Foundations of imprecise probabilities

Choice functions

Arthur Van Camp

SIPTA Summer School 2026

28 July 2026

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- understand their connection with sets of desirable gambles;
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- understand their connection with sets of desirable gambles;
- understand representation and how it can be used.

More importantly, I'd like to convince you that choice functions are useful and promising models of imprecise probabilities, with a clear expressive advantage. There are also still many interesting open questions.

History and future of choice functions in imprecise probabilities

Teddy Seidenfeld

First to consider imprecise-probabilistic choice functions.
Established representation in terms of arbitrary set of probability / utility pairs.

- A Rubinesque theory of decision, A Festschrift for Herman Rubin Institute of Mathematical Statistics Lecture Notes – Monograph Series, **2004**, doi:10.1214/lnms/1196285378
- Coherent Choice Functions under Uncertainty, Synthese, **2010**, doi:10.1007/s11229-009-9470-7

History and future of choice functions in imprecise probabilities

Gert de Cooman

First to consider choice functions on gambles.
Introduced sets of desirable gamble sets with **Jasper De Bock**.
Studied choice in general Banach spaces.
Studied connection with logic with **Arthur Van Camp** and **Jasper De Bock**.

- Interpreting, Axiomatising and Representing Coherent Choice Functions in Terms of Desirability, ISIPTA **2019**
- Coherent and Archimedean choice in general Banach spaces, International Journal of Approximate Reasoning, **2022**, doi:10.1016/j.ijar.2021.09.005
- The logic behind desirable sets of things, and its filter representation, International Journal of Approximate Reasoning, **2024**, doi:10.1016/j.ijar.2024.109241

History and future of choice functions in imprecise probabilities

Arthur Van Camp

PhD dissertation about imprecise-probabilistic choice functions, under the guidance of **Gert de Cooman** and **Enrique Miranda**.
Indifference with **Gert de Cooman**, **Enrique Miranda** and **Erik Quaeghebeur**.

Enrique Miranda

Erik Quaeghebeur

Jason Konek

Kevin Blackwell

Irrelevance, natural extension and marginal extension with **Enrique Miranda** and **Gert de Cooman**.
Epistemic independence with **Jason Konek** and **Kevin Blackwell**.

- Choice functions as a tool to model uncertainty, PhD dissertation, Ghent University **2018**
- Coherent choice functions, desirability and indifference, Fuzzy Sets and Systems, **2018**, doi:10.1016/j.fss.2017.05.019
- Modelling epistemic irrelevance with choice functions, International Journal of Approximate Reasoning, **2020**, doi:10.1016/j.ijar.2020.06.010
- Independent natural extension for choice functions, International Journal of Approximate Reasoning, **2023**, doi:10.1016/j.ijar.2022.11.003

History and future of choice functions in imprecise probabilities

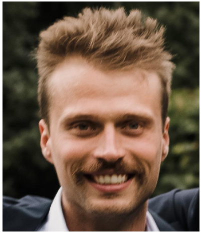
Jasper De Bock

Important generalisation to things with arbitrary **closure operators**.
Introduced sets of desirable gamble sets with useful axiomatisation, with **Gert de Cooman**.
First to study algorithms, with **Arne Decadt** and **Alexander Erreygers**.

- Interpreting, Axiomatising and Representing Coherent Choice Functions in Terms of Desirability, ISIPTA **2019**
- Archimedean Choice Functions: an Axiomatic Foundation for Imprecise Decision Making, IPMU **2020**, doi:10.1007/978-3-030-50143-3_15
- A theory of desirable things, ISIPTA **2023**

History and future of choice functions in imprecise probabilities

Arne Decadt



Alexander Erreygers

Study algorithms with choice functions.

- Extending choice assessments to choice functions: An algorithm for computing the natural extension, International Journal of Approximate Reasoning, **2025**, doi:10.1016/j.ijar.2024.109331
- Conservative decision-making with sets of probabilities: How to infer new choices from previous ones, Fuzzy Sets and Systems, **2025**, doi:10.1016/j.fss.2025.109612

History and future of choice functions in imprecise probabilities

Catrin Campbell-Moore

Studies probability filters as equivalent model to choice functions.

– Probability filters and desirable gambles, International Journal of Approximate Reasoning, **2026**, doi:10.1016/j.ijar.2026.109643

History and future of choice functions in imprecise probabilities

Kevin Blackwell

Studies connection between sets of desirable gamble sets and choice functions in a more general context.

- Coherent rejection functions for arbitrary things, ISIPTA **2025**

History and future of choice functions in imprecise probabilities

Justyna Dąbrowska

Studies marginal problem and information algebras with **Arthur Van Camp** and **Erik Quaeghebeur**.
Studies different types of independence with **Banda Norimarna** and **Arthur Van Camp**.

Banda Norimarna

- The marginal problem for sets of desirable gamble sets, ISIPTA **2025**
- A Preliminary Study of Notions of Independence for Choice Functions, IPMU **2026**, doi:10.1007/978-3-032-28994-0_17

Outline

DESIRABILITY

Desirability: review

We focus on desirability rather than acceptability
[see Erik Quaeghebeur's lecture and axiom variants].

The key difference is that

$$0 \notin \mathcal{D}.$$

for sets of desirable gambles $\mathcal{D}$.

Desirability: review

A set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$, with $\mathcal{X}$ a finite possibility space, is coherent when

$\mathcal{D}_1$. $0 \notin \mathcal{D}$;

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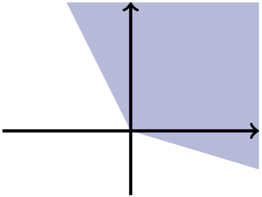

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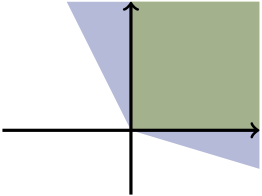

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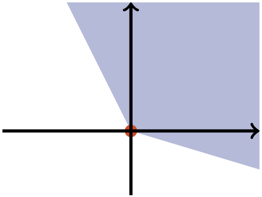

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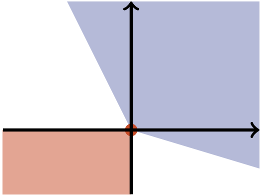

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Consequence of coherence: $\mathcal{D}$ has nothing in common with the negative orthant $\mathcal{L}_{\leq 0}$.

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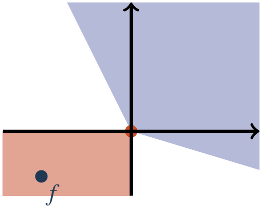

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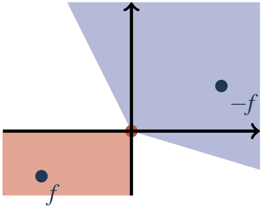

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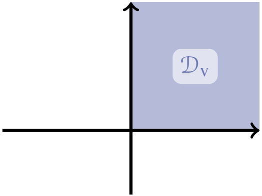

$\mathcal{D}_{\mathrm{v}}$ is the vacuous set of desirable gambles: the smallest coherent $\mathcal{D}$.

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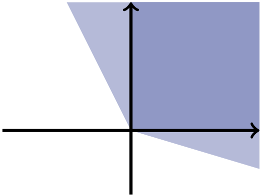

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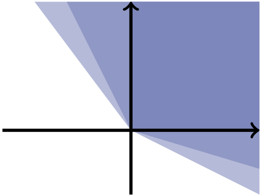

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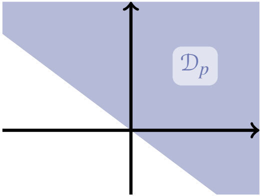

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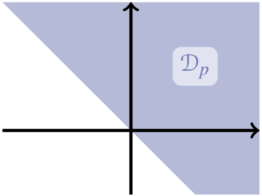

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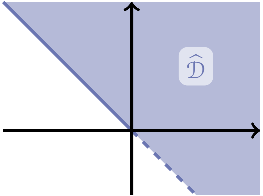

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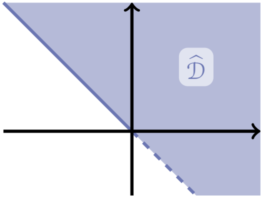

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A coherent set of desirable gambles $\mathcal{D}$ is equivalent to a partial preference order $\prec$:

$$f \prec g \Leftrightarrow g - f \in \mathcal{D} \quad \text{and} \quad \mathcal{D} = \{f : 0 \prec f\}.$$

satisfying

O_1. $f \not\prec f$;

O_2. if $f < g$ then $f \prec g$;

O_3. if $f \prec g$ and $g \prec h$, then $f \prec h$;

O_4. $f \prec g$ iff $\lambda f + h \prec \lambda g + h$, for all $\lambda \in \mathbb{R}_{>0}$.

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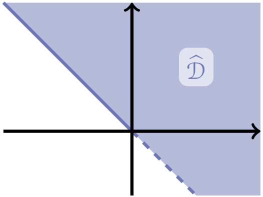

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$\mathcal{D}$ and $\prec$ are equivalent representations of the same information.

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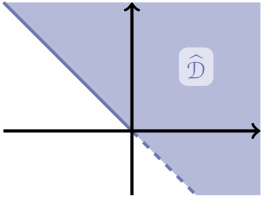


A maximal set of desirable gambles $\widehat{\mathcal{D}}$.

A coherent set of desirable gambles $\mathcal{D}$ is equivalent to a partial preference order $\prec$:

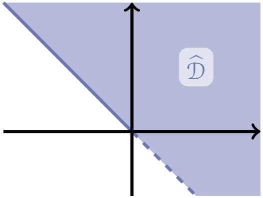
$$f \prec g \Leftrightarrow g - f \in \mathcal{D} \quad \text{and} \quad \mathcal{D} = \{f : 0 \prec f\}.$$

Example

1. If p is a pmf, then $\prec$ defined by $f \prec g \Leftrightarrow E_p(f) < E_p(g)$ or $f < g$ is a partial preference order.

Desirability: review

A set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$, with $\mathcal{X}$ a finite possibility space, is coherent when



A maximal set of desirable gambles $\widehat{\mathcal{D}}$.

$\mathcal{D}_1$. $0 \notin \mathcal{D}$;

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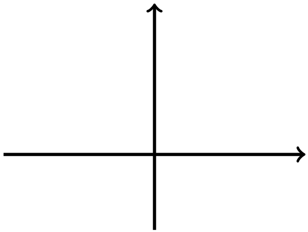
Example

1. If p is a pmf, then $\prec$ defined by $f \prec g \Leftrightarrow E_p(f) < E_p(g)$ or $f < g$ is a partial preference order.
2. If $\mathcal{C}$ is a credal set, then $\prec$ defined by $f \prec g \Leftrightarrow (\forall p \in \mathcal{C}) E_p(f) < E_p(g)$ or $f < g$ is a partial preference order.

Desirability: natural extension



You can make assessments about desirability of gambles.

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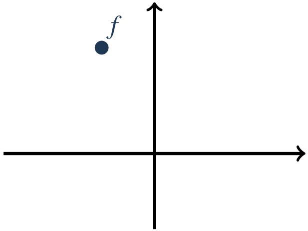

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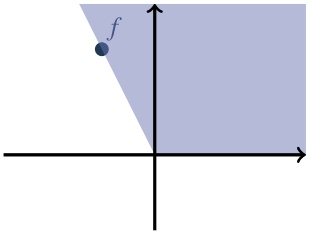

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If "f is desirable", then

$$\bigcap\{\mathcal{D} \text{ coherent} : f \in \mathcal{D}\} = \operatorname{posi}(\mathcal{L}_{>0} \cup \{f\})$$

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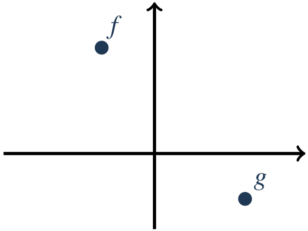

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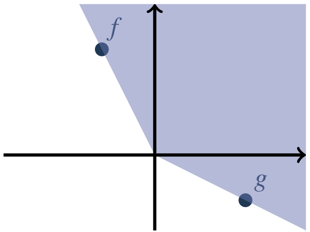

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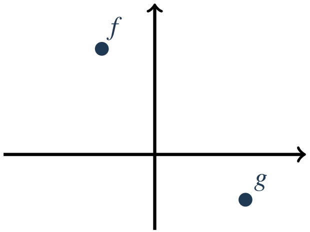
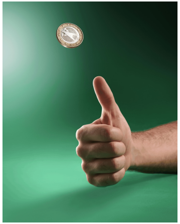


You can make assessments about desirability of gambles.

What if the assessment is “f is desirable or g is desirable”?

SETS OF DESIRABLE GAMBLE SETS

Running example: coin with bias in unknown direction



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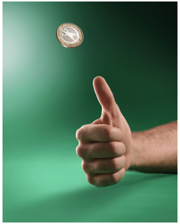

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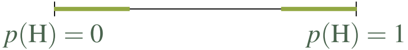

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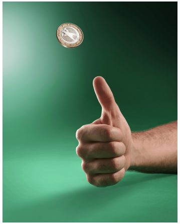

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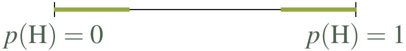

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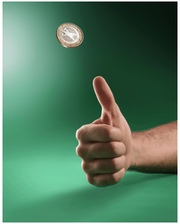

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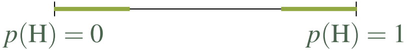

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If we use sets of desirable gambles, we may be inclined to use

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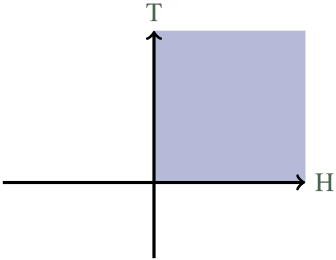

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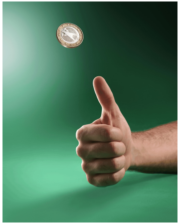

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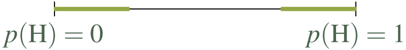

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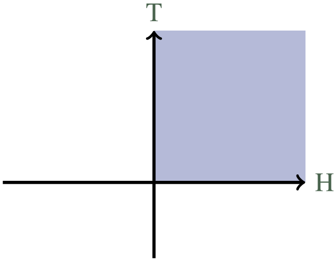

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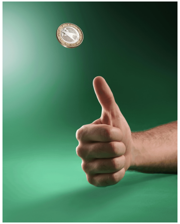

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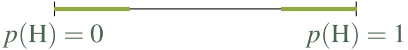

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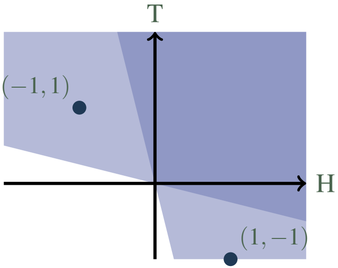

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Solution: use gamble sets. For each pmf $p \in \mathcal{C}$, there is a gamble in $\{(-1,1), (1,-1)\}$ with positive p-expectation.

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A gamble set F is a finite subset of $\mathcal{L}$:

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The intended interpretation of $\mathcal{K}$ is that a gamble set $F = \{f_1, \ldots, f_n\}$ belongs to $\mathcal{K}$

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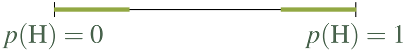

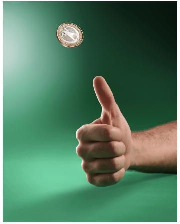

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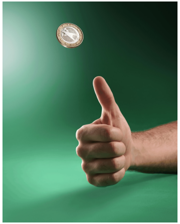

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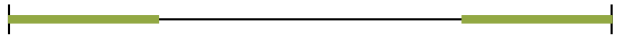
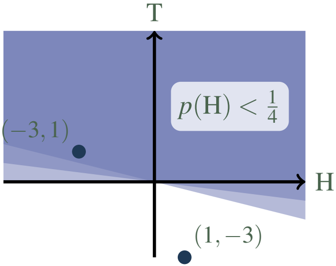
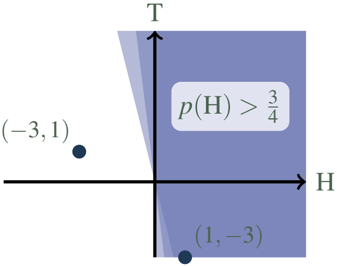


This means that the gamble set $\{f, g\} = \{(-3, 1), (1, -3)\}$ is desirable: $\{f, g\} \in \mathcal{K}$.





Sets of desirable gamble sets: coherence

Consider a finite possibility space $\mathcal{X}$. A set of desirable gamble sets $\mathcal{K} \subseteq \mathcal{Q}$ is coherent when for all $F, G \in \mathcal{Q}$ and $f \in \mathcal{L}$:

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Justification for Axiom $\mathcal{K}_2$: If F is a desirable gamble set, then some gamble $f \in F$ [I don't assume that you can identify f!] must be a desirable gamble. But then $f \neq 0$ by Axiom $\mathcal{D}_1$, so the set $F \setminus \{0\}$ contains a desirable gamble, and is hence a desirable gamble set.

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Justification for Axiom $\mathcal{K}_3$: Such f is desirable by Axiom $\mathcal{D}_2$.

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Justification for Axiom $\mathcal{K}_4$: If F is a desirable gamble set, it must contain some desirable gamble, and therefore so must G.

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$\mathcal{K}_5$. if $F, G \in \mathcal{K}$ and $(\lambda_{fg}, \mu_{fg}) > 0$ for all $f \in F$ and $g \in G$,[1] then

$$\{\lambda_{fg} f + \mu_{fg} g : f \in F, g \in G\} \in \mathcal{K}.$$ [productive axiom]

[1] We use for vectors $(\alpha_1, \dots, \alpha_n)$ the same convention as for gambles: $(\alpha_1, \dots, \alpha_n) > 0$ iff '$\alpha_k \geq 0$ for all k and $\alpha_\ell > 0$ for some ℓ'.

Sets of desirable gamble sets: coherence

Consider a finite possibility space $\mathcal{X}$. A set of desirable gamble sets $\mathcal{K} \subseteq \mathcal{Q}$ is coherent when for all $F, G \in \mathcal{Q}$ and $f \in \mathcal{L}$:

$\mathcal{K}_1$. $\emptyset \notin \mathcal{K}$; [destructive axiom]

$\mathcal{K}_2$. if $F \in \mathcal{K}$ then $F \setminus \{0\} \in \mathcal{K}$; [productive axiom]

$\mathcal{K}_3$. if $f > 0$ then $\{f\} \in \mathcal{K}$; [productive axiom]

$\mathcal{K}_4$. if $F \in \mathcal{K}$ and $F \subseteq G$, then $G \in \mathcal{K}$; [productive axiom]

$\mathcal{K}_5$. if $F, G \in \mathcal{K}$ and $(\lambda_{fg}, \mu_{fg}) > 0$ for all $f \in F$ and $g \in G$,[1] then

$$\{\lambda_{fg} f + \mu_{fg} g : f \in F, g \in G\} \in \mathcal{K}.$$ [productive axiom]

Justification for Axiom $\mathcal{K}_5$: If F and G are desirable gamble sets, then they contain some desirable gambles $f^\star \in F$ and $g^\star \in G$. Then the composite gamble $\lambda_{f^\star g^\star} f^\star + \mu_{f^\star g^\star} g^\star \in \{\lambda_{fg} f + \mu_{fg} g : f \in F, g \in G\}$, is desirable too, taking into account Axioms $\mathcal{D}_3$ and $\mathcal{D}_4$.

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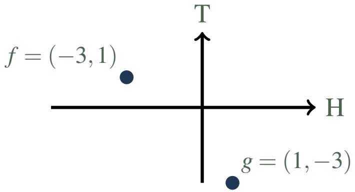
We found that the gamble set $\{f, g\} = \{(-3, 1), (1, -3)\}$ is desirable: $\{f, g\} \in \mathcal{K}$.



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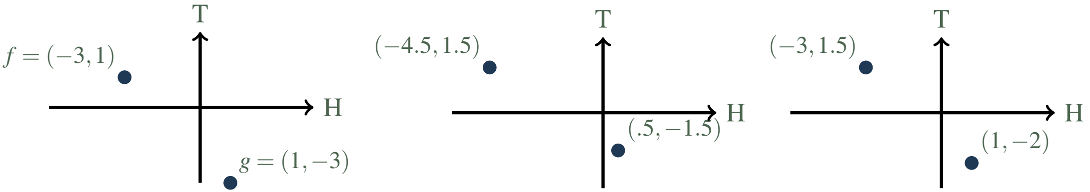

Exercise

Assume that there is a coherent set of desirable gamble sets $\mathcal{K}$ such that $\{f, g\} \in \mathcal{K}$. Show that

1) $\{\lambda f, \mu g\} \in \mathcal{K}$ for every $\lambda, \mu > 0$;
2) $\{f', g'\} \in \mathcal{K}$ for every $f' \geq f$ and $g' \geq g$;
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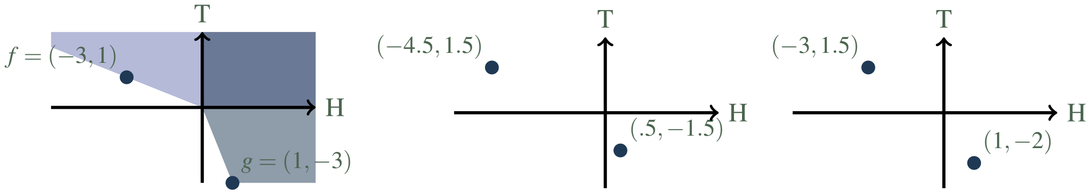

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Exercise 1): Use Axiom $\mathcal{K}_5$ with $F = G = \{f, g\} \in \mathcal{K}$ and, for instance, $(\lambda_{ff}, \mu_{ff}) = (\lambda, 0)$, $(\lambda_{fg}, \mu_{fg}) = (0, \mu)$, $(\lambda_{gf}, \mu_{gf}) = (0, \lambda)$ and $(\lambda_{gg}, \mu_{gg}) = (\mu, 0)$. Then Axiom $\mathcal{K}_5$ tells us that $\{\lambda_{ff} f + \mu_{ff} f, \lambda_{fg} f + \mu_{fg} g, \lambda_{gf} g + \mu_{gf} f, \lambda_{gg} g + \mu_{gg} g\} = \{\lambda f, \mu g\} \in \mathcal{K}$.

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Exercise 2): We first show that $\{f', g\} \in \mathcal{K}$. If $f' = f$ then we are done, so assume $f' > f$. Then $h := f' - f > 0$ so Axiom $\mathcal{K}_3$ tells us that $\{h\} \in \mathcal{K}$. Use Axiom $\mathcal{K}_5$ with $F = \{f, g\}$, $G = \{h\}$ and $(\lambda_{fh}, \mu_{fh}) = (1, 1)$ and $(\lambda_{gh}, \mu_{gh}) = (1, 0)$ to infer that $\{\lambda_{fh} f + \mu_{fh} h, \lambda_{gh} g + \mu_{gh} h\} = \{f + h, g + 0\} = \{f', g\} \in \mathcal{K}$. Similarly, we infer that $\{f', g'\} \in \mathcal{K}$.

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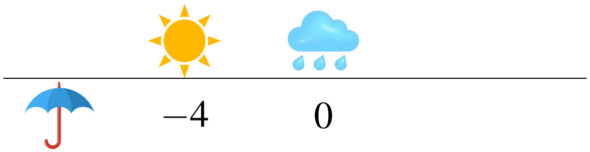
Exercise 3): Combine the two results above to infer that $\{\lambda f', \mu g'\} \in \mathcal{K}$ for every $\lambda, \mu > 0$ and $f' \geq f$ and $g' \geq g$. Then use Axiom $\mathcal{K}_4$ to infer that $F \in \mathcal{K}$ for every finite $F \supseteq \{\lambda f', \mu g'\}$ where $\lambda, \mu > 0$ and $f' \geq f$ and $g' \geq g$.

INTERMEZZO: CHOICE FUNCTIONS

Choice functions

Choice functions



Choice functions

		
	-4	0
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Gambles represent decisions under uncertainty.

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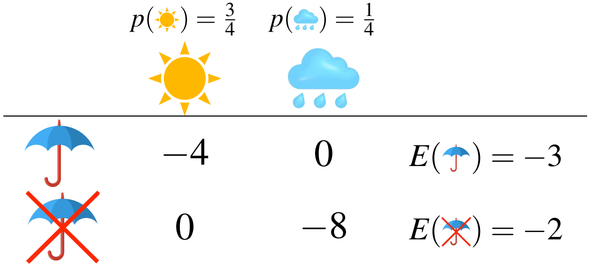

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We focus on the function that describes making decisions: the choice function.

Definition (Choice function & rejection function)

A choice function C is a map

$C\colon \mathcal{Q} \to \mathcal{Q}\colon F \mapsto C(F) \subseteq F$ — $C(F)$ collects the non-rejected or admissible gambles from the menu F.

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The choice function C and rejection function R are linked through

$$C(F) = F \setminus R(F) \qquad \text{and} \qquad R(F) = F \setminus C(F) \qquad \text{for all } F \in \mathcal{Q}.$$

Choice functions

Example 1: expected utility maximisation

If I have a pmf p, then $R_p(F) = \{f \in F\colon \text{ there is } g \in F \text{ such that } E_p(f) < E_p(g) \text{ or } f < g\}$ and $C_p(F) = F \setminus R_p(F)$.

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Example 2: Sen–Walley maximality

If I have a coherent set of desirable gambles $\mathcal{D}$, or a partial preference order $\prec$, then I can use it to make decisions:

$$R(F) = \{f \in F\colon \text{there is } g \in F \text{ such that } f \prec g\} = \{f \in F\colon \text{there is } g \in F \text{ such that } g - f \in \mathcal{D}\}$$

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Example 3: E-admissibility

If I have a credal set $\mathcal{C}$, then I can use E-admissibility to make decisions:

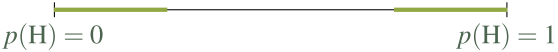
$$C_{\mathcal{C}}(F) := \{f \in F\colon \text{there is } p \in \mathcal{C} \text{ such that } f \in C_p(F)\} = \bigcup_{p \in \mathcal{C}} C_p(F) \qquad \text{for all } F \in \mathcal{Q}.$$

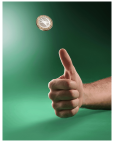
Unlike the first two examples, E-admissibility is an instance of non-binary choice: it may happen that

$$f \in C_{\mathcal{C}}(\{f,g\}), \quad f \in C_{\mathcal{C}}(\{f,h\}), \quad \text{even though} \quad f \in R_{\mathcal{C}}(\{f,g,h\}).$$

Running example: coin with bias in unknown direction

$\mathcal{C} := \{p \text{ pmf} : p(\mathrm{H}) < \frac{1}{4} \text{ or } p(\mathrm{H}) > \frac{3}{4}\}$.





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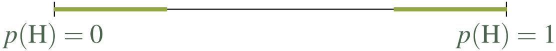

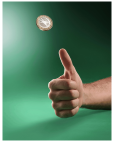

Let's consider the E-admissible choice function based on $\mathcal{C}$:

$$C_{\mathcal{C}}(F) = \{f \in F \colon \text{there is } p \in \mathcal{C} \text{ such that } f \in C_p(F)\} = \bigcup_{p \in \mathcal{C}} C_p(F) \quad \text{for all } F \in \mathcal{Q}.$$

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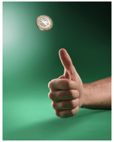

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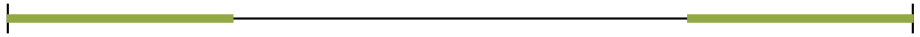

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Exercise

Show that $C_{\mathcal{C}}$ is non-binary: find gambles f, g and h such that $f \in C_{\mathcal{C}}(\{f,g\}), f \in C_{\mathcal{C}}(\{f,h\})$ but $f \notin C_{\mathcal{C}}(\{f,g,h\})$.

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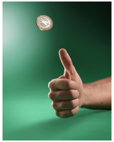

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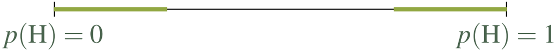

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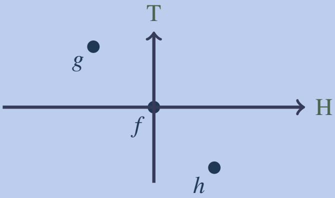
Solution

Consider for instance $f = 0$, $g = (-1,1)$ and $h = (1,-1)$.
Then $f \in C_{\mathcal{C}}(\{f,g\})$ because $f \in C_p(\{f,g\})$ for $p \in \mathcal{C}$ such that $p(\mathrm{H}) > \frac{3}{4}$.
Similarly, $f \in C_{\mathcal{C}}(\{f,h\})$, because $f \in C_p(\{f,h\})$ for $p \in \mathcal{C}$ such that $p(\mathrm{H}) < \frac{1}{4}$.
But $f \in R_{\mathcal{C}}(\{f,g,h\})$ because for every $p \in \mathcal{C}$, either $E_p(f) < E_p(g)$ [when $p(\mathrm{H}) < \frac{1}{4}$] or $E_p(f) < E_p(h)$ [when $p(\mathrm{H}) > \frac{3}{4}$].



Choice functions: coherence

A choice function $C\colon \mathcal{Q} \to \mathcal{Q}$, with $\mathcal{X}$ a finite possibility space, is coherent when, for all $F, G \in \mathcal{Q}$, $f, g \in \mathcal{L}$ and $\lambda \in \mathbb{R}_{>0}$:

C_1. $C(F) \neq \emptyset$; [non-empty choice]

C_2. if $F \subseteq G$ then $R(F) \subseteq R(G)$; [Sen α], [independence of irrelevant alternatives]

C_3. if $f \in R(F)$ then $f + g \in R(F + g)$;[2] [translation invariance]

C_4. if $0 \in R(\{f\} \cup F)$ and $0 \in R(\{g\} \cup F)$, then $0 \in R(\{f + g\} \cup F)$; [additivity]

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The following outcast property (also called Aizerman's property) is a consequence:

C_6. if $f,g \in R(F)$ and $f \neq g$, then $f \in R(F \setminus \{g\})$. [Aizerman's property], [outcast property]

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Example

Expected utility maximisation, Sen–Walley maximality and E-admissibility are coherent choice functions.

[2] We let $F + g := \{f + g : f \in F\}$.

Connection between choice functions and sets of desirable gamble sets

Given a translation invariant rejection function R, if $f \in R(F)$ then $0 \in R(F - f)$. Then $F - f$ is a gamble set from which 0 is rejected. Informally, this must be because some gamble in $F - f$ is desirable, so $F - f$ is a desirable gamble set.

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We let $\mathcal{K}_R := \{F \in \mathcal{Q} \colon 0 \in R(\{0\} \cup F)\}$ be the corresponding set of desirable gamble sets.

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Conversely, given a set of desirable gamble sets $\mathcal{K}$, we define a corresponding rejection function $R_{\mathcal{K}}$ by letting $f \in R_{\mathcal{K}}(F)$ if $F - f \in \mathcal{K}$, for all $F \in \mathcal{Q}$ and $f \in F$.

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Proposition

1. *If $\mathcal{K}$ is a coherent set of desirable gamble sets, then $R_{\mathcal{K}}$ is a coherent rejection function.*
2. *The mappings $\mathcal{K}_R$ and $R_{\mathcal{K}}$ commute.*

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2. The mappings $\mathcal{K}_R$ and $R_{\mathcal{K}}$ commute.

Conclusion: $\mathcal{K}$ is coherent iff corresponding R is coherent.

$\mathcal{K}$ and R are equivalent representations of the same information.

Examples of coherent sets of desirable gamble sets

Example 1: expected utility maximisation

If I have a pmf p, then $\mathcal{K}_p := \{F \in \mathcal{Q}\colon$ there is $f \in F$ such that $E_p(f) > 0$ or $f > 0\}$ is coherent.

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Example 2: Sen–Walley maximality

If I have a coherent set of desirable gambles $\mathcal{D}$, or a partial preference relation $\prec$, then

$$\mathcal{K} = \{F \in \mathcal{Q}\colon \text{there is } f \in F \text{ such that } f \in \mathcal{D}\} = \{F \in \mathcal{Q}\colon \text{there is } f \in F \text{ such that } 0 \prec f\}$$

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Example 3: E-admissibility

If I have a credal set $\mathcal{C}$, then

$$\mathcal{K}_{\mathcal{C}} \coloneqq \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\forall p \in \mathcal{C})(\exists f \in F) E_p(f) > 0\} = \bigcap_{p \in \mathcal{C}} \mathcal{K}_p$$

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Exercise

Derive $\mathcal{K}_{\mathcal{C}}$ from $C_{\mathcal{C}}$ and the connection between choice functions and sets of desirable gamble sets.

Solution

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Use the E-admissible rejection function $R_{\mathcal{C}}(F) = F \setminus C_{\mathcal{C}}(F)$:

$$f \in R_{\mathcal{C}}(F) \Leftrightarrow f \notin C_{\mathcal{C}}(F) \Leftrightarrow (\forall p \in \mathcal{C}) f \notin C_p(F) \Leftrightarrow (\forall p \in \mathcal{C})(\exists g \in F) E_p(f) < E_p(g) \text{ or } f < g$$

for all $F \in \mathcal{Q}$ and $f \in F$. Infer that

$$0 \in R_{\mathcal{C}}(F) \Leftrightarrow (\forall p \in \mathcal{C})(\exists g \in F) E_p(g) > 0 \text{ or } g > 0 \Leftrightarrow (\forall p \in \mathcal{C})(\exists g \in F) E_p(g) > 0 \text{ or } F \cap \mathcal{L}_{>0} \neq \emptyset$$

for all $F \in \mathcal{Q}$ such that $0 \in F$. Now use the connection between rejection functions and sets of desirable gamble sets to infer that

$$\mathcal{K}_{\mathcal{C}} = \{F \in \mathcal{Q} \colon 0 \in R_{\mathcal{C}}(F \cup \{0\})\} = \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\forall p \in \mathcal{C})(\exists f \in F) E_p(f) > 0\}.$$

INFERENCE WITH
SETS OF DESIRABLE GAMBLE SETS

Sets of desirable gamble sets: intersection structure

Recall the rationality axioms:

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Exercise

Each of the Axioms $\mathcal{K}_1$–$\mathcal{K}_5$ is closed under arbitrary non-empty intersections: if $\mathbb{K}$ is a non-empty set of sets of desirable gamble sets where each $\mathcal{K} \in \mathbb{K}$ satisfies Axiom $\mathcal{K}_\ell$, then $\bigcap \mathbb{K}$ is a set of desirable gamble sets that satisfies Axiom $\mathcal{K}_\ell$.

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Solution

Axiom $\mathcal{K}_1$. If $\emptyset \notin \mathcal{K}$ for every $\mathcal{K} \in \mathbb{K}$, then $\emptyset \notin \bigcap \mathbb{K}$.

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Solution

Axiom $\mathcal{K}_2$. For every $\mathcal{K} \in \mathbb{K}$, if $F \in \mathcal{K}$ then $F \setminus \{0\} \in \mathcal{K}$. Therefore $F \setminus \{0\} \in \bigcap \mathbb{K}$.

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Solution

Axiom $\mathcal{K}_3$. If $f > 0$, then $\{f\} \in \mathcal{K}$ for every $\mathcal{K} \in \mathbb{K}$. Therefore $\{f\} \in \bigcap \mathbb{K}$.

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Solution

Axiom $\mathcal{K}_4$. For every $\mathcal{K} \in \mathbb{K}$, if $F \in \mathcal{K}$ then $G \in \mathcal{K}$. Therefore $G \in \bigcap \mathbb{K}$.

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Axiom $\mathcal{K}_5$. For every $\mathcal{K} \in \mathbb{K}$, if $F, G \in \mathcal{K}$ then $\{\lambda_{fg} f + \mu_{fg} g : f \in F, g \in G\} \in \mathcal{K}$. Therefore $\{\lambda_{fg} f + \mu_{fg} g : f \in F, g \in G\} \in \bigcap \mathbb{K}$.

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We have just shown that $\overline{\mathbb{K}}$ is an intersection structure: if $\emptyset \neq \mathbb{K} \subseteq \overline{\mathbb{K}}$ then $\bigcap \mathbb{K} \in \overline{\mathbb{K}}$.

For any coherent $\mathcal{K}_\ell$, $\ell \in I$, their intersection $\bigcap_{\ell \in I} \mathcal{K}_\ell$ is coherent.

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Exercise

This implies that there is a smallest coherent $\mathcal{K}$, called vacuous $\mathcal{K}_{\mathrm{v}} := \bigcap \overline{\mathbb{K}}$. Show that $\mathcal{K}_{\mathrm{v}} = \{F \in \mathcal{Q} : F \cap \mathcal{L}_{>0} \neq \emptyset\}$.

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Solution

Call $\mathcal{K}^\star := \{F \in \mathcal{Q} : F \cap \mathcal{L}_{>0} \neq \emptyset\}$. We show (i) $\mathcal{K}^\star$ is coherent, and (ii) $\mathcal{K}^\star$ is the smallest element of $\overline{\mathbb{K}}$. The proof of (i) is a straightforward verification of all the axioms. For (ii), consider any $\mathcal{K} \in \overline{\mathbb{K}}$. Then $\{f\} \in \mathcal{K}$ for all $f > 0$ by Axiom $\mathcal{K}_3$, and therefore any superset of $\{f\}$ belongs to $\mathcal{K}$ too, taking into account Axiom $\mathcal{K}_4$. This implies that $\mathcal{K}^\star \subseteq \mathcal{K}$, as desired.

Natural extension

We have seen that the set of all coherent sets of desirable gamble sets $\overline{\mathbb{K}}$ is an intersection structure: if $\emptyset \neq \mathbb{K} \subseteq \overline{\mathbb{K}}$ then $\bigcap \mathbb{K} \in \overline{\mathbb{K}}$.

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There is an associated map $\mathrm{cl}_{\overline{\mathbb{K}}} \colon \mathcal{P}(\mathcal{Q}) \to \overline{\mathbb{K}} \cup \{\mathcal{Q}\}$, called natural extension, defined by

$$\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) := \bigcap \{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}, \qquad \text{where we let } \bigcap \emptyset = \mathcal{Q} \text{ by convention.}$$

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If $\mathcal{F} \not\subseteq \mathcal{K}$ for every $\mathcal{K} \in \overline{\mathbb{K}}$, then $\mathcal{F}$ is inconsistent: it is impossible to extend it coherently.

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Hint

Typically, you can prove the inconsistency of an assessment $\mathcal{F}$ by iteratively applying the productive axioms $\mathcal{K}_2$–$\mathcal{K}_5$ to infer the desirability of more and more gamble sets, until you find a contradiction with the destructive axiom $\mathcal{K}_1$.

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Example

The assessment $\mathcal{F} = \{\{f\}, \{-f\}\}$ is inconsistent: If there were a coherent superset $\mathcal{K}$, then by Axiom $\mathcal{K}_5$ $\{0\} = \{1f + 1(-f)\} \in \mathcal{K}$, so Axiom $\mathcal{K}_2$ would imply that $\{0\} \setminus \{0\} = \emptyset \in \mathcal{K}$, contradicting Axiom $\mathcal{K}_1$.

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If $\mathcal{F} \subseteq \mathcal{K}$ for some $\mathcal{K} \in \overline{\mathbb{K}}$, then $\mathcal{F}$ is consistent: it is possible to extend it coherently.

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Suppose I have an assessment or partially specified set of desirable gamble sets $\mathcal{F}$. This assessment $\mathcal{F}$ consists of gamble sets F that I consider desirable.

If $\mathcal{F} \not\subseteq \mathcal{K}$ for every $\mathcal{K} \in \overline{\mathbb{K}}$, then $\mathcal{F}$ is inconsistent: it is impossible to extend it coherently.

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I want to make it coherent, but I want to be as conservative as possible: I don't want to inject any additional information. So what I want is the least informative (smallest) coherent superset: $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$.

Natural extension

We have seen that the set of all coherent sets of desirable gamble sets $\overline{\mathbb{K}}$ is an intersection structure: if $\emptyset \neq \mathbb{K} \subseteq \overline{\mathbb{K}}$ then $\bigcap \mathbb{K} \in \overline{\mathbb{K}}$.

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If $\mathcal{F}$ is consistent, then $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$ is the smallest coherent superset of $\mathcal{F}$.

If $\mathcal{F}$ is inconsistent, then $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \mathcal{Q}$.

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Suppose I have an assessment or partially specified set of desirable gamble sets $\mathcal{F}$. This assessment $\mathcal{F}$ consists of gamble sets F that I consider desirable.

If $\mathcal{F} \nsubseteq \mathcal{K}$ for every $\mathcal{K} \in \overline{\mathbb{K}}$, then $\mathcal{F}$ is inconsistent: it is impossible to extend it coherently.

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If $\mathcal{F}$ is inconsistent, then $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \mathcal{Q}$.

I can do this for any theory of imprecise probabilities, as long as coherence forms an intersection structure.

Properties of natural extension

$\mathcal{F}$'s natural extension is defined by $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) := \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}$, where we let $\bigcap \emptyset = \mathcal{Q}$ by convention.

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Proposition

Consider any $\mathcal{F} \subseteq \mathcal{Q}$ and $\mathcal{K} \in \overline{\mathbb{K}}$. Then

1. *if $\mathcal{F}$ is coherent then $\mathcal{F}$ is consistent;*
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Proof.

For 1, if $\mathcal{F}$ is coherent then it is included in a coherent set of desirable gamble sets [namely itself], so it is consistent.

For 2, if $\mathcal{F}$ is coherent then $\mathcal{F}$ belongs to $\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}$, so $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} = \mathcal{F}$. Moreover, in 1 we have shown that then $\mathcal{F}$ is consistent. Conversely, if $\mathcal{F}$ is consistent then $\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} \neq \emptyset$, so $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}$ is a coherent set of desirable gamble sets that includes $\mathcal{F}$. If moreover $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \mathcal{F}$, then $\mathcal{F}$ is coherent.

For 3, if $\mathcal{F} \subseteq \mathcal{K}$ then $\mathcal{K} \in \{\mathcal{K}' \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}'\}$, so $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K}' \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}'\} \subseteq \mathcal{K}$. The converse implication follows from [extensivity] (see the next exercise). □

Properties of natural extension

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This argument does not depend on the content of the rationality axioms, but uses only the fact that they form an intersection structure! Therefore, it holds for any theory of imprecise probabilities where coherence forms an intersection structure.

Properties of natural extension

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Consider any $\mathcal{F} \subseteq \mathcal{Q}$ and $\mathcal{K} \in \overline{\mathbb{K}}$. Then

1. *if $\mathcal{F}$ is coherent then $\mathcal{F}$ is consistent;*
2. *$\mathcal{F}$ is coherent iff $\mathcal{F}$ is consistent and $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \mathcal{F}$;*
3. *$\mathcal{F} \subseteq \mathcal{K} \Leftrightarrow \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathcal{K}$.*

Exercise

The natural extension operator $\mathrm{cl}_{\overline{\mathbb{K}}}$ is a closure operator:

cl_1. $\mathcal{F} \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$; [extensivity]

cl_2. $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})) = \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$; [idempotency]

cl_3. if $\mathcal{F} \subseteq \mathcal{F}'$ then $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}')$. [monotonicity]

Hint: this also depend on the fact that coherence is an intersection structure, so it holds for any theory of imprecise probabilities where coherence forms an intersection structure.

Properties of natural extension: solution

Exercise

The natural extension operator $\mathrm{cl}_{\overline{\mathbb{K}}}$ is a closure operator:

cl_1. $\mathcal{F} \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$; [extensivity]

cl_2. $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})) = \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$; [idempotency]

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Solution

For cl_1, $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}$, so $\mathcal{F} \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$.

Properties of natural extension: solution

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The natural extension operator $\mathrm{cl}_{\overline{\mathbb{K}}}$ is a closure operator:

$\mathrm{cl}_1.$ $\mathcal{F} \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$; [extensivity]

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$\mathrm{cl}_3.$ if $\mathcal{F} \subseteq \mathcal{F}'$, then $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}')$. [monotonicity]

Solution

For cl_1, $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}$, so $\mathcal{F} \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$.

For cl_2, $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathcal{K}\} = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} = \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$, where we used that $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathcal{K} \Leftrightarrow \mathcal{F} \subseteq \mathcal{K}$ (established in Item 3 of the previous proposition) in the second equality.

Properties of natural extension: solution

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cl_1. $\mathcal{F} \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$; [extensivity]

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Solution

For cl_1, $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}$, so $\mathcal{F} \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$.

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For cl_3, if $\mathcal{F} \subseteq \mathcal{F}'$ then $\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} \supseteq \{\mathcal{K}' \in \overline{\mathbb{K}} \colon \mathcal{F}' \subseteq \mathcal{K}'\}$, so $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} \subseteq \bigcap\{\mathcal{K}' \in \overline{\mathbb{K}} \colon \mathcal{F}' \subseteq \mathcal{K}'\} = \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}')$.

BINARY
SETS OF DESIRABLE GAMBLE SETS

Binary part of a set of desirable gamble sets

If a set F belongs to a set of desirable gamble sets $\mathcal{K}$ then at least one gamble in F is desirable. Therefore, if $F = \{f\}$ is a singleton, then $\{f\} \in \mathcal{K}$ means that f is desirable.

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Definition

The binary part $\mathcal{D}_{\mathcal{K}}$ of a set of desirable gamble sets $\mathcal{K}$ is the set of desirable gambles

$$\mathcal{D}_{\mathcal{K}} := \{f \in \mathcal{L} \colon \{f\} \in \mathcal{K}\}.$$

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If the set of desirable gamble sets $\mathcal{K}$ is coherent, then so is the set of desirable gambles $\mathcal{D}_{\mathcal{K}}$.

Hint: Verify each of the axioms $\mathcal{D}_1$–$\mathcal{D}_4$ for $\mathcal{D}_{\mathcal{K}}$.

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Exercise

If the set of desirable gamble sets $\mathcal{K}$ is coherent, then so is the set of desirable gambles $\mathcal{D}_{\mathcal{K}}$.

Hint: Verify each of the axioms $\mathcal{D}_1$–$\mathcal{D}_4$ for $\mathcal{D}_{\mathcal{K}}$.

Solution

For Axiom $\mathcal{D}_1$, if towards contradiction $0 \in \mathcal{D}_{\mathcal{K}}$ then $\{0\} \in \mathcal{K}$. Then Axiom $\mathcal{K}_2$ would imply that $\emptyset = \{0\} \setminus \{0\} \in \mathcal{K}$, a contradiction with Axiom $\mathcal{K}_1$.

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Hint: Verify each of the axioms $\mathcal{D}_1$–$\mathcal{D}_4$ for $\mathcal{D}_{\mathcal{K}}$.

Solution

For Axiom $\mathcal{D}_2$, if $f > 0$ then $\{f\} \in \mathcal{K}$ by Axiom $\mathcal{K}_3$, so $f \in \mathcal{D}_{\mathcal{K}}$.

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Hint: Verify each of the axioms $\mathcal{D}_1$–$\mathcal{D}_4$ for $\mathcal{D}_{\mathcal{K}}$.

Solution

For Axiom $\mathcal{D}_3$, if $f \in \mathcal{D}_{\mathcal{K}}$ and $\lambda \in \mathbb{R}_{>0}$, then $\{f\} \in \mathcal{K}$, and hence $\{\lambda f\} \in \mathcal{K}$ by Axiom $\mathcal{K}_5$ [**Check this yourself!**], so $\lambda f \in \mathcal{D}_{\mathcal{K}}$.

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Exercise

If the set of desirable gamble sets $\mathcal{K}$ is coherent, then so is the set of desirable gambles $\mathcal{D}_{\mathcal{K}}$.

Hint: Verify each of the axioms $\mathcal{D}_1$–$\mathcal{D}_4$ for $\mathcal{D}_{\mathcal{K}}$.

Solution

For Axiom $\mathcal{D}_4$, if $f, g \in \mathcal{D}_{\mathcal{K}}$ then $\{f\}, \{g\} \in \mathcal{K}$, and hence $\{f+g\} = \{1f' + 1g' : f' \in \{f\}, g' \in \{g\}\} \in \mathcal{K}$ by Axiom $\mathcal{K}_5$, so $f + g \in \mathcal{D}_{\mathcal{K}}$.

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Why is $\mathcal{D}_{\mathcal{K}}$ called "the binary part of $\mathcal{K}$"?

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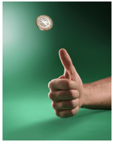
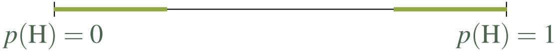
Why is $\mathcal{D}_{\mathcal{K}}$ called “the binary part of $\mathcal{K}$”?

Because it is the part of $\mathcal{K}$ that is determined by a binary preference ordering, namely the desirability ordering $\prec$ on gambles, defined by $f \prec g$ if $g - f \in \mathcal{D}_{\mathcal{K}}$. It is the binary relation (or set of desirable gambles) induced by $\mathcal{K}$.

Running example: coin with bias in unknown direction

$\mathcal{C} := \{p \text{ pmf}: p(\mathrm{H}) < \frac{1}{4} \text{ or } p(\mathrm{H}) > \frac{3}{4}\}$.





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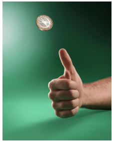

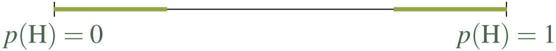

Let's consider the E-admissible choice function based on $\mathcal{C}$:

$$C_{\mathcal{C}}(F) = \{f \in F\colon \text{there is } p \in \mathcal{C} \text{ such that } f \in C_p(F)\} = \bigcup_{p \in \mathcal{C}} C_p(F) \quad \text{for all } F \in \mathcal{Q}.$$

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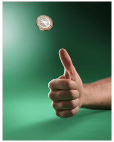

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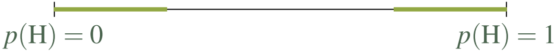

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Exercise

Find the set of desirable gamble sets $\mathcal{K}_{\mathcal{C}}$ corresponding to $C_{\mathcal{C}}$, and then find the binary part $\mathcal{D}_{\mathcal{K}_{\mathcal{C}}}$ of it.

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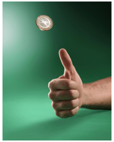
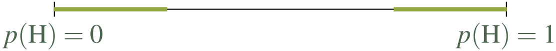


$\mathcal{C} := \{p \text{ pmf} \colon p(\mathrm{H}) < \frac{1}{4} \text{ or } p(\mathrm{H}) > \frac{3}{4}\}$.



Let's consider the E-admissible choice function based on $\mathcal{C}$:

$$C_{\mathcal{C}}(F) = \{f \in F \colon \text{there is } p \in \mathcal{C} \text{ such that } f \in C_p(F)\} = \bigcup_{p\in\mathcal{C}} C_p(F) \quad \text{for all } F \in \mathcal{Q}.$$

Exercise

Find the set of desirable gamble sets $\mathcal{K}_{\mathcal{C}}$ corresponding to $C_{\mathcal{C}}$, and then find the binary part $\mathcal{D}_{\mathcal{K}_{\mathcal{C}}}$ of it.

Solution

We have seen in a previous exercise that $\mathcal{K}_{\mathcal{C}} = \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\forall p \in \mathcal{C})(\exists f \in F) E_p(f) > 0\} = \bigcap_{p\in\mathcal{C}} \mathcal{K}_p$.
Consider any gamble f and infer that

$$f \in \mathcal{D}_{\mathcal{K}_{\mathcal{C}}} \Leftrightarrow \{f\} \in \mathcal{K}_{\mathcal{C}} \Leftrightarrow (\forall p \in \mathcal{C})\{f\} \in \mathcal{K}_p \Leftrightarrow \underbrace{(\forall p \in \mathcal{C}) E_p(f) > 0}_{\substack{\Rightarrow E_{p_{\mathrm{H}}}(f) = f(\mathrm{H}) > 0 \text{ and} \\ E_{p_{\mathrm{T}}}(f) = f(\mathrm{T}) > 0}} \text{ or } f \in \mathcal{L}_{>0} \Leftrightarrow f \in \mathcal{L}_{>0}.$$

The binary part of $\mathcal{K}_{\mathcal{C}}$ is vacuous!

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$\mathcal{K}_{\mathcal{D}} = \{F \in \mathcal{Q} : F \cap \mathcal{L}_{>0} \neq \emptyset\} = \{F \in \mathcal{Q} : (\exists f \in F) f > 0\}$, which is the vacuous set of desirable gamble sets.

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Solution

If $\{f, g\} \in \mathcal{K}_{\mathcal{D}}$ then $\{f, g\} \cap \mathcal{D} \neq \emptyset$, so $f \in \mathcal{D}$ or $g \in \mathcal{D}$, whence $\{f\} \in \mathcal{K}_{\mathcal{D}}$ or $\{g\} \in \mathcal{K}_{\mathcal{D}}$.

REPRESENTATION

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Working with sets of desirable gambles:

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Theorem (Representation theorem)

A set of desirable gamble sets $\mathcal{K} \subseteq \mathcal{Q}$ is coherent iff there is a non-empty collection $\mathbb{D}$ such that

$$\mathcal{K} = \bigcap_{\mathcal{D} \in \mathbb{D}} \mathcal{K}_{\mathcal{D}}.$$

In that case, the largest such representing set is given by

$$\mathbb{D}(\mathcal{K}) := \{\mathcal{D} \textit{ coherent} \colon \mathcal{K} \subseteq \mathcal{K}_{\mathcal{D}}\}.$$

Representation: exercise

Exercise

Consider the vacuous set of desirable gamble sets $\mathcal{K}_{\mathrm{v}} = \bigcap \overline{\mathbb{K}} = \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset\}$. Find a representation of $\mathcal{K}_{\mathrm{v}}$.

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Solution
It follows from $\mathcal{K}_{\mathrm{v}} = \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset\}$ that $\{\mathcal{D}_{\mathrm{v}}\} = \{\mathcal{L}_{>0}\}$ is a representation of $\mathcal{K}_{\mathrm{v}}$.

We can add supersets of elements of $\mathbb{D}$ without changing the intersection $\bigcap_{\mathcal{D} \in \mathbb{D}} \mathcal{K}_{\mathcal{D}}$, so moreover every superset of $\{\mathcal{D}_{\mathrm{v}}\}$ [every subset of $\overline{\mathbb{D}}$ that includes $\mathcal{D}_{\mathrm{v}}$] is a representation of $\mathcal{K}_{\mathrm{v}}$, too. In particular, $\overline{\mathbb{D}}$ is a representation of $\mathcal{K}_{\mathrm{v}}$.

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Theorem (Natural extension theorem using representation)

Consider any assessment $\mathcal{F} \subseteq \mathcal{Q}$. Then

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Exercise

Prove the Natural extension theorem using representation, using the representation of $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$.

Solution

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If $\mathcal{F}$ is inconsistent, then $\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} = \emptyset$ so in particular $\{\mathcal{K}_{\mathcal{D}} \colon \mathcal{D}$ coherent and $\mathcal{F} \subseteq \mathcal{K}_{\mathcal{D}}\} = \emptyset$. Then $\mathrm{cl}_{\mathbb{K}}(\mathcal{F}) = \mathcal{Q} = \bigcap\{\mathcal{K}_{\mathcal{D}} \colon \mathcal{D}$ coherent and $\mathcal{F} \subseteq \mathcal{K}_{\mathcal{D}}\}$ using the convention that $\bigcap\emptyset = \mathcal{Q}$, and we are done, so assume that $\mathcal{F}$ is consistent.

Use the representation theorem for $\mathrm{cl}_{\mathbb{K}}(\mathcal{F})$ and the earlier established fact that $\mathrm{cl}_{\mathbb{K}}(\mathcal{F}) \subseteq \mathcal{K}_{\mathcal{D}}$ iff $\mathcal{F} \subseteq \mathcal{K}_{\mathcal{D}}$ to infer that, indeed

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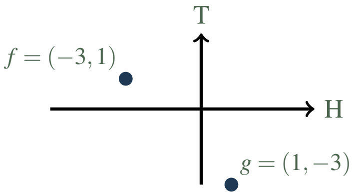
This result holds for any theory of imprecise probabilities with natural extension and representation. It does not depend on the exact formulation of the representation.

Running example: coin with bias in unknown direction

We found that the gamble set $\{f, g\} = \{(-3, 1), (1, -3)\}$ is desirable: $\{f, g\} \in \mathcal{K}$.



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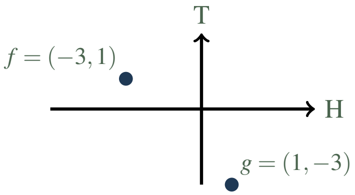

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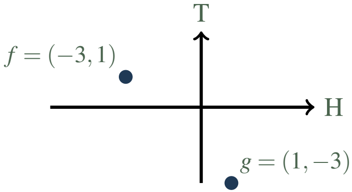

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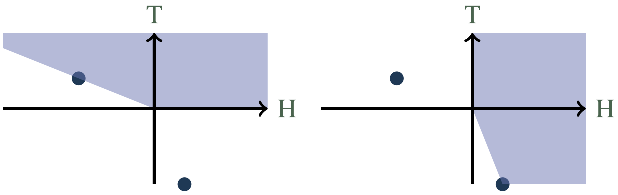

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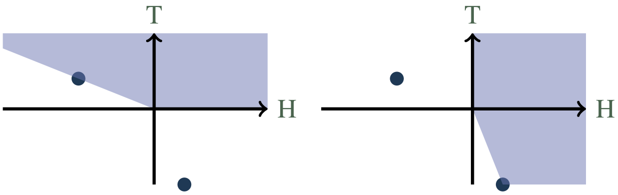

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Use the natural extension theorem using representation:

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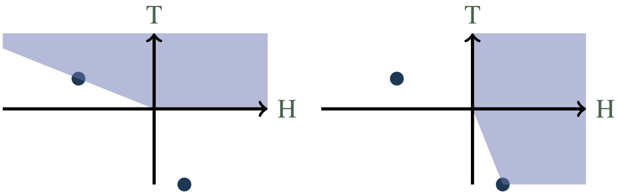

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So we find that $F \in \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$ iff $F \cap \mathrm{posi}(\{f\} \cup \mathcal{L}_{>0}) \neq \emptyset$ and $F \cap \mathrm{posi}(\{g\} \cup \mathcal{L}_{>0}) \neq \emptyset$.

SPECIAL CASES

E-admissibility

If I have a credal set $\mathcal{C}$, then the E-admissible set of desirable gamble sets based on $\mathcal{C}$ is

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E-admissibility $\Leftrightarrow$ represented by a set $\{\mathcal{D}_p \colon p \text{ pmf}\}$

Sen–Walley maximality

If I have a coherent set of desirable gambles $\mathcal{D}$, then the Sen–Walley maximal set of desirable gamble sets based on $\mathcal{D}$ is

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(Sen-Walley) maximality $\Leftrightarrow$ represented by a singleton $\{\mathcal{D}\}$

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Exercise

Give an example of a coherent set of desirable gambles that is both E-admissible and (Sen–Walley) maximal.

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Solution

Consider any pmf q. Then $\mathcal{K}_p = \mathcal{K}_{\mathcal{D}_q}$ is E-admissible [because it has a representation $\{\mathcal{D}_q\}$ that consists only of elements of the form $\mathcal{D}_p$] and (Sen-Walley) maximal [because it has a singleton representation $\{\mathcal{D}_q\}$.]

What are coherent sets of desirable gamble sets?

Recall the representation theorem:
A set of desirable gamble sets $\mathcal{K} \subseteq \mathcal{Q}$ is coherent iff there is a non-empty collection $\mathbb{D}$ such that

$$\mathcal{K} = \bigcap_{\mathcal{D} \in \mathbb{D}} \mathcal{K}_{\mathcal{D}}.$$

In that case, the largest such representing set is given by

$$\mathbb{D}(\mathcal{K}) := \{\mathcal{D} \text{ coherent} \colon \mathcal{K} \subseteq \mathcal{K}_{\mathcal{D}}\}.$$

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Coherent sets of desirable gamble sets have the E-admissibility structure,
but with potentially imprecise elements in $\mathbb{D}$.

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