comparisons

Many examples of coherent choice functions, such as the M-admissible or E-admissible ones can be written as the infima of choice functions determined by binary comparisons.

By considering $B := \{0, f, \lambda f\}$ with f a gamble and λ an element of $\mathbb{R}_{>0}$ and different from 1, we obtain that its natural extension $R_{\mathcal{B}}$ is coherent but is NOT the infima of binary choice functions.

A consequence of this is that either (i) choice functions do NOT form a strong belief structure, or (ii) there are maximal (=maximally informative) choice functions that are NOT determined by binary comparisons.

Overview

Sets of indifferent options

In addition to the set D of options that we prefer to the status quo, we can also consider the set I that we consider indifferent to it. We say that this set is coherent if for all u, v in $\mathcal{V}$ and λ in $\mathbb{R}$:

I_1. $0 \in I$;

I_2. if $u \in \mathcal{V}_{\succ 0} \cup \mathcal{V}_{\prec 0}$ then $u \notin I$;

I_3. if $u \in I$ then $\lambda u \in I$;

I_4. if $u, v \in I$ then $u + v \in I$.

Compatibility with choice functions

A coherent set of indifferent options I determines a quotient space

$$\mathcal{V}/I \coloneqq \{[u] : u \in \mathcal{V}\} = \{\{u\} + I : u \in \mathcal{V}\} = \{u/I : u \in \mathcal{V}\}.$$

R is compatible with I if there exists R' on $\mathcal{Q}(\mathcal{V}/I)$ such that

$$R(A) = \{u \in A : [u] \in R'(A/I)\} \; \forall A \in \mathcal{Q}(\mathcal{V}).$$

Compatibility with natural extension

Given $\mathcal{B} \subseteq \mathcal{Q}_0(\mathcal{V})$ and any coherent set of indifferent options I, the natural extension of $\mathcal{B}$ under I is the rejection function

$$R_{\mathcal{B},I}(A) \coloneqq \{u \in A : [u] \in R_{\mathcal{B}/I}(A/I)\} \text{ for all } A \text{ in } \mathcal{Q}(\mathcal{V}),$$

where $\mathcal{B}/I \coloneqq \{B/I : B \in \mathcal{B}\} \subseteq \mathcal{Q}_{[0]}(\mathcal{V}/I)$.

$R_{\mathcal{B},I}$ is the least informative rejection function that is coherent, extends $\mathcal{B}$, and is compatible with I, if there is one.

Overview

Conclusions and open problems

Conclusions:

- The notion of natural extension can be extended to the theory of coherent choice functions.
- Binary comparisons (=sets of desirable options) follow as a particular case.
- Coherent choice functions are not a strong belief structure.

Open problems:

↪ Compatibility with other structural assessments: irrelevance, exchangeability.

References

- T. Seidenfeld, M.J. Schervish, J.B. Kadane, Coherent choice functions under uncertainty. Synthese 172(1), 157-176 (2010).
- A. Van Camp, G. de Cooman, E. Miranda, E. Quaeghebeur, Coherent choice functions, desirability and indifference. Fuzzy Sets and Systems, 341C, 1-36 (2018).
- P. Walley, Statistical reasoning with imprecise probabilities. Chapman and Hall (1991).