Choice functions

Version of 28th July 2026

Contents

0.1 Goals

Choice functions, and their equivalent guise of sets of desirable gamble sets are very *expressive* and *general* models of imprecise probabilities.

After this 3 hour session, you should be able to:

– understand what *types of beliefs* choice functions can model;

– perform *conservative reasoning* with them;

– understand their connection with *sets of desirable gambles*;

– understand *representation* and how it can be used.

More importantly, I'd like to convince you that choice functions are *useful* and *promising* models of imprecise probabilities, with a clear *expressive advantage*. There are also still *many interesting open questions*.

0.2 History and future of choice functions in imprecise probabilities

Teddy Seidenfeld

First to consider imprecise-probabilistic choice functions.
Established representation in terms of arbitrary set of probability / utility pairs.

– *A Rubinesque theory of decision*, A Festschrift for Herman Rubin Institute of Mathematical Statistics Lecture Notes – Monograph Series, **2004**, doi:10.1214/lnms/1196285378

– *Coherent Choice Functions under Uncertainty*, Synthese, **2010**, doi:10.1007/s11229-009-9470-7

Gert de Cooman

First to consider choice functions on *gambles*.
Introduced *sets of desirable gamble sets* with **Jasper De Bock**.
Studied choice in general *Banach spaces*.
Studied *connection with logic* with **Arthur Van Camp** and **Jasper De Bock**.

- *Interpreting, Axiomatising and Representing Coherent Choice Functions in Terms of Desirability*, ISIPTA **2019**
- *Coherent and Archimedean choice in general Banach spaces*, International Journal of Approximate Reasoning, **2022**, doi:10.1016/j.ijar.2021.09.005
- *The logic behind desirable sets of things, and its filter representation*, International Journal of Approximate Reasoning, **2024**, doi:10.1016/j.ijar.2024.109241

Arthur Van Camp

PhD dissertation about imprecise-probabilistic choice functions, under the guidance of **Gert de Cooman** and **Enrique Miranda**.
Indifference with **Gert de Cooman**, **Enrique Miranda** and **Erik Quaeghebeur**.




Enrique Miranda ***Erik Quaeghebeur*** ***Jason Konek*** ***Kevin Blackwell***

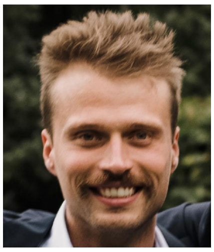
Irrelevance, *natural extension* and *marginal extension* with **Enrique Miranda** and **Gert de Cooman**.
Epistemic independence with **Jason Konek** and **Kevin Blackwell**.

- *Choice functions as a tool to model uncertainty*, PhD dissertation, Ghent University **2018**
- *Coherent choice functions, desirability and indifference*, Fuzzy Sets and Systems, **2018**, doi:10.1016/j.fss.2017.05.019
- *Modelling epistemic irrelevance with choice functions*, International Journal of Approximate Reasoning, **2020**, doi:10.1016/j.ijar.2020.06.010
- *Independent natural extension for choice functions*, International Journal of Approximate Reasoning, **2023**, doi:10.1016/j.ijar.2022.11.003

Jasper De Bock

Important generalisation to *things* with arbitrary **closure operators**.
Introduced *sets of desirable gamble sets* with useful axiomatisation, with **Gert de Cooman**.
First to study *algorithms*, with **Arne Decadt** and **Alexander Erreygers**.

- *Interpreting, Axiomatising and Representing Coherent Choice Functions in Terms of Desirability*, ISIPTA **2019**
- *Archimedean Choice Functions: an Axiomatic Foundation for Imprecise Decision Making*, IPMU **2020**, doi:10.1007/978-3-030-50143-3_15
- *A theory of desirable things*, ISIPTA **2023**

Arne Decadt ***Alexander Erreygers***

Study *algorithms* with choice functions.

- *Extending choice assessments to choice functions: An algorithm for computing the natural extension*, International Journal of Approximate Reasoning, **2025**, doi:10.1016/j.ijar.2024.109331
- *Conservative decision-making with sets of probabilities: How to infer new choices from previous ones*, Fuzzy Sets and Systems, **2025**, doi:10.1016/j.fss.2025.109612

Catrin Campbell-Moore

Studies *probability filters* as equivalent model to choice functions.

- *Probability filters and desirable gambles*, International Journal of Approximate Reasoning, **2026**, doi:10.1016/j.ijar.2026.109643

Kevin Blackwell

Studies connection between sets of *desirable gamble sets* and *choice functions* in a more general context.

– *Coherent rejection functions for arbitrary things*, ISIPTA **2025**

Justyna Dąbrowska

Studies *marginal problem* and *information algebras* with **Arthur Van Camp** and **Erik Quaeghebeur**.
Studies different types of *independence* with **Banda Norimarna** and **Arthur Van Camp**.

Banda Norimarna

– *The marginal problem for sets of desirable gamble sets*, ISIPTA **2025**

– *A Preliminary Study of Notions of Independence for Choice Functions*, IPMU **2026**, doi:10.1007/978-3-032-28994-0_17

1 Desirability

1.1 Desirability: review

We focus on *desirability* rather than *acceptability*
[see Erik Quaeghebeur's lecture and axiom variants].

The key difference is that

$$0 \notin \mathcal{D}.$$

for sets of *desirable gambles* $\mathcal{D}$.

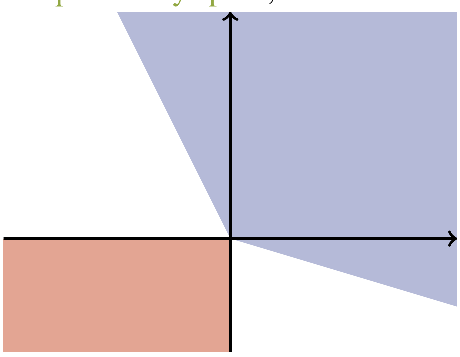
A set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{X})$, with $\mathcal{X}$ a finite possibility space, is *coherent* when

$\mathcal{D}_1$. $0 \notin \mathcal{D}$;

$\mathcal{D}_2$. if $f > 0$ then $f \in \mathcal{D}$;

$\mathcal{D}_3$. if $f, g \in \mathcal{D}$ then $f + g \in \mathcal{D}$;

$\mathcal{D}_4$. if $f \in \mathcal{D}$ and $\lambda \in \mathbb{R}_{>0}$, then $\lambda f \in \mathcal{D}$.



Axioms $\mathcal{D}_3$ and $\mathcal{D}_4$ make coherent $\mathcal{D}$ a convex cone.
Axiom $\mathcal{D}_2$ ensures that $\mathcal{D}$ contains the positive orthant $\mathcal{L}_{>0}$.
Axiom $\mathcal{D}_1$ guarantees that 0 is not a desirable gamble.

Exercise
Consequence of coherence: $\mathcal{D}$ has nothing in common with the negative orthant $\mathcal{L}_{\leq 0}$.

Solution
If, towards contradiction, $f \in \mathcal{D}$ for some $f \in \mathcal{L}_{<0}$, then $-f \in \mathcal{L}_{>0}$, so $-f \in \mathcal{D}$ by Axiom $\mathcal{D}_2$. By Axiom $\mathcal{D}_3$, $f+(-f)=0 \in \mathcal{D}$, contradicting Axiom $\mathcal{D}_1$.

A coherent set of desirable gambles $\mathcal{D}$ is equivalent to a partial preference order $\prec$:

$$f \prec g \Leftrightarrow g-f \in \mathcal{D} \quad \text{and} \quad \mathcal{D} = \{f \colon 0 \prec f\}.$$

satisfying

O_1. $f \not\prec f$;

O_2. if $f < g$ then $f \prec g$;

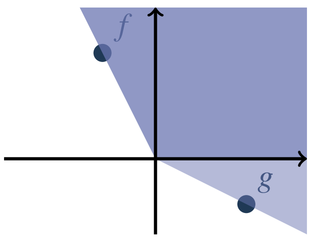
O_3. if $f \prec g$ and $g \prec h$, then $f \prec h$;

O_4. $f \prec g$ iff $\lambda f + h \prec \lambda g + h$, for all $\lambda \in \mathbb{R}_{>0}$.

$\mathcal{D}$ and $\prec$ are equivalent representations of the same information.

Example
1. If p is a *pmf*, then $\prec$ defined by $f \prec g \Leftrightarrow E_p(f) < E_p(g)$ or $f < g$ is a partial preference order.
2. If $\mathcal{C}$ is a *credal set*, then $\prec$ defined by $f \prec g \Leftrightarrow (\forall p \in \mathcal{C}) E_p(f) < E_p(g)$ or $f < g$ is a partial preference order.

1.2 Desirability: natural extension



You can make assessments about desirability of gambles.

If "*f is desirable*", then

$$\bigcap\{\mathcal{D} \text{ coherent} \colon f \in \mathcal{D}\} = \text{posi}(\mathcal{L}_{>0} \cup \{f\})$$

is the *smallest coherent set of desirable gambles that includes* f (the *natural extension of* $\{f\}$), if $f \in \mathcal{D}$ for some coherent $\mathcal{D}$.
If "f *is desirable* and g *is desirable*", then

$$\bigcap\{\mathcal{D} \text{ coherent}\colon \{f,g\} \subseteq \mathcal{D}\} = \operatorname{posi}(\mathcal{L}_{>0} \cup \{f,g\})$$

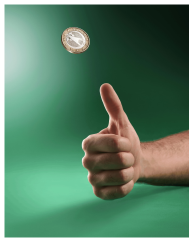
is the *smallest coherent set of desirable gambles that includes* f *and* g (the *natural extension of* $\{f,g\}$), if $f,g \in \mathcal{D}$ for some coherent $\mathcal{D}$.

What if the assessment is "f *is desirable or* g *is desirable*"?

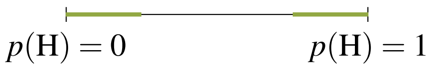
2 Sets of desirable gamble sets

2.1 Running example: coin with bias in unknown direction

A friend flips a coin. The outcome is either heads (H) or tails (T), so $\mathcal{X} = \{\mathrm{H},\mathrm{T}\}$.

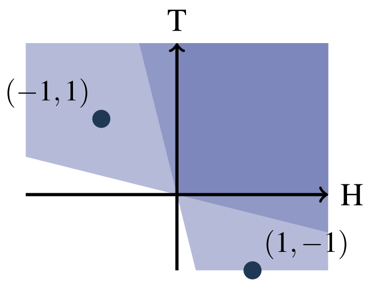


I belief that the coin is *biased but I don't know in which direction*: my credal set is $\mathcal{C} := \{p \text{ pmf}\colon p(\mathrm{H}) < \frac{1}{4} \text{ or } p(\mathrm{H}) > \frac{3}{4}\}$.



How can we describe this belief in terms of *desirable gambles*?
If we use *sets of desirable gambles*, we may be inclined to use

$$\mathcal{D} = \{f \in \mathcal{L}\colon (\forall p \in \mathcal{C}) E_p(f) > 0\} \cup \mathcal{L}_{>0} = \mathcal{L}_{>0}.$$



Problem: with *sets of desirable gambles* we cannot distinguish with the *vacuous model*, $\mathcal{D}_{\mathrm{v}} = \mathcal{L}_{>0}$.

Solution: use gamble sets. For each pmf $p \in \mathcal{C}$, there is a gamble in $\{(-1,1),(1,-1)\}$ with *positive* p*-expectation*.

2.2 (Sets of) desirable gamble sets

The assessment is "f *is desirable or* g *is desirable*".

Then we call the gamble set $\{f,g\}$ a desirable gamble set, because f is desirable or g is desirable.
We collect all desirable gamble sets in the set of desirable gamble sets $\mathcal{K}$.

Definition 1 (Gamble sets)**.** A gamble set F is a *finite subset* of $\mathcal{L}$:

$$F \Subset \mathcal{L}.$$

We collect all gamble sets in $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{L})$.

Definition 2 (Set of desirable gamble sets)**.** A set of desirable gamble sets $\mathcal{K}$ collects all desirable gamble sets, so it is a *set of gamble sets*:

$$\mathcal{K} \subseteq \mathcal{Q}.$$

2.3 Sets of desirable gamble sets: interpretation

The intended *interpretation* of $\mathcal{K}$ is that a gamble set $F = \{f_1, \dots, f_n\}$ belongs to $\mathcal{K}$

if f_1 *is desirable* or f_2 *if desirable* or ... or f_n *is desirable*,

equivalently, if *at least one gamble in* F *is desirable*.

Hint
Be careful:

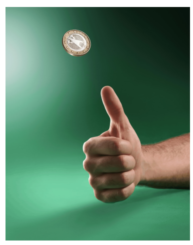
- $\{f, g\} \in \mathcal{K}$ means f is desirable or g is desirable;
- $\{f\}, \{g\} \in \mathcal{K}$ means f is desirable and g is desirable.

Exercise
If (f_1 is des. and g is des.) or (f_2 is des. and g is des.), what gamble sets belong to $\mathcal{K}$?

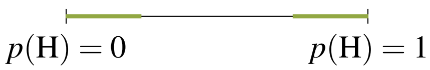
Solution
(f_1 is des. and g is des.) or (f_2 is des. and g is des.) is equivalent to
(f_1 is des. or f_2 is des.) and (f_1 is des. or g is des.) and (g is des. or f_2 is des.) and (g is des.), which is equivalent to
(f_1 is des. or f_2 is des.) and (g is des.). This translates to $\{f_1, f_2\}, \{g\} \in \mathcal{K}$.

2.4 Running example: coin with bias in unknown direction

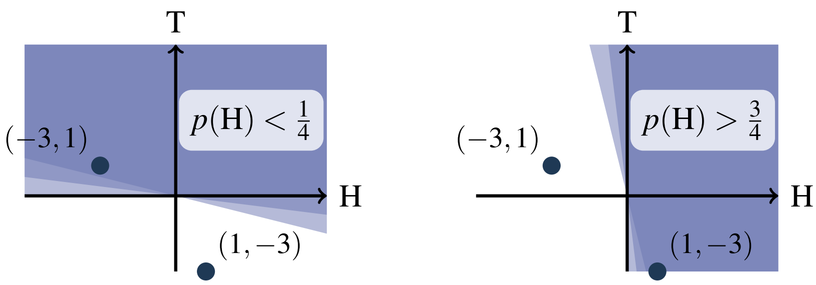
A friend flips a coin. The outcome is either heads (H) or tails (T), so $\mathcal{X} = \{\mathrm{H}, \mathrm{T}\}$.



I belief that the coin is *biased but I don't know in which direction*: my credal set is $\mathcal{C} := \{p \text{ pmf}\colon p(\mathrm{H}) < \frac{1}{4} \text{ or } p(\mathrm{H}) > \frac{3}{4}\}$.



This means that the gamble set $\{f, g\} = \{(-3, 1), (1, -3)\}$ is desirable: $\{f, g\} \in \mathcal{K}$.



2.5 Sets of desirable gamble sets: coherence

Consider a *finite* possibility space $\mathcal{X}$. A set of desirable gamble sets $\mathcal{K} \subseteq \mathcal{Q}$ is coherent when for all $F, G \in \mathcal{Q}$ and $f \in \mathcal{L}$:

$\mathcal{K}_1$. $\emptyset \notin \mathcal{K}$; [destructive axiom]

$\mathcal{K}_2$. if $F \in \mathcal{K}$ then $F \setminus \{0\} \in \mathcal{K}$; [productive axiom]

$\mathcal{K}_3$. if $f > 0$ then $\{f\} \in \mathcal{K}$; [productive axiom]

$\mathcal{K}_4$. if $F \in \mathcal{K}$ and $F \subseteq G$, then $G \in \mathcal{K}$; [productive axiom]

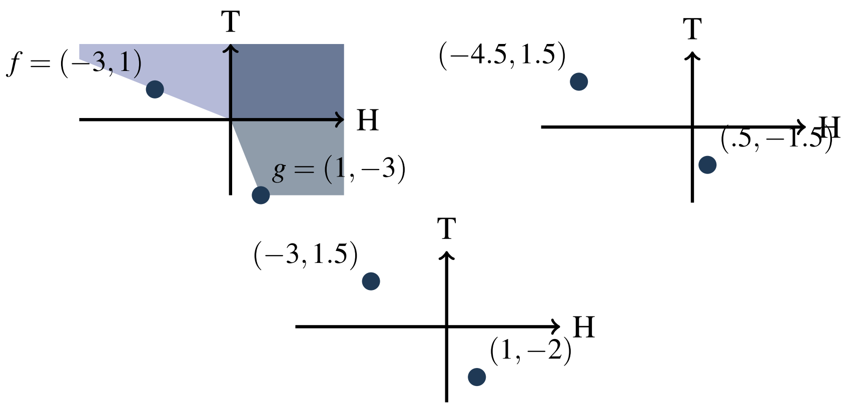
$\mathcal{K}_5$. if $F, G \in \mathcal{K}$ and $(\lambda_{fg}, \mu_{fg}) > 0$ for all $f \in F$ and $g \in G$,[1] then

$$\{\lambda_{fg} f + \mu_{fg} g \colon f \in F, g \in G\} \in \mathcal{K}.$$ [productive axiom]

Justification for Axiom $\mathcal{K}_1$: $\emptyset$ cannot contain a desirable gamble.
Justification for Axiom $\mathcal{K}_2$: If F is a desirable gamble set, then some gamble $f \in F$ [*I don't assume that you can identify* f!] must be a desirable gamble. But then $f \neq 0$ by Axiom $\mathcal{D}_1$, so the set $F \setminus \{0\}$ contains a desirable gamble, and is hence a desirable gamble set.
Justification for Axiom $\mathcal{K}_3$: Such f is desirable by Axiom $\mathcal{D}_2$.
Justification for Axiom $\mathcal{K}_4$: If F is a desirable gamble set, it must contain some desirable gamble, and therefore so must G.
Justification for Axiom $\mathcal{K}_5$: If F and G are desirable gamble sets, then they contain some desirable gambles $f^\star \in F$ and $g^\star \in G$. Then the *composite gamble* $\lambda_{f^\star g^\star} f^\star + \mu_{f^\star g^\star} g^\star \in \{\lambda_{fg} f + \mu_{fg} g \colon f \in F, g \in G\}$, is desirable too, taking into account Axioms $\mathcal{D}_3$ and $\mathcal{D}_4$.

2.6 Running example: coin with bias in unknown direction

We found that the gamble set $\{f, g\} = \{(-3,1), (1,-3)\}$ is desirable: $\{f, g\} \in \mathcal{K}$.



[1] We use for vectors $(\alpha_1, \dots, \alpha_n)$ the same convention as for gambles: $(\alpha_1, \dots, \alpha_n) > 0$ iff '$\alpha_k \geq 0$ for all k and $\alpha_\ell > 0$ for some ℓ'.

Exercise
Assume that there is a *coherent set of desirable gamble sets* $\mathcal{K}$ such that $\{f,g\} \in \mathcal{K}$. Show that
1) $\{\lambda f, \mu g\} \in \mathcal{K}$ for every $\lambda, \mu > 0$;

2) $\{f', g'\} \in \mathcal{K}$ for every $f' \geq f$ and $g' \geq g$;

3) $F \in \mathcal{K}$ for every finite $F \supseteq \{\lambda f', \mu g'\}$ where $\lambda, \mu > 0$ and $f' \geq f$ and $g' \geq g$.

2.7 Solution

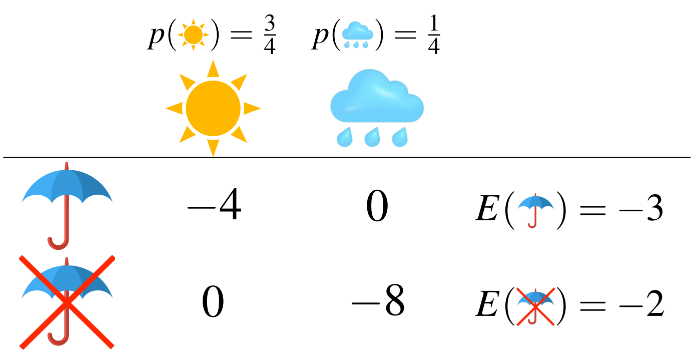
Solution
Exercise 1)*:* Use Axiom $\mathcal{K}_5$ with $F = G = \{f,g\} \in \mathcal{K}$ and, for instance, $(\lambda_{ff}, \mu_{ff}) = (\lambda, 0)$, $(\lambda_{fg}, \mu_{fg}) = (0, \mu)$, $(\lambda_{gf}, \mu_{gf}) = (0, \lambda)$ and $(\lambda_{gg}, \mu_{gg}) = (\mu, 0)$. Then Axiom $\mathcal{K}_5$ tells us that $\{\lambda_{ff} f + \mu_{ff} f, \lambda_{fg} f + \mu_{fg} g, \lambda_{gf} g + \mu_{gf} f, \lambda_{gg} g + \mu_{gg} g\} = \{\lambda f, \mu g\} \in \mathcal{K}$.

Exercise 2)*:* We first show that $\{f', g\} \in \mathcal{K}$. If $f' = f$ then we are done, so assume $f' > f$. Then $h := f' - f > 0$ so Axiom $\mathcal{K}_3$ tells us that $\{h\} \in \mathcal{K}$. Use Axiom $\mathcal{K}_5$ with $F = \{f,g\}$, $G = \{h\}$ and $(\lambda_{fh}, \mu_{fh}) = (1,1)$ and $(\lambda_{gh}, \mu_{gh}) = (1,0)$ to infer that $\{\lambda_{fh} f + \mu_{fh} h, \lambda_{gh} g + \mu_{gh} h\} = \{f + h, g + 0\} = \{f', g\} \in \mathcal{K}$. Similarly, we infer that $\{f', g'\} \in \mathcal{K}$.

Exercise 3)*:* Combine the two results above to infer that $\{\lambda f', \mu g'\} \in \mathcal{K}$ for every $\lambda, \mu > 0$ and $f' \geq f$ and $g' \geq g$. Then use Axiom $\mathcal{K}_4$ to infer that $F \in \mathcal{K}$ for every finite $F \supseteq \{\lambda f', \mu g'\}$ where $\lambda, \mu > 0$ and $f' \geq f$ and $g' \geq g$.

3 Intermezzo: Choice functions

3.1 Choice functions



Gambles represent decisions under uncertainty.
What if we don't have a pmf p?
We focus on the function that describes making decisions: the *choice function*.

Definition 3 (Choice function & rejection function)**.** A choice function C is a map

$C\colon \mathcal{Q} \to \mathcal{Q}\colon F \mapsto C(F) \subseteq F$ $\quad$ $C(F)$ collects the non-rejected or admissible gambles from the menu F.

A rejection function R is a map

$R\colon \mathcal{Q} \to \mathcal{Q}\colon F \mapsto R(F) \subseteq F$ $\quad$ $R(F)$ collects the rejected or inadmissible gambles from the menu F.

The choice function C and rejection function R are linked through

$$C(F) = F \setminus R(F) \qquad \text{and} \qquad R(F) = F \setminus C(F) \qquad \text{for all } F \in \mathcal{Q}.$$

Example 1: *expected utility maximisation*
If I have a pmf p, then $R_p(F) = \{f \in F\colon \text{ there is } g \in F \text{ such that } E_p(f) < E_p(g) \text{ or } f < g\}$ and $C_p(F) = F \setminus R_p(F)$.

Example 2: *Sen–Walley maximality*
If I have a *coherent set of desirable gambles* $\mathcal{D}$, or a partial preference order $\prec$, then I can use it to make decisions:

$$R(F) = \{f \in F\colon \text{there is } g \in F \text{ such that } f \prec g\} = \{f \in F\colon \text{there is } g \in F \text{ such that } g - f \in \mathcal{D}\} \qquad \text{(Sen–Walley maximality)}$$

and then $f \prec g \Leftrightarrow f \in R(\{f,g\})$.

Example 3: *E-admissibility*
If I have a *credal set* $\mathcal{C}$, then I can use *E-admissibility* to make decisions:

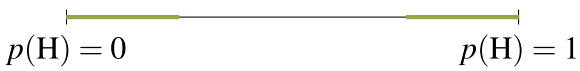
$$C_{\mathcal{C}}(F) \coloneqq \{f \in F\colon \text{there is } p \in \mathcal{C} \text{ such that } f \in C_p(F)\} = \bigcup_{p \in \mathcal{C}} C_p(F) \qquad \text{for all } F \in \mathcal{Q}.$$

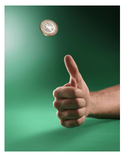
Unlike the first two examples, E-admissibility is an instance of non-binary choice: it may happen that

$$f \in C_{\mathcal{C}}(\{f,g\}), \quad f \in C_{\mathcal{C}}(\{f,h\}), \quad \text{even though} \quad f \in R_{\mathcal{C}}(\{f,g,h\}).$$

3.2 Running example: coin with bias in unknown direction

$\mathcal{C} \coloneqq \{p \text{ pmf}\colon\ p(\mathrm{H}) < \frac{1}{4} \text{ or } p(\mathrm{H}) > \frac{3}{4}\}$.



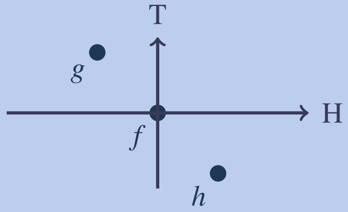


Let's consider the *E-admissible choice function* based on $\mathcal{C}$:

$$C_{\mathcal{C}}(F) = \{f \in F\colon \text{there is } p \in \mathcal{C} \text{ such that } f \in C_p(F)\} = \bigcup C_p(F) \quad \text{for all } F \in \mathcal{Q}.$$

Exercise
Show that $C_{\mathcal{C}}$ is *non-binary*: find gambles f, g and h such that $f \in C_{\mathcal{C}}(\{f,g\})$, $f \in C_{\mathcal{C}}(\{f,h\})$ but $f \notin C_{\mathcal{C}}(\{f,g,h\})$.

Solution
Consider for instance $f = 0$, $g = (-1, 1)$ and $h = (1, -1)$.
Then $f \in C_{\mathcal{C}}(\{f, g\})$ because $f \in C_p(\{f, g\})$ for $p \in \mathcal{C}$ such that $p(\mathrm{H}) > \frac{3}{4}$.
Similarly, $f \in C_{\mathcal{C}}(\{f, h\})$, because $f \in C_p(\{f, h\})$ for $p \in \mathcal{C}$ such that $p(\mathrm{H}) < \frac{1}{4}$.
But $f \in R_{\mathcal{C}}(\{f, g, h\})$ because for every $p \in \mathcal{C}$, either $E_p(f) < E_p(g)$ [when $p(\mathrm{H}) < \frac{1}{4}$] or $E_p(f) < E_p(h)$ [when $p(\mathrm{H}) > \frac{3}{4}$].



3.3 Choice functions: coherence

A choice function $C\colon \mathcal{Q} \to \mathcal{Q}$, with $\mathcal{X}$ a finite possibility space, is *coherent* when, for all $F, G \in \mathcal{Q}$, $f, g \in \mathcal{L}$ and $\lambda \in \mathbb{R}_{>0}$:

C_1. $C(F) \neq \emptyset$; [non-empty choice]

C_2. if $F \subseteq G$ then $R(F) \subseteq R(G)$; [Sen α], [independence of irrelevant alternatives]

C_3. if $f \in R(F)$ then $f + g \in R(F + g)$;[2] [translation invariance]

C_4. if $0 \in R(\{f\} \cup F)$ and $0 \in R(\{g\} \cup F)$, then $0 \in R(\{f + g\} \cup F)$; [additivity]

C_5. if $0 \in R(\{f\} \cup F)$ then $0 \in R(\{\lambda f\} \cup F)$. [scaling invariance]

The following *outcast property* (also called *Aizerman's property*) is a consequence:

C_6. if $f, g \in R(F)$ and $f \neq g$, then $f \in R(F \setminus \{g\})$. [Aizerman's property], [outcast property]

Example
Expected utility maximisation, *Sen–Walley maximality* and *E-admissibility* are coherent choice functions.

3.4 Connection between choice functions and sets of desirable gamble sets

Given a translation invariant rejection function R, if $f \in R(F)$ then $0 \in R(F - f)$. Then $F - f$ is a gamble set from which 0 is rejected. Informally, this must be because some gamble in $F - f$ is desirable, so $F - f$ is a *desirable gamble set*.

We let $\mathcal{K}_R := \{F \in \mathcal{Q} \colon 0 \in R(\{0\} \cup F)\}$ be the corresponding set of desirable gamble sets.

Proposition 1. *If R is a coherent rejection function, then $\mathcal{K}_R$ is a coherent set of desirable gamble sets.*

[2] We let $F + g := \{f + g \colon f \in F\}$.

Conversely, given a set of desirable gamble sets $\mathcal{K}$, we define a corresponding rejection function $R_{\mathcal{K}}$ by letting $f \in R_{\mathcal{K}}(F)$ if $F - f \in \mathcal{K}$, for all $F \in \mathcal{Q}$ and $f \in F$.

Proposition 2. *1. If $\mathcal{K}$ is a coherent set of desirable gamble sets, then $R_{\mathcal{K}}$ is a coherent rejection function.*
2. The mappings $\mathcal{K}_R$ and $R_{\mathcal{K}}$ commute.

Conclusion: $\mathcal{K}$ is coherent iff corresponding R is coherent.

$\mathcal{K}$ and R are equivalent representations of the same information.

3.5 Examples of coherent sets of desirable gamble sets

Example 1: ***expected utility maximisation***
If I have a pmf p, then $\mathcal{K}_p := \{F \in \mathcal{Q}\colon$ there is $f \in F$ such that $E_p(f) > 0$ or $f > 0\}$ is coherent.

Example 2: ***Sen–Walley maximality***
If I have a *coherent set of desirable gambles* $\mathcal{D}$, or a partial preference relation $\prec$, then

$$\mathcal{K} = \{F \in \mathcal{Q}\colon \text{there is } f \in F \text{ such that } f \in \mathcal{D}\} = \{F \in \mathcal{Q}\colon \text{there is } f \in F \text{ such that } 0 \prec f\}$$

is a coherent set of desirable gambles, for which $\{f\} \in \mathcal{K} \Leftrightarrow f \in \mathcal{D}$.

Example 3: ***E-admissibility***
If I have a *credal set* $\mathcal{C}$, then

$$\mathcal{K}_{\mathcal{C}} := \{F \in \mathcal{Q}\colon F \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\forall p \in \mathcal{C})(\exists f \in F) E_p(f) > 0\} = \bigcap_{p \in \mathcal{C}} \mathcal{K}_p$$

is a coherent set of desirable gamble sets.

Exercise
Derive $\mathcal{K}_{\mathcal{C}}$ from $C_{\mathcal{C}}$ and the connection between *choice functions* and *sets of desirable gamble sets*.

3.6 Solution

Solution
Use the E-admissible rejection function $R_{\mathcal{C}}(F) = F \setminus C_{\mathcal{C}}(F)$:

$$f \in R_{\mathcal{C}}(F) \Leftrightarrow f \notin C_{\mathcal{C}}(F) \Leftrightarrow (\forall p \in \mathcal{C}) f \notin C_p(F) \Leftrightarrow (\forall p \in \mathcal{C})(\exists g \in F) E_p(f) < E_p(g) \text{ or } f < g$$

for all $F \in \mathcal{Q}$ and $f \in F$. Infer that

$$\begin{aligned} 0 \in R_{\mathcal{C}}(F) &\Leftrightarrow (\forall p \in \mathcal{C})(\exists g \in F) E_p(g) > 0 \text{ or } g > 0 \\ &\Leftrightarrow (\forall p \in \mathcal{C})(\exists g \in F) E_p(g) > 0 \text{ or } F \cap \mathcal{L}_{>0} \neq \emptyset \end{aligned}$$

for all $F \in \mathcal{Q}$ such that $0 \in F$. Now use the connection between *rejection functions* and *sets of desirable gamble sets* to infer that

$$\mathcal{K}_{\mathcal{C}} = \{F \in \mathcal{Q} \colon 0 \in R_{\mathcal{C}}(F \cup \{0\})\} = \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\forall p \in \mathcal{C})(\exists f \in F) E_p(f) > 0\}.$$

4 Inference with sets of desirable gamble sets

4.1 Sets of desirable gamble sets: intersection structure

Recall the rationality axioms:

$\mathcal{K}_1$. $\emptyset \notin \mathcal{K}$ [destructive axiom]

$\mathcal{K}_2$. if $F \in \mathcal{K}$ then $F \setminus \{0\} \in \mathcal{K}$ [productive axiom]

$\mathcal{K}_3$. if $f > 0$ then $\{f\} \in \mathcal{K}$ [productive axiom]

$\mathcal{K}_4$. if $F \in \mathcal{K}$ and $F \subseteq G$, then $G \in \mathcal{K}$ [productive axiom]

$\mathcal{K}_5$. if $F, G \in \mathcal{K}$ and $(\lambda_{fg}, \mu_{fg}) > 0$ for all $f \in F, g \in G$, then $\{\lambda_{fg} f + \mu_{fg} g \colon f \in F, g \in G\} \in \mathcal{K}$ [productive axiom]

Exercise
Each of the Axioms $\mathcal{K}_1$–$\mathcal{K}_5$ is *closed under arbitrary non-empty intersections*: if $\mathbb{K}$ is a non-empty set of sets of desirable gamble sets where each $\mathcal{K} \in \mathbb{K}$ satisfies Axiom $\mathcal{K}_\ell$, then $\bigcap \mathbb{K}$ is a set of desirable gamble sets that satisfies Axiom $\mathcal{K}_\ell$.

Solution
Axiom $\mathcal{K}_1$. If $\emptyset \notin \mathcal{K}$ for every $\mathcal{K} \in \mathbb{K}$, then $\emptyset \notin \bigcap \mathbb{K}$. ***Axiom*** $\mathcal{K}_2$. For every $\mathcal{K} \in \mathbb{K}$, if $F \in \mathcal{K}$ then $F \setminus \{0\} \in \mathcal{K}$. Therefore $F \setminus \{0\} \in \bigcap \mathbb{K}$. ***Axiom*** $\mathcal{K}_3$. If $f > 0$, then $\{f\} \in \mathcal{K}$ for every $\mathcal{K} \in \mathbb{K}$. Therefore $\{f\} \in \bigcap \mathbb{K}$. ***Axiom*** $\mathcal{K}_4$. For every $\mathcal{K} \in \mathbb{K}$, if $F \in \mathcal{K}$ then $G \in \mathcal{K}$. Therefore $G \in \bigcap \mathbb{K}$. ***Axiom*** $\mathcal{K}_5$. For every $\mathcal{K} \in \mathbb{K}$, if $F, G \in \mathcal{K}$ then $\{\lambda_{fg} f + \mu_{fg} g \colon f \in F, g \in G\} \in \mathcal{K}$. Therefore $\{\lambda_{fg} f + \mu_{fg} g \colon f \in F, g \in G\} \in \bigcap \mathbb{K}$.

We collect *all coherent* $\mathcal{K}$ in the set $\overline{\mathbb{K}}$. Call $\mathcal{K}_1$ more informative than $\mathcal{K}_2$ if $\mathcal{K}_1 \supseteq \mathcal{K}_2$.

We have just shown that $\overline{\mathbb{K}}$ is an intersection structure: if $\emptyset \neq \mathbb{K} \subseteq \overline{\mathbb{K}}$ then $\bigcap \mathbb{K} \in \overline{\mathbb{K}}$.

For any coherent $\mathcal{K}_\ell$, $\ell \in I$, their intersection $\bigcap_{\ell \in I} \mathcal{K}_\ell$ is coherent.

Exercise
This implies that there is a smallest coherent $\mathcal{K}$, called *vacuous* $\mathcal{K}_\mathrm{v} := \bigcap \overline{\mathbb{K}}$. Show that $\mathcal{K}_\mathrm{v} = \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset\}$.

Solution
Call $\mathcal{K}^\star := \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset\}$. We show (i) $\mathcal{K}^\star$ is coherent, and (ii) $\mathcal{K}^\star$ is the *smallest* element of $\overline{\mathbb{K}}$. The proof of (i) is a straightforward verification of all the axioms. For (ii), consider any $\mathcal{K} \in \overline{\mathbb{K}}$. Then $\{f\} \in \mathcal{K}$ for all $f > 0$ by Axiom $\mathcal{K}_3$, and therefore any superset of $\{f\}$ belongs to $\mathcal{K}$ too, taking into account Axiom $\mathcal{K}_4$. This implies that $\mathcal{K}^\star \subseteq \mathcal{K}$, as desired.

4.2 Natural extension

We have seen that the set of all coherent sets of desirable gamble sets $\overline{\mathbb{K}}$ is an *intersection structure*: if $\emptyset \neq \mathbb{K} \subseteq \overline{\mathbb{K}}$ then $\bigcap \mathbb{K} \in \overline{\mathbb{K}}$.

There is an associated *map* $\mathrm{cl}_{\overline{\mathbb{K}}} \colon \mathcal{P}(\mathcal{Q}) \to \overline{\mathbb{K}} \cup \{\mathcal{Q}\}$, called natural extension, defined by

$$\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) := \bigcap \{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}, \qquad \text{where we let } \bigcap \emptyset = \mathcal{Q} \text{ by convention.}$$

Suppose I have an assessment or partially specified set of desirable gamble sets $\mathcal{F}$. This assessment $\mathcal{F}$ consists of gamble sets F that I consider *desirable*.

If $\mathcal{F} \nsubseteq \mathcal{K}$ for every $\mathcal{K} \in \overline{\mathbb{K}}$, then $\mathcal{F}$ is *inconsistent*: it is impossible to extend it coherently.

Hint
Typically, you can prove the inconsistency of an assessment $\mathcal{F}$ by iteratively applying the productive axioms $\mathcal{K}_2$–$\mathcal{K}_5$ to infer the desirability of more and more gamble sets, until you find a contradiction with the destructive axiom $\mathcal{K}_1$.

Example
The assessment $\mathcal{F} = \{\{f\}, \{-f\}\}$ is *inconsistent*: If there were a coherent superset $\mathcal{K}$, then by Axiom $\mathcal{K}_5$ $\{0\} = \{1f + 1(-f)\} \in \mathcal{K}$, so Axiom $\mathcal{K}_2$ would imply that $\{0\} \setminus \{0\} = \emptyset \in \mathcal{K}$, contradicting Axiom $\mathcal{K}_1$.

If $\mathcal{F} \subseteq \mathcal{K}$ for some $\mathcal{K} \in \overline{\mathbb{K}}$, then $\mathcal{F}$ is *consistent*: it is possible to extend it coherently.

I want to make it coherent, but I want to be *as conservative as possible*: I don't want to inject any additional information. So what I want is the least informative (smallest) coherent superset: $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$.

If $\mathcal{F}$ is consistent, then $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$ is the *smallest coherent superset of* $\mathcal{F}$.

If $\mathcal{F}$ is inconsistent, then $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \mathcal{Q}$.

I can do this for any theory of imprecise probabilities, as long as *coherence forms an intersection structure*.

4.3 Properties of natural extension

$\mathcal{F}$'s natural extension is defined by $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) := \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}$, where we let $\bigcap\emptyset = \mathcal{Q}$ by convention.

Proposition 3. *Consider any $\mathcal{F} \subseteq \mathcal{Q}$ and $\mathcal{K} \in \overline{\mathbb{K}}$. Then*
1. *if $\mathcal{F}$ is coherent then $\mathcal{F}$ is consistent;*
2. *$\mathcal{F}$ is coherent iff $\mathcal{F}$ is consistent and $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \mathcal{F}$;*
3. *$\mathcal{F} \subseteq \mathcal{K} \Leftrightarrow \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathcal{K}$.*

Proof. For 1, if $\mathcal{F}$ is *coherent* then it is included in a coherent set of desirable gamble sets [namely itself], so it is *consistent*.

For 2, if $\mathcal{F}$ is *coherent* then $\mathcal{F}$ belongs to $\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}$, so $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} = \mathcal{F}$. Moreover, in 1 we have shown that then $\mathcal{F}$ is *consistent*. Conversely, if $\mathcal{F}$ is *consistent* then $\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} \neq \emptyset$, so $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}$ is a coherent set of desirable gamble sets that includes $\mathcal{F}$. If moreover $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \mathcal{F}$, then $\mathcal{F}$ is *coherent*.

For 3, if $\mathcal{F} \subseteq \mathcal{K}$ then $\mathcal{K} \in \{\mathcal{K}' \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}'\}$, so $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K}' \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}'\} \subseteq \mathcal{K}$. The converse implication follows from [extensivity] (see the next exercise). $\square$

This argument does not depend on the content of the rationality axioms, but uses only the fact that they form an intersection structure! Therefore, it holds for *any theory of imprecise probabilities* where coherence forms an intersection structure.

Exercise
The natural extension operator $\mathrm{cl}_{\overline{\mathbb{K}}}$ is a closure operator:
cl_1. $\mathcal{F} \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$; [extensivity]
cl_2. $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})) = \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$; [idempotency]
cl_3. if $\mathcal{F} \subseteq \mathcal{F}'$ then $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}')$. [monotonicity]
Hint: this also depend on the fact that coherence is an intersection structure, so it holds for any theory of imprecise probabilities where coherence forms an intersection structure.

4.4 Properties of natural extension: solution

Exercise
The natural extension operator $\mathrm{cl}_{\overline{\mathbb{K}}}$ is a closure operator:
cl_1. $\mathcal{F} \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$; [extensivity]

cl_2. $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})) = \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$; [idempotency]

cl_3. if $\mathcal{F} \subseteq \mathcal{F}'$, then $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}')$. [monotonicity]

Solution
For cl_1, $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\}$, so $\mathcal{F} \subseteq \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$.

For cl_2, $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathcal{K}\} = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} = \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$, where we used that $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathcal{K} \Leftrightarrow \mathcal{F} \subseteq \mathcal{K}$ (established in Item 3 of the previous proposition) in the second equality.

For cl_3, if $\mathcal{F} \subseteq \mathcal{F}'$ then $\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} \supseteq \{\mathcal{K}' \in \overline{\mathbb{K}} \colon \mathcal{F}' \subseteq \mathcal{K}'\}$, so $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} \subseteq \bigcap\{\mathcal{K}' \in \overline{\mathbb{K}} \colon \mathcal{F}' \subseteq \mathcal{K}'\} = \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}')$.

5 Binary sets of desirable gamble sets

5.1 Binary part of a set of desirable gamble sets

If a set F belongs to a set of desirable gamble sets $\mathcal{K}$ then *at least one gamble in F is desirable*. Therefore, if $F = \{f\}$ is a singleton, then $\{f\} \in \mathcal{K}$ means that f is desirable.

Definition 4. The binary part $\mathcal{D}_{\mathcal{K}}$ of a set of desirable gamble sets $\mathcal{K}$ is the set of desirable gambles

$$\mathcal{D}_{\mathcal{K}} := \{f \in \mathcal{L} \colon \{f\} \in \mathcal{K}\}.$$

Exercise
If the *set of desirable gamble sets* $\mathcal{K}$ is *coherent*, then so is the *set of desirable gambles* $\mathcal{D}_{\mathcal{K}}$.

Hint: Verify each of the axioms $\mathcal{D}_1$–$\mathcal{D}_4$ for $\mathcal{D}_{\mathcal{K}}$.

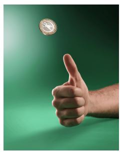
Solution
For Axiom $\mathcal{D}_1$, if towards contradiction $0 \in \mathcal{D}_{\mathcal{K}}$ then $\{0\} \in \mathcal{K}$. Then Axiom $\mathcal{K}_2$ would imply that $\emptyset = \{0\} \setminus \{0\} \in \mathcal{K}$, a *contradiction* with Axiom $\mathcal{K}_1$. For Axiom $\mathcal{D}_2$, if $f > 0$ then $\{f\} \in \mathcal{K}$ by Axiom $\mathcal{K}_3$, so $f \in \mathcal{D}_{\mathcal{K}}$. For Axiom $\mathcal{D}_3$, if $f \in \mathcal{D}_{\mathcal{K}}$ and $\lambda \in \mathbb{R}_{>0}$, then $\{f\} \in \mathcal{K}$, and hence $\{\lambda f\} \in \mathcal{K}$ by Axiom $\mathcal{K}_5$ [**Check this yourself!**], so $\lambda f \in \mathcal{D}_{\mathcal{K}}$. For Axiom $\mathcal{D}_4$, if $f, g \in \mathcal{D}_{\mathcal{K}}$ then $\{f\}, \{g\} \in \mathcal{K}$, and hence $\{f+g\} = \{1f' + 1g' \colon f' \in \{f\}, g' \in \{g\}\} \in \mathcal{K}$ by Axiom $\mathcal{K}_5$, so $f + g \in \mathcal{D}_{\mathcal{K}}$.

Why is $\mathcal{D}_{\mathcal{K}}$ called "the *binary part* of $\mathcal{K}$"?

Because it is the part of $\mathcal{K}$ that is *determined by a binary preference ordering*, namely the *desirability ordering* $\prec$ on gambles, defined by $f \prec g$ if $g - f \in \mathcal{D}_{\mathcal{K}}$. It is the *binary relation* (or *set of desirable gambles*) induced by $\mathcal{K}$.

5.2 Running example: coin with bias in unknown direction

$\mathcal{C} := \{p \text{ pmf} \colon p(\mathrm{H}) < \frac{1}{4} \text{ or } p(\mathrm{H}) > \frac{3}{4}\}$.



$p(\mathrm{H}) = 0$ $\qquad\qquad$ $p(\mathrm{H}) = 1$

Let's consider the *E-admissible choice function* based on $\mathcal{C}$:

$$C_{\mathcal{C}}(F) = \{f \in F \colon \text{there is } p \in \mathcal{C} \text{ such that } f \in C_p(F)\} = \bigcup_{p \in \mathcal{C}} C_p(F) \quad \text{for all } F \in \mathcal{Q}.$$

Exercise
Find the set of desirable gamble sets $\mathcal{K}_{\mathcal{C}}$ corresponding to $C_{\mathcal{C}}$, and then find the binary part $\mathcal{D}_{\mathcal{K}_{\mathcal{C}}}$ of it.

Solution
We have seen in a *previous exercise* that $\mathcal{K}_{\mathcal{C}} = \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\forall p \in \mathcal{C})(\exists f \in F) E_p(f) > 0\} = \bigcap_{p \in \mathcal{C}} \mathcal{K}_p$. Consider any *gamble* f and infer that

$$f \in \mathcal{D}_{\mathcal{K}_{\mathcal{C}}} \Leftrightarrow \{f\} \in \mathcal{K}_{\mathcal{C}} \Leftrightarrow (\forall p \in \mathcal{C})\{f\} \in \mathcal{K}_p \Leftrightarrow \underbrace{(\forall p \in \mathcal{C}) E_p(f) > 0}_{\substack{\Rightarrow E_{p_{\mathrm{H}}}(f) = f(\mathrm{H}) > 0 \text{ and} \\ E_{p_{\mathrm{T}}}(f) = f(\mathrm{T}) > 0}} \text{ or } f \in \mathcal{L}_{>0} \Leftrightarrow f \in \mathcal{L}_{>0}.$$

The binary part of $\mathcal{K}_{\mathcal{C}}$ is *vacuous*!

5.3 Binary sets of desirable gamble sets

Suppose I have a set of desirable gambles $\mathcal{D}$. I want to express the information in $\mathcal{D}$ using a set of desirable gamble sets.

Every $f \in \mathcal{D}$ defines a *desirable gamble set (singleton)* $\{f\}$. My *assessment* is $\mathcal{F}_{\mathcal{D}} := \{\{f\} \colon f \in \mathcal{D}\}$. I want to make it coherent, but I want to be *as conservative as possible*: I don't want to inject any additional information, so I'm looking for the *natural extension* $\mathcal{K}_{\mathcal{D}} := \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}_{\mathcal{D}})$.

Proposition 4. *If $\mathcal{D}$ is a coherent set of desirable gambles, then $\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F}_{\mathcal{D}} \subseteq \mathcal{K}\} = \{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{D}_{\mathcal{K}} = \mathcal{D}\}$. Moreover,*

$$\mathcal{K}_{\mathcal{D}} = \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}_{\mathcal{D}}) = \{F \in \mathcal{Q} \colon F \cap \mathcal{D} \neq \emptyset\} = \{F \in \mathcal{Q} \colon (\exists f \in F) 0 \prec f\}.$$

(Sen–Walley maximality)

Because $\mathcal{K}_{\mathcal{D}}$ is completely determined by a binary preference ordering, we call them binary.

Exercise
Find $\mathcal{K}_{\mathcal{D}}$ when $\mathcal{D} = \mathcal{L}_{>0}$ (the *vacuous* set of desirable gambles).

Solution
$\mathcal{K}_{\mathcal{D}} = \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset\} = \{F \in \mathcal{Q} \colon (\exists f \in F) f > 0\}$, which is the *vacuous* set of desirable gamble sets.

Exercise
Show that it is *impossible* to have $\{f\}, \{g\} \notin \mathcal{K}_{\mathcal{D}}$ but $\{f, g\} \in \mathcal{K}_{\mathcal{D}}$. This means that *being binary* here coincides with *being binary* for rejection functions.

Solution
If $\{f, g\} \in \mathcal{K}_{\mathcal{D}}$ then $\{f, g\} \cap \mathcal{D} \neq \emptyset$, so $f \in \mathcal{D}$ or $g \in \mathcal{D}$, whence $\{f\} \in \mathcal{K}_{\mathcal{D}}$ or $\{g\} \in \mathcal{K}_{\mathcal{D}}$.

6 Representation

6.1 Representation

Working with *sets of desirable gambles*:

– avoids problems with conditioning on sets of probability zero;

– is simple, intuitive and elegant;

– gives a geometrical flavour to probabilistic inference.

Working with *sets of desirable gamble sets* also avoids problems with conditioning on sets of probability zero (not in this lecture), but they are less simple. They are more expressive and general.

There are simple sets of desirable gamble sets, namely the binary ones $\mathcal{K}_{\mathcal{D}} = \{F \in \mathcal{Q} \colon F \cap \mathcal{D} \neq \emptyset\}$. Can we exploit their simplicity?

Theorem 5 (Representation theorem)**.** *A set of desirable gamble sets $\mathcal{K} \subseteq \mathcal{Q}$ is coherent iff there is a non-empty collection $\mathbb{D}$ such that*

$$\mathcal{K} = \bigcap_{\mathcal{D} \in \mathbb{D}} \mathcal{K}_{\mathcal{D}}.$$

In that case, the largest such representing set is given by

$$\mathbb{D}(\mathcal{K}) := \{\mathcal{D} \textit{ coherent} \colon \mathcal{K} \subseteq \mathcal{K}_{\mathcal{D}}\}.$$

6.2 Representation: exercise

Exercise
Consider the *vacuous set of desirable gamble sets* $\mathcal{K}_{\mathrm{v}} = \bigcap \overline{\mathbb{K}} = \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset\}$. Find a *representation* of $\mathcal{K}_{\mathrm{v}}$.

Solution
It follows from $\mathcal{K}_{\mathrm{v}} = \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset\}$ that $\{\mathcal{D}_{\mathrm{v}}\} = \{\mathcal{L}_{>0}\}$ is a representation of $\mathcal{K}_{\mathrm{v}}$.

We can add supersets of elements of $\mathbb{D}$ without changing the intersection $\bigcap_{\mathcal{D} \in \mathbb{D}} \mathcal{K}_{\mathcal{D}}$, so moreover *every superset of* $\{\mathcal{D}_{\mathrm{v}}\}$ [every subset of $\overline{\mathbb{D}}$ that includes $\mathcal{D}_{\mathrm{v}}$] is a representation of $\mathcal{K}_{\mathrm{v}}$, too. In particular, $\overline{\mathbb{D}}$ is a representation of $\mathcal{K}_{\mathrm{v}}$.

6.3 Natural extension using representation

Theorem 6 (Representation)**.** *A set of desirable gamble sets $\mathcal{K} \subseteq \mathcal{Q}$ is coherent iff there is a non-empty collection $\mathbb{D}$ such that*

$$\mathcal{K} = \bigcap_{\mathcal{D} \in \mathbb{D}} \mathcal{K}_{\mathcal{D}}.$$

In that case, the largest such representing set is given by

$$\mathbb{D}(\mathcal{K}) := \{\mathcal{D} \text{ coherent} \colon \mathcal{K} \subseteq \mathcal{K}_{\mathcal{D}}\}.$$

Can we use this powerful result for the *natural extension*?

Theorem 7 (Natural extension theorem using representation)**.** *Consider any assessment $\mathcal{F} \subseteq \mathcal{Q}$. Then*

$$\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap \{\mathcal{K}_{\mathcal{D}} \colon \mathcal{D} \text{ coherent and } \mathcal{F} \subseteq \mathcal{K}_{\mathcal{D}}\} = \bigcap \{\mathcal{K}_{\mathcal{D}} \colon \mathcal{D} \in \mathbb{D}(\mathcal{F})\}.$$

Exercise
Prove the *Natural extension theorem using representation*, using the *representation* of $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$.

6.4 Solution

Solution
If $\mathcal{F}$ is *inconsistent*, then $\{\mathcal{K} \in \overline{\mathbb{K}} \colon \mathcal{F} \subseteq \mathcal{K}\} = \emptyset$ so in particular $\{\mathcal{K}_{\mathcal{D}} \colon \mathcal{D}$ coherent and $\mathcal{F} \subseteq \mathcal{K}_{\mathcal{D}}\} = \emptyset$. Then $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \mathcal{Q} = \bigcap\{\mathcal{K}_{\mathcal{D}} \colon \mathcal{D}$ coherent and $\mathcal{F} \subseteq \mathcal{K}_{\mathcal{D}}\}$ using the convention that $\bigcap \emptyset = \mathcal{Q}$, and we are done, so assume that $\mathcal{F}$ is *consistent*.

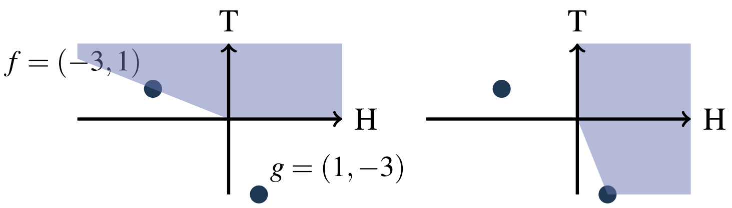
Use the *representation theorem* for $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$ and the earlier established fact that $\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathcal{K}_{\mathcal{D}}$ iff $\mathcal{F} \subseteq \mathcal{K}_{\mathcal{D}}$ to infer that, indeed

$$\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K}_{\mathcal{D}} \colon \mathcal{D} \text{ coherent and } \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathcal{K}_{\mathcal{D}}\} = \bigcap\{\mathcal{K}_{\mathcal{D}} \colon \mathcal{D} \text{ coherent and } \mathcal{F} \subseteq \mathcal{K}_{\mathcal{D}}\}.$$

This result holds for any theory of imprecise probabilities with natural extension and representation. It does not depend on the exact formulation of the representation.

6.5 Running example: coin with bias in unknown direction

We found that the gamble set $\{f, g\} = \{(-3, 1), (1, -3)\}$ is desirable: $\{f, g\} \in \mathcal{K}$.



Exercise
Find the natural extension of $\{f, g\}$, or in other words, the smallest coherent $\mathcal{K}$ such that $\{f, g\} \in \mathcal{K}$.

Solution
We need to find $\mathrm{cl}_{\overline{\mathbb{K}}}(\{\{f, g\}\})$. Consider any coherent $\mathcal{D}$ and infer that

$$\{\{f, g\}\} \subseteq \mathcal{K}_{\mathcal{D}} \Leftrightarrow \{f, g\} \cap \mathcal{D} \neq \emptyset \Leftrightarrow f \in \mathcal{D} \text{ or } g \in \mathcal{D} \Leftrightarrow \mathcal{D} \supseteq \mathrm{posi}(\{f\} \cup \mathcal{L}_{>0}) \text{ or } \mathcal{D} \supseteq \mathrm{posi}(\{g\} \cup \mathcal{L}_{>0}).$$

Use the *natural extension theorem using representation*:

$$\mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) = \bigcap\{\mathcal{K}_{\mathcal{D}} \colon \mathcal{D} \text{ coherent and } \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F}) \subseteq \mathcal{K}_{\mathcal{D}}\} = \mathcal{K}_{\mathrm{posi}(\{f\} \cup \mathcal{L}_{>0})} \cap \mathcal{K}_{\mathrm{posi}(\{g\} \cup \mathcal{L}_{>0})}.$$

So we find that $F \in \mathrm{cl}_{\overline{\mathbb{K}}}(\mathcal{F})$ iff $F \cap \mathrm{posi}(\{f\} \cup \mathcal{L}_{>0}) \neq \emptyset$ and $F \cap \mathrm{posi}(\{g\} \cup \mathcal{L}_{>0}) \neq \emptyset$.

7 Special cases

7.1 E-admissibility

If I have a *credal set* $\mathcal{C}$, then the *E-admissible set of desirable gamble sets* based on $\mathcal{C}$ is

$$\mathcal{K}_{\mathcal{C}} = \{F \in \mathcal{Q} \colon F \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\forall p \in \mathcal{C})(\exists f \in F) E_p(f) > 0\} = \bigcap_{p \in \mathcal{C}} \mathcal{K}_p,$$

where

$$\mathcal{K}_p = \{F \in \mathcal{Q}\colon \text{ there is } f \in F \text{ such that } E_p(f) > 0 \text{ or } f > 0\}.$$

Recall that $\mathcal{D}_p = \{f \in \mathcal{L}\colon E_p(f) > 0 \text{ or } f > 0\}$, so that $\mathcal{K}_{\mathcal{D}_p} = \mathcal{K}_p$.

This gives us a representation of $\mathcal{K}_{\mathcal{C}}$:

$$\mathcal{K}_{\mathcal{C}} = \bigcap_{p \in \mathcal{C}} \mathcal{K}_{\mathcal{D}_p} \quad \text{so } \mathbb{D}_{\mathcal{C}} := \{\mathcal{D}_p\colon p \in \mathcal{C}\} \textit{ represents } \mathcal{K}_{\mathcal{C}}.$$

E-admissibility $\Leftrightarrow$ represented by a set $\{\mathcal{D}_p\colon p \text{ pmf}\}$

7.2 Sen–Walley maximality

If I have a *coherent set of desirable gambles* $\mathcal{D}$, then the *Sen–Walley maximal set of desirable gamble sets* based on $\mathcal{D}$ is

$$\{F \in \mathcal{Q}\colon \text{there is } f \in F \text{ such that } f \in \mathcal{D}\} = \{F \in \mathcal{Q}\colon F \cap \mathcal{D} \neq \emptyset\} = \mathcal{K}_{\mathcal{D}}.$$

This gives us an immediate representation of $\mathcal{K}_{\mathcal{D}}$,

$$\mathcal{K}_{\mathcal{D}} = \bigcap_{\mathcal{D}' \in \{\mathcal{D}\}} \mathcal{K}_{\mathcal{D}'} \quad \text{so } \{\mathcal{D}\} \textit{ represents } \mathcal{K}_{\mathcal{D}}.$$

(Sen-Walley) maximality $\Leftrightarrow$ represented by a singleton $\{\mathcal{D}\}$

Exercise
Give an example of a coherent set of desirable gambles that is both E-admissible and (Sen–Walley) maximal.

Solution
Consider any pmf q. Then $\mathcal{K}_p = \mathcal{K}_{\mathcal{D}_q}$ is *E-admissible* [because it has a representation $\{\mathcal{D}_q\}$ that *consists only of elements of the form* $\mathcal{D}_p$] and *(Sen-Walley) maximal* [because it has a *singleton representation* $\{\mathcal{D}_q\}$.]

7.3 What are coherent sets of desirable gamble sets?

Recall the *representation theorem*:
A set of desirable gamble sets $\mathcal{K} \subseteq \mathcal{Q}$ is *coherent* iff there is a non-empty collection $\mathbb{D}$ such that

$$\mathcal{K} = \bigcap_{\mathcal{D} \in \mathbb{D}} \mathcal{K}_{\mathcal{D}}.$$

In that case, the largest such representing set is given by

$$\mathbb{D}(\mathcal{K}) := \{\mathcal{D} \text{ coherent} \colon \mathcal{K} \subseteq \mathcal{K}_{\mathcal{D}}\}.$$

- If $(\forall \mathcal{D} \in \mathbb{D})(\exists p)\mathcal{D} = \mathcal{D}_p$ then $\mathcal{K}$ is *E-admissible*.
- If $\mathbb{D} = \{\mathcal{D}\}$ then $\mathcal{K}$ is *(Sen-Walley) maximal*.

Coherent sets of desirable gamble sets have the *E-admissibility* structure, but with potentially imprecise elements in $\mathbb{D}$.

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