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The marginal problem for sets of desirable gambles sets with a finite representation

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ABSTRACT

We study the marginal problem for sets of desirable gamble sets (SoDGSes), which is equivalent to studying this problem for choice functions. More specifically, given a number of marginal SoDGSes on overlapping domains, we establish conditions under which they are compatible in the sense that they can be derived from a common joint SoDGS. We do so for SoDGSes that admit a concrete finite representation. Our main result is that such SoDGSes are compatible if they are pairwise compatible and if a running intersection property is satisfied. Additionally, we study representations of the smallest compatible joint and, provided the representation is finite, we derive an iterative method of constructing such joint and its representation.

1. Introduction

Consider a probability mass function p_1 describing an agent's beliefs about the uncertain variables X_1 and X_2, and a probability mass function p_2 describing her beliefs about the uncertain variables X_2 and X_3. When are they *compatible*, in the sense that they are derived from a common joint p about X_1, X_2 and X_3? This problem is called the 'marginal problem': the compatibility of a number of marginal uncertainty models with a joint model. The difficulty here is that p_1 and p_2 both represent the variable X_2: their respective domains 'overlap'. So one necessary condition for a positive answer to the marginal problem, is that p_1 and p_2 marginalise to a common probability mass function for X_2. The marginal problem has been studied extensively for probabilities in the past [1,2] and more recently and references therein [3].

Our inspiration for this paper is the work of [3], who studied the marginal problem for sets of desirable gambles, and obtained sufficient conditions that guarantee a positive answer. We study the marginal problem for sets of desirable gamble sets ('SoDGSes' – singular 'SoDGS'), thereby generalising some of their results. SoDGSes attribute desirability to *sets of gambles* rather than to gambles. When an agent finds a set $\{f, g\}$ desirable – preferred over the status quo indicated by 0 – this means that one of f or g is preferred over 0, but she might not be able to identify which of f or g is preferred over 0. As such, SoDGSes are capable of modelling disjunctions of preference statements [4], and this makes them among the most expressive uncertainty models in the literature of imprecise probabilities. They are equivalent to imprecise-probabilistic choice functions [5,6]. Choice functions and SoDGSes are gaining popularity, and various investigations of foundational aspects have recently been carried out [4,7–12].

One advantage of SoDGSes and choice functions is that they are easy to work with from a theoretical point of view, thanks to their representation in terms of a collection of partial preference orders – a collection of sets of desirable gambles. In this paper we solve the marginal problem for an interesting subclass of SoDGSes, namely those that admit a finite representation.

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In their study of the marginal problem for sets of desirable gambles, Casanova et al. [13] followed a different approach than the one of [3]: they solved the marginal problem for any *information algebra*, and have shown that coherent sets of desirable gambles form such an algebra. In our current study of the marginal problem for SoDGSes, we do not follow this approach, as it would not lead to a representation of the SoDGS compatible with the given SoDGSes in terms of a collection of sets of desirable gambles. Instead, we follow a more direct approach, which, as we will see, will lead to a representation of the compatible joint.

While for our purposes the *running intersection property (RIP)* will be used purely as a technical tool to establish compatibility, it has been extensively studied in the context of *hypergraphs* – graphs in which edges can join any number of vertices. In particular, several works on polynomial and multilinear optimisation [14–16] exploit RIP-like structures to decompose large problems into smaller, consistent subproblems, enabling stronger relaxations, sparse semidefinite hierarchies, and convergence guarantees. Beyond hypergraphs, portfolio optimisation with overlapping marginals can similarly exploit RIP to decompose the problem efficiently while preserving robustness to dependency [17].

We first recall the theory of SoDGSes (Section 2). This is followed by our contributions concerning the finite representation (Section 3). Then we discuss multivariate SoDGSes in the by then established context (Section 4) in preparation of our formulation and solution of the marginal problem, together with the detailed study of the representations of the smallest compatible joint (Section 5).

This paper is an extended journal version of an earlier conference paper [18], and contains an additional study of the smallest compatible joint in the new Section 5.3, together with additional results in Sections 3 and 4.

2. Sets of desirable gamble sets

2.1. Preliminaries

Consider an uncertain variable X, taking values in a finite possibility space $\mathcal{X}$. A gamble is a real-valued function on $\mathcal{X}$. A gamble f commits its owner to the following transaction: first, X's real outcome x in $\mathcal{X}$ is revealed, and then the owner receives $f(x)$ units of utile on a predetermined utility scale. This value $f(x)$ may be negative, so gambles can be regarded as risky transactions.

We denote the set of all gambles on $\mathcal{X}$ by $\mathcal{L}(\mathcal{X})$, and sometimes also simply by $\mathcal{L}$ if it is unambiguous from the context what the possibility space $\mathcal{X}$ is. The set of all gambles $\mathcal{L}$ forms an $|\mathcal{X}|$-dimensional real linear space under the pointwise addition and scalar multiplication of gambles. We collect the "positive" gambles in $\mathcal{L}_{>0}(\mathcal{X}) := \{f \in \mathcal{L}(\mathcal{X}) : f > 0\}$, and the "non-positive" ones in $\mathcal{L}_{\leq 0}(\mathcal{X}) := \{f \in \mathcal{L}(\mathcal{X}) : f \leq 0\}$, or $\mathcal{L}_{>0}$ and $\mathcal{L}_{\leq 0}$ when unambiguous.[1]

2.2. Sets of desirable gambles

An agent might have beliefs about the uncertain variable X, which can lead her to prefer some gambles over others. She might prefer a gamble f over the status quo 0, in which case we call f *desirable*. The agent's *set of desirable gambles* $D \subseteq \mathcal{L}$ collects all the gambles that she finds desirable. They have been introduced by [19] and [20], and have been extensively studied e.g., Walley [21, 22], De Cooman and Quaeghebeur [23], De Cooman and Miranda [24], Quaeghebeur [25].

Definition 1 (Coherent set of desirable gambles). A set of desirable gambles $D \subseteq \mathcal{L}$ is called *coherent* if:

D_1. $0 \notin D$;
D_2. $\mathcal{L}_{>0} \subseteq D$;
D_3. if $f, g \in D$ and $(\lambda, \mu) > 0$, then $\lambda f + \mu g \in D$.

We collect all the coherent sets of desirable gambles in $\overline{\mathbf{D}}(\mathcal{X}) \subseteq \wp(\mathcal{L})$, or $\overline{\mathbf{D}}$ for short, where $\wp$ denotes the powerset of its argument.

We say that the set of desirable gambles D_1 is at least as informative (or committal, precise) as the set of desirable gambles D_2 when $D_2 \subseteq D_1$, simply because an agent with D_1 finds more gambles desirable. The partially ordered set $(\overline{\mathbf{D}}, \subseteq)$ is a complete meet-semilattice. De Cooman [26] showed that such a structure is strong enough to yield a closure operator

$$\mathrm{cl}_{\overline{\mathbf{D}}} : \wp(\mathcal{L}) \to \overline{\mathbf{D}} \cup \{\mathcal{L}\} : A \mapsto \mathrm{cl}_{\overline{\mathbf{D}}}(A) := \bigcap_{A \subseteq D \in \overline{\mathbf{D}}} D$$

that maps any partially specified set of desirable gambles $A \subseteq \mathcal{L}$ that is *consistent* – has a coherent superset – to its unique least informative coherent extension $\mathrm{cl}_{\overline{\mathbf{D}}}(A)$. The sets of desirable gambles that are not consistent are mapped to $\mathcal{L}$. We call $\mathrm{cl}_{\overline{\mathbf{D}}}(A)$ the *natural extension* of the assessment A. So, the natural extension is the unique set of desirable gambles that is the consequence of finding every gamble in A desirable and taking into account the rationality axioms 1–1, but nothing else.

There is a useful characterisation of $\mathrm{cl}_{\overline{\mathbf{D}}}$ in terms of the positive linear hull operator "posi", defined for all $B \subseteq \mathcal{L}$ as follows

$$\mathrm{posi}(B) := \left\{ \sum_{f \in F} \lambda_f f : F \Subset B, \lambda \in (\mathbb{R}^{|F|})_{>0} \right\},$$

where $\Subset$ denotes the finite subset relation.

Theorem 1. *[23][Theorem 1] Consider any assessment $A \subseteq \mathcal{L}$. Then A is consistent if and only if $0 \notin \mathrm{posi}(\mathcal{L}_{>0} \cup A)$. If this is the case, then $\mathrm{cl}_{\overline{\mathbf{D}}}(A) = \mathrm{posi}(\mathcal{L}_{>0} \cup A)$.*

[1] For all elements u and v of a finite-dimensional vector space, we write $u \leq v$ if $u(z) \leq v(z)$ (or $u_z \leq v_z$) for all arguments (indices) z, and $u < v$ if $u \leq v$ but $u \neq v$. These definitions are used both for gambles and sequences of coefficients.

The smallest coherent set of desirable gambles – the *vacuous* set of desirable gambles – can be obtained from the empty assessment. We call it $D_{\mathrm{v}} := \mathrm{cl}_{\overline{\mathcal{D}}}(\emptyset) = \mathcal{L}_{>0}$.

2.3. Sets of desirable gamble sets

The agent might have beliefs that allow her to state "gamble f is desirable or gamble g is desirable", but she might not have sufficient knowledge to decide which of f and g are desirable. In other words, she knows that the set $\{f,g\}$ contains a desirable gamble, in which case we call $\{f,g\}$ a *desirable gamble set*.

More generally, a *gamble set* on $\mathcal{X}$ is a finite set of gambles on $\mathcal{X}$. We denote the set of all gamble sets on $\mathcal{X}$ by $\mathcal{Q}(\mathcal{X})$, or $\mathcal{Q}$ when unambiguous. A gamble set is *desirable* when it contains at least one desirable gamble. We collect an agent's desirable gamble sets in her *set of desirable gamble sets (SoDGS)* $K \subseteq \mathcal{Q}$. If the domain of the gambles $\mathcal{X}$ is important, we sometimes call $K \subseteq \mathcal{Q}(\mathcal{X})$ a SoDGS *on* $\mathcal{X}$.

De Bock and De Cooman [27] introduced a well-justified definition and axiomatisation of *coherent* SoDGSes – SoDGSes of rational agents.

Definition 2 (Coherent SoDGS)**.** An SoDGS $K \subseteq \mathcal{Q}$ is called *coherent* if:

K_0. $\emptyset \notin K$;
K_1. $F \in K \Rightarrow F \setminus \{0\} \in K$;
K_2. $\{f\} \in K$ for all f in $\mathcal{L}_{>0}$;
K_3. if $F, G \in K$ and $(\lambda^{f,g}, \mu^{f,g}) > 0$ for every pair (f,g) in $F \times G$, then $\{\lambda^{f,g} f + \mu^{f,g} g : f \in F, g \in G\} \in K$;
K_4. if $F \in K$ and $F \subseteq G$, then $G \in K$.

We collect all the coherent SoDGSes in the collection $\overline{\mathcal{K}}(\mathcal{X})$, or $\overline{\mathcal{K}}$ when unambiguous.

Similarly as we did for sets of desirable gambles, we say that the SoDGS K_1 is at least as informative as the SoDGS K_2 when $K_2 \subseteq K_1$, simply because an agent with K_1 finds more gamble sets desirable. The partially ordered set $(\overline{\mathcal{K}}, \subseteq)$ is again a complete meet-semilattice, so it, too, yields a closure operator

$$\mathrm{cl}_{\overline{\mathcal{K}}} : \wp(\mathcal{Q}) \to \overline{\mathcal{K}} \cup \{\mathcal{Q}\} : \mathcal{A} \mapsto \mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A}) := \textstyle\bigcap_{\mathcal{A} \subseteq K \in \overline{\mathcal{K}}} K$$

that maps any partially specified SoDGS $\mathcal{A} \subseteq \mathcal{Q}$ that is *consistent* – has a coherent superset – to its unique least informative coherent extension $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A})$. We call $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A})$ the *natural extension* of the assessment $\mathcal{A}$. Here too, the natural extension is the consequence of finding every gamble set in $\mathcal{A}$ desirable and the rationality axioms 2–2, and nothing else.

De Bock and De Cooman [27] showed that there is a useful characterisation of $\mathrm{cl}_{\overline{\mathcal{K}}}$ in terms of the positive linear hull operator 'posi' lifted to all $\mathcal{B} \subseteq \mathcal{Q}$ as follows

$$\mathrm{Posi}(\mathcal{B}) := \left\{ \left\{ \textstyle\sum_{F \in \mathcal{F}} \lambda_F^h h_F : h \in H \right\} : \mathcal{F} \Subset \mathcal{B}, \left(\forall h \in H := \bigtimes_{F \in \mathcal{F}} F\right) \lambda^h \in (\mathbb{R}^{|\mathcal{F}|})_{>0} \right\}.$$

We also use a version of $\mathcal{L}_{>0}(\mathcal{X})$ lifted to gamble sets, namely $\mathcal{L}^{\mathrm{s}}_{>0}(\mathcal{X}) := \{\{f\} : f \in \mathcal{L}_{>0}(\mathcal{X})\}$, or $\mathcal{L}^{\mathrm{s}}_{>0}$ when unambiguous, and use the transformation $\mathrm{Rs} : \wp(\mathcal{Q}) \to \wp(\mathcal{Q}) : \mathcal{B} \mapsto \mathrm{Rs}(\mathcal{B}) := \{F \in \mathcal{Q} : (\exists G \in \mathcal{B}) G \setminus \mathcal{L}_{\leq 0} \subseteq F\}$, which is the abbreviation of 'Remove negative gambles from gamble sets in its input and take supersets'.

Theorem 2. *[27][Theorem 10] Consider any assessment $\mathcal{A} \subseteq \mathcal{Q}$. Then $\mathcal{A}$ is consistent if and only if $\emptyset \notin \mathcal{A}$ and $\{0\} \notin \mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A})$. If this is the case, then $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A}) = \mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A}))$.*

The smallest coherent SoDGS – the *vacuous* SoDGS – is obtained from the empty assessment. We call it $K_{\mathrm{v}} := \mathrm{cl}_{\overline{\mathcal{K}}}(\emptyset) = \mathrm{Rs}(\mathcal{L}^{\mathrm{s}}_{>0}) = \{F \in \mathcal{Q} : (\exists f \in \mathcal{L}_{>0}) f \in F\}$.

If the agent has a set of desirable gambles D, then she finds desirable all the gamble sets $\{f\}$ for f in D. In other words, her SoDGS that corresponds to D is the natural extension of the assessment $\mathcal{A} := \{\{f\} : f \in D\}$, which is given by $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A})$. Whenever D is coherent, it follows that $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A})$ is coherent and equal to $K_D := \{F \in \mathcal{Q} : F \cap D \neq \emptyset\}$ [9][Proposition 5].

We use the notation K_D for $\{F \in \mathcal{Q} : F \cap D \neq \emptyset\}$ even when D is not coherent. De Bock and De Cooman [28][Proposition 8] showed that K_D is coherent if and only if D is. We call an SoDGS K *binary* if there is a set of desirable gambles D such that $K_D = K$, because K is then characterised by the preference order determined by D, which is a *binary* order. The vacuous SoDGS K_{v} is binary, and $D_{\mathrm{v}} = \mathcal{L}_{>0}$ is the set of desirable gambles that determines it: $K_{\mathrm{v}} = \{F \in \mathcal{Q} : F \cap \mathcal{L}_{>0} \neq \emptyset\} = K_{D_{\mathrm{v}}}$.

Conversely, if the agent has an SoDGS K, then $D_K := \{f : \{f\} \in K\}$ is a set of desirable gambles for her. Moreover, De Bock and De Cooman [27][Lemma 13] showed that D_K is coherent whenever K is.

2.4. Choice functions

An SoDGS can be equivalently represented by a choice function, which is a map

$$C : \mathcal{Q} \to \mathcal{Q} : F \mapsto C(F) \subseteq F.$$

The idea is that C identifies the non-rejected (or choiceworthy) gambles $C(F)$ in any gamble set F, which represents a finite decision problem. Choice functions are intimately related to the fundamental problem of decision theory: how to make decisions from a set of available options. von Neumann and Morgenstern [29] introduced rationality axioms for choice functions based on binary comparisons, which was generalised to more general choice functions by [30,31], among others.

When a choice function is determined by maximising expected utility, then if the agent is indecisive between f and g – meaning that $\{f, g\} = C(\{f, g\})$ – then this must be because both f and g have the same expected utility, so the agent is indifferent between f and g. In such a case, the choice function C has the property that adding any positive ϵ to f renders g not choiceworthy: $g \notin C(\{f + \epsilon, g\})$, and *vice versa*. Seidenfeld et al. [5], Kadane et al. [6] have introduced imprecise-probabilistic choice functions, which have the characteristic feature that an agent can be indecisive between two gambles f and g, without being necessarily indifferent between f and g. This is characteristic of most theories of imprecise probabilities.

A choice function C determines an SoDGS as follows. When a gamble set F satisfies $0 \notin C(\{0\} \cup F)$ – in other words, when the status quo is rejected from $\{0\} \cup F$ – then we call F *desirable*, and collect all the desirable gamble sets in $K_C := \{F \in \mathcal{Q} : 0 \notin C(\{0\} \cup F)\}$. The idea is that, if 0 is not choiceworthy in $\{0\} \cup F$, then F must contain a gamble preferred to 0.

The other way around, an SoDGS K determines a choice function C_K as follows: For any gamble set F, we define $C_K(F) := \{f \in F : \{g - f : g \in F\} \notin K\}$.

Choice function have rationality axioms similar to Axioms 2–2 [27][Definition 4]. They can also be ordered: If $C_1(F) \subseteq C_2(F)$ for all F in $\mathcal{Q}$, then we call C_1 more informative than C_2.

Under very mild conditions, much weaker than coherence,[2] these two constructions commute ($K_{C_K} = K$ and $C_{K_C} = C$) and preserve the order and coherence, so choice functions and SoDGSes are equivalent representations of the same information. We refer to [28][Section 4] for more information. In this paper, for notational reasons we stick to SoDGSes, but due to this equivalence, *all the results can be translated to choice functions*.

3. Finite representation

Coherent SoDGSes have an interesting representation in terms of collections of sets of desirable gambles.

Theorem 3. *Representation [28][Theorem 9] For any SoDGS K, the following two expressions are equivalent:*

(i) *K is coherent;*
(ii) *there is a non-empty set $\mathcal{D} \subseteq \overline{\mathbf{D}}$ of coherent sets of desirable gambles such that $K = K_{\mathcal{D}} := \bigcap\{K_D : D \in \mathcal{D}\}$, in which case we say that* $\mathcal{D}$ represents K.

Moreover, K's (unique) largest representing set is $\mathcal{D}_K := \{D \in \overline{\mathbf{D}} : K \subseteq K_D\}$.

Note that $\mathcal{D}_K$ is an *upset*: if $D_1 \in \mathcal{D}_K$ and $D_1 \subseteq D_2$, then $D_2 \in \mathcal{D}_K$, for any D_1 and D_2 in $\overline{\mathbf{D}}$. In other words, $\mathcal{D}_K = \uparrow\mathcal{D}_K$, where $\uparrow\mathcal{D} := \{D \in \overline{\mathbf{D}} : (\exists D' \in \mathcal{D}) D' \subseteq D\}$ is the smallest upset containing $\mathcal{D}$, for any $\mathcal{D} \subseteq \overline{\mathbf{D}}$.

The notion of an upset allows for a simple but interesting way to derive new representations from old ones.

Proposition 1. *Consider any coherent SoDGS K and any non-empty $\mathcal{D} \subseteq \overline{\mathbf{D}}$. Then $\mathcal{D}$ represents K if and only if $\uparrow\mathcal{D}$ represents K.*

Proof. The result follows by realising that for any D in $(\uparrow\mathcal{D}) \setminus \mathcal{D}$ there is a $D' \in \mathcal{D}$ such that $D' \subseteq D$, and therefore also $K_{D'} \subseteq K_D$, so the element K_D plays no role in the intersection $\bigcap_{D^\star \in \uparrow\mathcal{D}} K_{D^\star}$, and therefore can be removed from it: $K_{\uparrow\mathcal{D}} = \bigcap_{D^\star \in \uparrow\mathcal{D}} K_{D^\star} = \bigcap_{D^\star \in (\uparrow\mathcal{D}) \setminus D} K_{D^\star}$. This can be done for every D in $(\uparrow\mathcal{D}) \setminus \mathcal{D}$, from which $K_{\uparrow\mathcal{D}} = K_{\mathcal{D}}$ and the result follows. □

Theorem 3 also allows for a simpler expression for the natural extension:

Theorem 4 (De Bock & De Cooman, private communication). *An assessment $\mathcal{A} \subseteq \mathcal{Q}$ is consistent if and only if there is some D in $\overline{\mathbf{D}}$ such that $\mathcal{A} \subseteq K_D$. In that case $\mathrm{cl}_{\overline{\mathbf{K}}}(\mathcal{A}) = \bigcap\{K_D : D \in \overline{\mathbf{D}} \text{ and } \mathcal{A} \subseteq K_D\}$.*

The representation of coherent K in terms of $\mathcal{D}_K \subseteq \overline{\mathbf{D}}$ in Theorem 3 has been crucial for deriving several theoretical properties of SoDGSes, among which the irrelevant natural extension [9], independent natural extension [8] and marginal extension [32].

Despite the success of the representation of Theorem 3, we here focus our attention, for technical reasons,[3] on a special subclass of coherent SoDGSes: the ones that admit a *finite representation*.

Definition 3 (Finite representation). Consider a coherent SoDGS K. We say that K has a *finite representation* if there is a finite subset $\mathcal{D} \subseteq \overline{\mathbf{D}}$ that represents K, or in other words, if $K = K_{\mathcal{D}}$ for a finite $\mathcal{D} \subseteq \overline{\mathbf{D}}$.

Example 1. SoDGSes with a finite representation are still fairly general. For instance, consider any set of probability mass functions (pmfs) $\mathcal{M}$. This set $\mathcal{M}$ might for instance correspond to a non-additive measure, a belief function, or a lower probability. Denoting the p-expectation by E_p, then the coherent SoDGS based on $\mathcal{M}$ using *Sen–Walley maximality* [21,33] is

$$K^{\mathrm{m}}_{\mathcal{M}} := K_{\mathrm{v}} \cup \{F \in \mathcal{Q} : (\exists f \in F)(\forall p \in \mathcal{M}) E_p(f) > 0\},$$

which is a binary SoDGS, and hence has a finite – singleton – representation.

[2] The condition for SoDGSes K is: $F \in K \Leftrightarrow \{0\} \cup F \in K$, for all F in $\mathcal{Q}$. The condition for choice functions C is: $C(F + \{f\}) = C(F) + \{f\}$ for all F in $\mathcal{Q}$ and f in $\mathcal{L}$.

[3] More specifically, our proof of Proposition 5 depends crucially on this assumption, on which Theorem 7 builds.

Running example 1. *Consider a finite set of pmfs $\mathcal{M}$, then the SoDGS based on $\mathcal{M}$ using* E-admissibility *[5,6,34] is*

$$K_{\mathcal{M}} := K_{\mathrm{v}} \cup \{F \in \mathcal{Q} : (\forall p \in \mathcal{M})(\exists f \in F)E_p(f) > 0\}.$$

By [8][Lemma 5], $K_{\mathcal{M}}$ is represented by $\{D_p : p \in \mathcal{M}\}$, where

$$D_p := \mathcal{L}_{>0} \cup \{f \in \mathcal{L} : E_p(f) > 0\}$$

is the smallest set of desirable gambles that contains all the gambles with positive p-expectation. Since $\mathcal{M}$ is finite, $K_{\mathcal{M}}$ has a finite representation.

We show that the largest representation $\mathcal{D}_{K_{\mathcal{M}}}$ of $K_{\mathcal{M}}$ is $\uparrow\{D_p : p \in \mathcal{M}\}$. Proposition 1 implies that this is indeed a representation of $K_{\mathcal{M}}$, so $\uparrow\{D_p : p \in \mathcal{M}\} \subseteq \mathcal{D}_{K_{\mathcal{M}}}$. To show the converse inclusion, consider any $D \notin \uparrow\{D_p : p \in \mathcal{M}\}$. This implies, for all pmfs p in $\mathcal{M}$, that $D_p \nsubseteq D$. Therefore, for every $p \in \mathcal{M}$, there is some $f_p \in D_p$ such that $f_p \notin D$. Collect these f_p in the set $F := \{f_p : p \in \mathcal{M}\}$, which is a valid gamble set because it is finite. Then F belongs to $K_{\mathcal{M}}$, but does not belong to K_D, so $D \notin \mathcal{D}_{K_{\mathcal{M}}}$, indeed.

An interesting characterisation of an SoDGS K with a finite representation involves the existence of minimal elements in the poset $(\mathcal{D}_K, \subseteq)$ of K's largest representation. In order to discuss it, for every $\mathcal{D} \subseteq \overline{\mathbf{D}}$, define $\min \mathcal{D} := \{D \in \mathcal{D} : (\forall D' \in \mathcal{D})(D' \subseteq D \Rightarrow D' = D)\}$ as $\mathcal{D}$'s minimal elements.

Theorem 5. *For any coherent SoDGS K, we have that $\min \mathcal{D}_K \neq \emptyset$ so the poset $(\mathcal{D}_K, \subseteq)$ has minimal elements. Moreover, $\mathcal{D}_K = \uparrow \min \mathcal{D}_K$. As a consequence $K = K_{\min \mathcal{D}_K}$ so $\min \mathcal{D}_K$ represents K.*

Proof. We first show that $\min \mathcal{D}_K \neq \emptyset$. To this end, consider any non-empty chain $\mathcal{C} \subseteq \mathcal{D}_K$, meaning that $\mathcal{C}$ is totally ordered by $\subseteq$. By Zorn's Lemma it suffices to show that $\mathcal{C}$ has a lower bound in $\mathcal{D}_K$. We will show that the lower bound $D_\star := \bigcap \mathcal{C}$ of $\mathcal{C}$ belongs to $\mathcal{D}_K$. In order to do so, note already that $D_\star$ is a coherent set of desirable gambles because every element of $\mathcal{C}$ is coherent. To show that $D_\star$ belongs to $\mathcal{D}_K$ – or equivalently, that $K \subseteq K_{D_\star}$ – consider any F in $\mathcal{Q}$ such that $F \notin K_{D_\star}$, implying that $F \cap D_\star = \emptyset$, and we will show that $F \notin K$. From $F \cap D_\star = \emptyset$, it follows that $F \subseteq D_\star^c = \bigcup_{D \in \mathcal{C}} D^c$, so for every gamble f in F there is some set of desirable gambles D_f in $\mathcal{C}$ such that $f \notin D_f$. Defining $D^\star := \bigcap_{f \in F} D_f$, we infer that $F \cap D^\star = \emptyset$, whence $F \notin K_{D^\star}$. But since F is finite and $\mathcal{C}$ is a chain, $D^\star = \bigcap_{f \in F} D_f$ simply is the smallest element of the finite $\{D_f : f \in F\} \subseteq \mathcal{C}$, which therefore belongs to $\mathcal{C} \subseteq \mathcal{D}_K$. This implies that $F \notin K_{\mathcal{D}_K} = K$, where the equality follows from Theorem 3. So $\mathcal{C}$ has a lower bound $D_\star$ that belongs to $\mathcal{D}_K$. Therefore, since the choice of $\mathcal{C} \subseteq \mathcal{D}_K$ was arbitrary, using Zorn's Lemma we know that $\min \mathcal{D}_K$ is non-empty.

We now use this to show that $\mathcal{D}_K = \uparrow \min \mathcal{D}_K$. Because $\min \mathcal{D}_K \subseteq \mathcal{D}_K$, and the fact that $\mathcal{D}_K$ is an upset, we already know that $\mathcal{D}_K \supseteq \uparrow \min \mathcal{D}_K$, so let us prove the converse set inclusion $\mathcal{D}_K \subseteq \uparrow \min \mathcal{D}_K$. To this end, consider any $D^\star$ in $\mathcal{D}_K$, and assume *ex absurdo* that $D' \nsubseteq D^\star$ for every D' in $\min \mathcal{D}_K$. This would imply that the non-empty $\downarrow D^\star := \{D \in \mathcal{D}_K : D \subseteq D^\star\}$ has no minimal element. Indeed, if $\downarrow D^\star$ had a minimal element $D_\star$, then $D_\star$ would also be a minimal element of $\mathcal{D}_K$, because otherwise there would be an element $D' \subset D_\star$ in $\mathcal{D}_K$ which would then also belong to $\downarrow D^\star$.

We apply Zorn's Lemma to the poset $(\downarrow D^\star, \subseteq)$, so consider any non-empty chain $\mathcal{C} \subseteq \downarrow D^\star$. Then $D_\star := \bigcap \mathcal{C}$ is a coherent set of desirable gambles, and by a similar argument as above we find that $K \subseteq K_{D_\star}$. This implies that $D_\star$ belongs to $\downarrow D^\star$, so the chain $\mathcal{C}$ has a lower bound in $\downarrow D^\star$. But since the choice of the chain $\mathcal{C} \subseteq \downarrow D^\star$ was arbitrary, Zorn's Lemma tells us that then $\downarrow D^\star$ has at least one minimal element, contradicting that $\downarrow D^\star$ has no minimal elements. We conclude that it is impossible that $D' \nsubseteq D^\star$ for every D' in $\min \mathcal{D}_K$, and hence indeed $D^\star \in \uparrow \min \mathcal{D}_K$.

We now turn to the last statement, that $K = K_{\min \mathcal{D}_K}$. Using Theorem 3 we know that $K = K_{\mathcal{D}_K}$, and since $\min \mathcal{D}_K \subseteq \mathcal{D}_K$ we know that $K = K_{\mathcal{D}_K} \subseteq K_{\min \mathcal{D}_K}$, so it suffices to show the converse set inclusion $K_{\mathcal{D}_K} \supseteq K_{\min \mathcal{D}_K}$. To this end, consider any F in $K_{\min \mathcal{D}_K}$, meaning that $F \cap D \neq \emptyset$ for every D in $\min \mathcal{D}_K$, and hence also for every D in $\uparrow \min \mathcal{D}_K$. Since we have shown above that $\uparrow \min \mathcal{D}_K = \mathcal{D}_K$, we infer that $F \cap D \neq \emptyset$ for every D in $\mathcal{D}_K$, whence indeed $F \in K_{\mathcal{D}_K}$. □

Running example 2. *The set of minimal elements $\min \mathcal{D}_{K_{\mathcal{M}}}$ of $\mathcal{D}_{K_{\mathcal{M}}}$ is exactly $\{D_p : p \in \mathcal{M}\}$: indeed, recall from the previous instalment of this running example that $\mathcal{D}_{K_{\mathcal{M}}} = \uparrow\{D_p : p \in \mathcal{M}\}$. Clearly $\min \mathcal{D}_{K_{\mathcal{M}}} \subseteq \{D_p : p \in \mathcal{M}\}$, so to show that $\min \mathcal{D}_{K_{\mathcal{M}}} = \{D_p : p \in \mathcal{M}\}$, it suffices to note that no two distinct elements D_p and D_q of $\{D_p : p \in \mathcal{M}\}$ are ordered by $\subseteq$.*

Theorem 5 tells us that every coherent SoDGS K is represented by an antichain – a set with the property that no two distinct elements are ordered by $\subseteq$ – namely $\min \mathcal{D}_K$, but there might be several different antichains that represent the same K. However, we will establish in Proposition 3 later on that any coherent SoDGS with a finite representation has a unique finite antichain that represents it.

Lemma 1. *Consider any coherent SoDGS K with a finite representation $\mathcal{D}$. Then $\min \mathcal{D}$ is also a representation of K.*

Proof. Since $(\mathcal{D}, \subseteq)$ is a *finite* poset, its set of minimal elements $\min \mathcal{D}$ is non-empty, and every D in $\mathcal{D}$ dominates – is a superset of – a minimal element in $\min \mathcal{D}$. Consider any gamble set F and infer that

$$\begin{aligned} F \in K &\Leftrightarrow (\forall D \in \mathcal{D}) F \cap D \neq \emptyset \\ &\Leftrightarrow (\forall D \in \min \mathcal{D}) F \cap D \neq \emptyset \Leftrightarrow F \in K_{\min \mathcal{D}}, \end{aligned}$$

where the first equivalence follows since $\mathcal{D}$ represents K, and the second equivalence since every element of $\mathcal{D}$ is a superset of an element of $\min \mathcal{D}$. Since the choice of F was arbitrary, this implies that $K = K_{\min \mathcal{D}}$, and hence, indeed, $\min \mathcal{D}$ represents K. □

Proposition 2. *Consider any coherent SoDGS K. Then K has a finite representation if and only if $\min \mathcal{D}_K$ is finite.*

Proof. Sufficiency follows at once from the fact that $\min \mathcal{D}_K$ is a representation of K, established in Theorem 5.

For necessity, assume that K has a finite representation $\mathcal{D}$. Then by Lemma 1 $\min \mathcal{D}$, which is a subset of $\mathcal{D}$ and hence finite, also represents K. We show that $\min \mathcal{D}_K \subseteq \min \mathcal{D}$, ensuring that the former is finite. To do so, consider any D in $\overline{\mathcal{D}}$ such that $D \notin \min \mathcal{D}$, and we show that $D \notin \min \mathcal{D}_K$. If $D \supseteq D'$ for some $D' \in \min \mathcal{D} \subseteq \mathcal{D}_K$, then $D \neq D'$ and hence D is not a minimal element of the poset $(\mathcal{D}_K, \subseteq)$, so $D \notin \min \mathcal{D}_K$ and we are done. So assume $(\forall D' \in \min \mathcal{D}) D' \nsubseteq D$, so $(\forall D' \in \min \mathcal{D})(\exists f_{D'} \in D') f_{D'} \notin D$. Collect all these $f_{D'} \in D' \setminus D$ in the set $F := \{f_{D'} : D' \in \min \mathcal{D}\}$ which is finite and therefore a valid gamble set. Then $F \cap D' \neq \emptyset$ for every $D' \in \min \mathcal{D}$, so $F \in K$ because $\min \mathcal{D}$ represents K, as established earlier. But $F \cap D = \emptyset$, so $F \notin K_D$ and hence $K \nsubseteq K_D$, whence $D \notin \mathcal{D}_K$, and therefore in particular $D \notin \min \mathcal{D}_K$, indeed. □

The main importance, for our purpose, of SoDGSes with a finite representation lies in the following property.

Proposition 3. *Consider two coherent SoDGSes K_1 and K_2 with finite representation $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Then $K_1 = K_2$ if and only if $\min \mathcal{D}_1 = \min \mathcal{D}_2$.*

Proof. For necessity, assume that $\min \mathcal{D}_1 \neq \min \mathcal{D}_2$, and we show that $K_1 \neq K_2$. That $\min \mathcal{D}_1 \neq \min \mathcal{D}_2$ implies that $\min \mathcal{D}_1 \nsubseteq \min \mathcal{D}_2$ or $\min \mathcal{D}_1 \nsupseteq \min \mathcal{D}_2$ – we will assume that $\min \mathcal{D}_1 \nsubseteq \min \mathcal{D}_2$; the other case will follow by a very similar argument. This implies that there is some D_1 in $\min \mathcal{D}_1$ such that $D_1 \neq D_2$ for all D_2 in $\min \mathcal{D}_2$.

If there is some D_2 in $\min \mathcal{D}_2$ such that $D_2 \subseteq D_1$, then also $D_2 \subset D_1$, and hence also $D'_1 \nsubseteq D_2$ for all $D'_1 \in \min \mathcal{D}_1$ since $\min \mathcal{D}_1$ is an antichain. So for every D'_1 in $\min \mathcal{D}_1$ there is a gamble $f_{D'_1}$ in $D'_1 \setminus D_2$. Collect all these gambles in $F := \{f_{D'_1} : D'_1 \in \min \mathcal{D}_1\}$ which is finite and hence a valid gamble set. Then $F \cap D'_1 \neq \emptyset$ for every D'_1 in $\min \mathcal{D}_1$, so $F \in K_1$ because $\min \mathcal{D}_1$ represents K_1 by Lemma 1. Also, $F \cap D_2 = \emptyset$, so $F \notin K_2$ because $\mathcal{D}_2$ represents K_2, and hence $K_1 \neq K_2$ and we are done.

So assume that $D_2 \nsubseteq D_1$ for all D_2 in $\mathcal{D}_2$. Then for every D_2 in $\min \mathcal{D}_2$ there is a gamble f_{D_2} in $D_2 \setminus D_1$. Collect all these gambles in $F := \{f_{D_2} : D_2 \in \min \mathcal{D}_2\}$ which is finite and hence a valid gamble set. Then $F \cap D_2 \neq \emptyset$ for every D_2 in $\min \mathcal{D}_2$, so $F \in K_2$ because $\min \mathcal{D}_2$ represents K_2 by Lemma 1. Also, $F \cap D_1 = \emptyset$, so $F \notin K_1$ because $\mathcal{D}_1$ represents K_1 and hence, indeed, $K_1 \neq K_2$.

For sufficiency, assume that $\min \mathcal{D}_1 = \min \mathcal{D}_2$. Since $\mathcal{D}_1$ (finitely) represents K_1 and $\mathcal{D}_2$ (finitely) represents K_2, use Lemma 1 to infer that also $\min \mathcal{D}_1$ represents K_1, and, similarly, $\min \mathcal{D}_2$ represents K_2. But since $\min \mathcal{D}_1 = \min \mathcal{D}_2$ we find that, indeed, $K_1 = K_2$. □

Proposition 3 implies that the coherent SoDGSes with finite representation are *in a one-to-one relation with the finite antichains in $\overline{\mathcal{D}}$*. Lemma 1, and therefore also Proposition 3, fail to hold for infinite representations $\mathcal{D}$, as the following example shows.

Example 2. Consider the vacuous SoDGS K_v. We show that each of the three sets $\mathcal{D}_1 := \overline{\mathcal{D}}$, $\mathcal{D}_2 := \{D_v\}$ and $\mathcal{D}_3 := \overline{\mathcal{D}} \setminus \{D_v\}$ represents K_v. To see this, observe that $\mathcal{D}_2$ represents K_v by definition, and then since $\mathcal{D}_1 = \uparrow\mathcal{D}_2$, also $\mathcal{D}_1$ represents K_v. To show that $\mathcal{D}_3$ also represents K_v, since $\mathcal{D}_3 \subseteq \mathcal{D}_1$ it suffices to show that $K_{\mathcal{D}_3} \subseteq K_v$. To this end, consider any $F \notin K_v$, and we will infer that $F \notin K_D$ for some D in $\mathcal{D}_3$. That $F \notin K_v$ implies $F \cap \mathcal{L}_{>0} = \emptyset$. If $F \subseteq \mathcal{L}_{\leq 0}$ then F cannot intersect any coherent set of desirable gambles D: if it did, then there is some $f \in D$ such that $f \leq 0$. By Axiom 1, $f \neq 0$, so $-f > 0$ and therefore $-f \in D$ by Axiom 1, whence $0 = f - f \in D$ by Axiom 1, contradicting Axiom 1. So $F \notin K_D$ for any coherent D and we are done, so assume that F contains at least one gamble in $\mathcal{L} \setminus (\mathcal{L}_{>0} \cup \mathcal{L}_{\leq 0})$. The idea is now to consider the gamble (or one of them) $f \in F$ whose ray is closest to $\mathcal{L}_{>0}$, which exists since F is finite. Then $f(x) < 0$ for some x, and by letting $\epsilon := f(x)/2$ we find that $f^\star := f + \epsilon \notin \mathcal{L}_{>0}$. $f^\star$ is contained on a ray that is (strictly) closer to $\mathcal{L}_{>0}$ than the ray through any gamble in F. Consider now $D := \mathrm{cl}_{\overline{\mathcal{D}}}(\{f^\star\}) \in \mathcal{D}_3$, whose rays all are closer to $\mathcal{L}_{>0}$ than the rays of gambles in F. This implies that $F \cap D = \emptyset$, whence, indeed, $F \notin K_D$.

So we see that the representation of K_v is not unique: in this case, as $\mathcal{D}_2 = \mathcal{D}_1 \setminus \mathcal{D}_3$, there is even a partition $\{\mathcal{D}_2, \mathcal{D}_3\}$ of representations. The representation $\mathcal{D}_3$ does not have minimal elements, demonstrating that the requirement that $\mathcal{D}$ is a finite representation in Lemma 1 is not superfluous.

Note that K_v has a finite representation, so Proposition 3 predicts that it has a unique finite antichain that represents it, which in this case is $\{D_v\}$.

4. Multivariate sets of desirable gamble sets

This section introduces the notation and concepts necessary for multivariate SoDGSes. It is heavily based on the work of [8][Section 5]. Along the way, we add some specialised results for SoDGSes that admit a finite representation, needed for Theorem 7.

4.1. Preliminaries

Consider $n \in \mathbb{N}$ uncertain variables $X_1, \ldots, X_n$, each assuming values in the finite possibility spaces $\mathcal{X}_1, \ldots, \mathcal{X}_n$, respectively. For brevity, we use the notation $N := \{1, \ldots, n\}$ for the global index set. To express beliefs – sets of desirable gambles or SoDGSes – about the uncertain variables $X_1, \ldots, X_n$ together, we consider gambles on the Cartesian product $\mathcal{X} := \bigtimes_{k=1}^n \mathcal{X}_k$, which belong to the $\prod_{k=1}^n |\mathcal{X}_k|$-dimensional linear space $\mathcal{L}(\mathcal{X})$.

For any subset $I \subseteq N$ of indices, we let X_I be the tuple of uncertain variables that assumes values in $\mathcal{X}_I := \bigtimes_{k \in I} \mathcal{X}_k$. If $I = \emptyset$, then $\mathcal{X}_\emptyset$ contains only one element: the empty map. In this case, there is no uncertainty about the variable $X_\emptyset$.

For any subset $I \subseteq N$, any gamble f on $\mathcal{X}$ and any $x_I \in \mathcal{X}_I$, we can regard the partial map $f(x_I, \bullet)$ as a gamble on $\mathcal{X}_{I^c}$, where we let $I^c := N \setminus I$ be the indices outside I. Conversely, it will be useful to relate a gamble f on $\mathcal{X}_I$ to a gamble on $\mathcal{X}$.

Definition 4 (Cylindrical extension). Given two disjoint subsets I and J of N and any gamble f on $\mathcal{X}_I$, we let its *cylindrical extension* f^J to $\mathcal{X}_{I \cup J}$ be defined by $f^J(x_I, x_J) := f(x_I)$ for all x_I in $\mathcal{X}_I$ and x_J in $\mathcal{X}_J$. Similarly, given any set of gambles $F \subseteq \mathcal{L}(\mathcal{X}_I)$, we let its *cylindrical extension* $F^J \subseteq \mathcal{L}(\mathcal{X}_{I \cup J})$ be defined as $F^J := \{f^J : f \in F\}$.

Formally, f belongs to $\mathcal{L}(\mathcal{X}_I)$ while f^J belongs to $\mathcal{L}(\mathcal{X}_{I\cup J})$. However, for any x_I in $\mathcal{X}_I$, the partial map $f^J(x_I,\bullet) \in \mathcal{L}(\mathcal{X}_J)$ is actually constant, so f^J depends only on the value of X_I. So we see that f and f^J are indistinguishable from a behavioural point of view.

Remark 1. We do *not* notationally distinguish between f and f^J, and identify a gamble f on $\mathcal{X}_I$ with its cylindrical extension f^J [8,24].

Using this convention allows us to regard $\mathcal{L}(\mathcal{X}_I)$ as a subset of $\mathcal{L}(\mathcal{X}_{I\cup J})$ and, similarly, $\mathcal{Q}(\mathcal{X}_I)$ as a subset of $\mathcal{Q}(\mathcal{X}_{I\cup J})$.

4.2. Marginalisation

Assume that we have an SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$ that models the agent's beliefs about the uncertain variable X_N. We are interested in the information about X_S present in K, where $S \subseteq N$. This information can be obtained by collecting the gamble sets that belong to K but depend only on X_S.

Definition 5 (Marginalisation [9]). For any SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$ and any $S \subseteq N$, its *S-marginal* $\operatorname{Marg}_S K \subseteq \mathcal{Q}(\mathcal{X}_S)$ is defined as

$$\operatorname{Marg}_S K := K \cap \mathcal{Q}(\mathcal{X}_S).$$

Note that marginalisation preserves the order: if $K_1 \subseteq K_2$ then $\operatorname{Marg}_S K_1 \subseteq \operatorname{Marg}_S K_2$ [9][Section 4.2]. Note also that this marginalisation operator satisfies the usual requirement of successive marginalisation: for all $S_1 \subseteq S_2 \subseteq N$.

$$\operatorname{Marg}_{S_1}(\operatorname{Marg}_{S_2} K) = K \cap \mathcal{Q}(\mathcal{X}_{S_2}) \cap \mathcal{Q}(\mathcal{X}_{S_1}) = K \cap \mathcal{Q}(\mathcal{X}_{S_1}) = \operatorname{Marg}_{S_1} K. \tag{1}$$

This definition of marginalisation generalises the one for sets of desirable gambles, in that $\operatorname{Marg}_S K_D = K_{\operatorname{marg}_S D}$ [9][Proposition 10], where $\operatorname{marg}_S D := D \cap \mathcal{L}(\mathcal{X}_S)$ for all $D \subseteq \mathcal{L}(\mathcal{X})$, as defined by [24]. It will be convenient to lift the operator marg_S on $\wp(\mathcal{L}(\mathcal{X}))$ to a version on $\wp(\wp(\mathcal{L}(\mathcal{X})))$, defined by $\operatorname{marg}_S \mathcal{D} := \{\operatorname{marg}_S D : D \in \mathcal{D}\}$ for every $\mathcal{D} \subseteq \wp(\mathcal{L}(\mathcal{X}))$.

Proposition 4. *[8][Proposition 11] Consider any coherent SoDGS K, any representation $\mathcal{D}$ of it, and any $S \subseteq N$. Then* $\operatorname{Marg}_S K$ *is coherent. Moreover,* $\operatorname{Marg}_S K$ *is represented by* $\operatorname{marg}_S \mathcal{D}$*, meaning that* $\operatorname{Marg}_S K = K_{\operatorname{marg}_S \mathcal{D}}$.

Running example 3. *Suppose that the coherent SoDGS $K_{\mathcal{M}}$ is determined by a finite set $\mathcal{M}$ of pmfs on $\mathcal{X}_1 \times \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are two finite possibility spaces. Recall from Running Example 1 that $K_{\mathcal{M}}$ is represented by $\{D_p : p \in \mathcal{M}\}$. Using Proposition 4, we infer that* $\operatorname{Marg}_1 K_{\mathcal{M}}$ *is represented by* $\operatorname{marg}_1\{D_p : p \in \mathcal{M}\}$*. In this instalment, we set out to find another representation of* $\operatorname{Marg}_1 K_{\mathcal{M}}$ *that is more direct in terms of the marginal pmfs on $\mathcal{X}_1$.*

Note that

$$\operatorname{marg}_1 D_p = D_p \cap \mathcal{L}(\mathcal{X}_1) = \mathcal{L}_{>0}(\mathcal{X}_1) \cup \{f \in \mathcal{L}(\mathcal{X}_1) : E_p(f) > 0\}$$

and since $E_p(f) = \sum_{x_1\in\mathcal{X}_1, x_2\in\mathcal{X}_2} p(x_1,x_2) f(x_1) = E_{p_{X_1}}(f)$, where $p_{X_1} := \sum_{x_2\in\mathcal{X}_2} p(\bullet, x_2)$ is the X_1-marginal of p, we find that

$$\operatorname{marg}_1 D_p = D_{p_{X_1}},$$

and hence that $\operatorname{marg}_1\{D_p : p \in \mathcal{M}\} = \{D_{p_{X_1}} : p \in \mathcal{M}\} = \{D_p : p \in \operatorname{marg}_1 \mathcal{M}\}$*, where* $\operatorname{marg}_1 \mathcal{M} := \{p_{X_1} : p \in \mathcal{M}\}$ *is the X_1-marginal of $\mathcal{M}$. So we have found that* $\{D_p : p \in \operatorname{marg}_1 \mathcal{M}\}$ *represents* $\operatorname{Marg}_1 K_{\mathcal{M}}$*, and hence that* $\operatorname{Marg}_1 K_{\mathcal{M}} = K_{\operatorname{marg}_1 \mathcal{M}}$.

Theorem 3 and Proposition 4 imply that $\operatorname{marg}_S \mathcal{D}_K$ is a representation of $\operatorname{Marg}_S K$, but it does not imply that $\operatorname{marg}_S \mathcal{D}_K$ is equal to the unique largest representation $\mathcal{D}_{\operatorname{Marg}_S K}$ of $\operatorname{Marg}_S K$. However, if K has a finite representation then this turns out to be the case, which will be a useful property further on. This is what we now set out to prove in the rest of this section. We need the following two technical lemmas, because they connect a coherent set of desirable gambles with one on a smaller domain.

Lemma 2. *Consider any $S \subseteq N$, any coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X})$ and any coherent set of desirable gambles $D_S \subseteq \mathcal{L}(\mathcal{X}_S)$ such that* $\operatorname{marg}_S D \subseteq D_S$*. Then* $D^\star := \operatorname{posi}(D \cup D_S)$ *is a coherent set of desirable gambles, that marginalises to D_S, in the sense that* $\operatorname{marg}_S D^\star = D_S$.

Proof. For the first statement, it suffices to show that $0 \notin \operatorname{posi}(D \cup D_S)$, taking into account Theorem 1 and the fact that $\mathcal{L}_{>0}(\mathcal{X}) \subseteq D$. To this end, assume *ex absurdo* that $0 \in \operatorname{posi}(D \cup D_S)$, so we would find f in D and g in D_S such that $f + g = 0$, taking into account the coherence of D and D_S. But then $f = -g$, and hence f would belong to $\mathcal{L}(\mathcal{X}_S)$, since g belongs to $\mathcal{L}(\mathcal{X}_S)$. So we would find that $f \in D \cap \mathcal{L}(\mathcal{X}_S) = \operatorname{marg}_S D$, and hence, since $\operatorname{marg}_S D \subseteq D_S$, also that $-g = f \in D_S$. But also $g \in D_S$, contradicting D_S's coherence: Axiom 1 implies that $0 = g - g \in D_S$, which contradicts Axiom 1.

For the second statement, note that $D_S \subseteq D^\star$ by construction, whence $D_S \subseteq \operatorname{marg}_S D^\star$, so it suffices to show that $\operatorname{marg}_S D^\star \subseteq D_S$. To this end, consider any f in $\operatorname{marg}_S D^\star$, so that $f = g + g_S$ for some $g \in D \cup \{0\}$ and $g_S \in D_S \cup \{0\}$, taking the coherence of D and D_S into account. Since both f and g_S belong to $\mathcal{L}(\mathcal{X}_S)$, so does g, and hence $g \in \operatorname{marg}_S D \cup \{0\} \subseteq D_S \cup \{0\}$. So we find that $f = g + g_S$ for some g and g_S in $D_S \cup \{0\}$, so by coherence [more specifically, Axiom 1] we infer that $f \in D_S \cup \{0\}$, and, taking the coherence of $D^\star$ into account, which we have established above, even that $f \in D_S$, indeed. $\square$

Lemma 3. *Consider any coherent SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$, any $S \subseteq N$, and any coherent set of desirable gambles $D_S \subseteq \mathcal{L}(\mathcal{X}_S)$. Then $D_S \in$* $\operatorname{marg}_S \mathcal{D}_K \Leftrightarrow (\exists D \in \mathcal{D}_K) \operatorname{marg}_S D \subseteq D_S$.

Proof. For necessity, assume that $D_S \in \operatorname{marg}_S \mathcal{D}_K$, implying that $\operatorname{marg}_S D = D_S$ – and hence indeed $\operatorname{marg}_S D \subseteq D_S$ – for some $D \in \mathcal{D}_K$.

For sufficiency, assume that $\operatorname{marg}_S D \subseteq D_S$ for some $D \in \mathcal{D}_K$, and let $D^\star := \operatorname{posi}(D \cup D_S)$. Then Lemma 2 tells us that $D^\star$ is coherent and marginalises to D_S. Moreover, $D^\star \supseteq D$ by construction, so $D^\star$ also belongs to $\mathcal{D}_K$. So we have found $D^\star$ in $\mathcal{D}_K$ such that $\operatorname{marg}_S D^\star = D_S$, whence, indeed, $D_S \in \operatorname{marg}_S \mathcal{D}_K$. □

This allows us to establish a property connecting the marginalisation on SoDGSes with their largest representations, when they obtain a finite representation.

Proposition 5. *Consider any coherent SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$ with a finite representation, and any $S \subseteq N$. Then $\operatorname{Marg}_S K$ has a finite representation too, and $\operatorname{marg}_S \mathcal{D}_K = \mathcal{D}_{\operatorname{Marg}_S K}$.*

Proof. For the first statement, use Theorem 3 to infer that $\operatorname{Marg}_S K$ is represented by $\mathcal{D}_{\operatorname{Marg}_S K}$. Since K has a finite representation, it is represented by a finite set $\mathcal{D}$, so Proposition 4 tells us that $\operatorname{Marg}_S K$ is represented by the set $\operatorname{marg}_S \mathcal{D}$ which is finite, implying that $\operatorname{Marg}_S K$ has a finite representation, too.

For the second statement, Proposition 4 tells us that $\operatorname{Marg}_S K$ is represented by $\operatorname{marg}_S \mathcal{D}_K$, which then is necessarily a subset of $\mathcal{D}_{\operatorname{Marg}_S K}$ by Theorem 3. It therefore suffices to show that $\mathcal{D}_{\operatorname{Marg}_S K} \subseteq \operatorname{marg}_S \mathcal{D}_K$. To this end, we consider any D_S in $\overline{\mathcal{D}}(\mathcal{X}_S)$ such that $D_S \notin \operatorname{marg}_S \mathcal{D}_K$, and will show that then $D_S \notin \mathcal{D}_{\operatorname{Marg}_S K}$. Use Lemma 3 to infer that $\operatorname{marg}_S D \nsubseteq D_S$ for all D in $\mathcal{D}_K$, and therefore, for the finite representation $\mathcal{D}$ of K, in particular $(\forall D \in \mathcal{D}) \operatorname{marg}_S D \nsubseteq D_S$. This implies that, for every D in $\mathcal{D}$, there is some $f_D \in \operatorname{marg}_S D$ such that $f_D \notin D_S$. Collect all these gambles f_D in $F := \{f_D : D \in \mathcal{D}\}$, which is finite because $\mathcal{D}$ is, and therefore a valid gamble set on $\mathcal{X}_S$. Then $f_D \in D$ – and therefore $F \cap D \neq \emptyset$ – for every D in $\mathcal{D}$, so $F \in K_{\mathcal{D}} = K$, where we took into account that $\mathcal{D}$ is a representation of K. Moreover, since $F \in \mathcal{Q}(\mathcal{X}_S)$, we find that $F \in K \cap \mathcal{Q}(\mathcal{X}_S) = \operatorname{Marg}_S K$. Since $f_D \notin D_S$ for every D in $\mathcal{D}$, we find also that $F \cap D_S = \emptyset$, so $F \notin K_{D_S}$ and therefore $\operatorname{Marg}_S K \nsubseteq K_{D_S}$, whence, indeed, $D_S \notin \mathcal{D}_{\operatorname{Marg}_S K}$. □

Running example 4. *Using Running Example 3, we know that $\{D_p : p \in \operatorname{marg}_1 \mathcal{M}\}$ represents $\operatorname{Marg}_1 K_{\mathcal{M}}$. Taking into account Proposition 5 and Running Example 1, we infer that $\operatorname{Marg}_1 K_{\mathcal{M}}$'s largest representation $\mathcal{D}_{\operatorname{Marg}_1 K_{\mathcal{M}}}$ is equal to $\uparrow\{D_p : p \in \operatorname{marg}_1 \mathcal{M}\} = \uparrow \operatorname{marg}_1\{D_p : p \in \mathcal{M}\}$.*

5. The marginal problem

Let us now turn to the main topic of this paper: the marginal problem. Suppose that we are given $m \in \mathbb{N}$ coherent SoDGSes $K_\ell \subseteq \mathcal{Q}(\mathcal{X}_{S_\ell})$ for some non-empty index sets $S_\ell \subseteq N$, where these index sets may overlap. The marginal problem is this:

> "When is there a joint SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$ that marginalises to the given SoDGSes K_1, …, K_m?"

This question has been solved for sets of desirable gambles by miranda2020, who also discussed an interesting connection with information algebras. This connection has been further developed by casanova2023,casanova2022383. In this work, we build on these results to provide a solution for SoDGSes. At this point, we would like to stress that the marginal problem, as we understand it, is a *satisfiability problem*, which naturally arises in different subfields in artificial intelligence – we refer to [3][Section 1] for an overview.

5.1. Pairwise compatibility and compatibility

Let us first make this problem more precise. We generalise [3][Definitions 9 and 10] from sets of desirable gambles to the current setting.

Definition 6 (Pairwise compatibility). Two coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$ and $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$ are called *pairwise compatible* if

$$\operatorname{Marg}_{S_1 \cap S_2} K_1 = \operatorname{Marg}_{S_1 \cap S_2} K_2.$$

The m coherent SoDGSes $K_\ell \subseteq \mathcal{Q}(\mathcal{X}_{S_\ell})$, $\ell \in \{1, \dots, m\}$, are called *pairwise compatible* if any two of them are pairwise compatible. Similarly, two coherent sets of desirable gambles $D_1 \subseteq \mathcal{L}(\mathcal{X}_{S_1})$ and $D_2 \subseteq \mathcal{L}(\mathcal{X}_{S_2})$ are *pairwise compatible* if $\operatorname{marg}_{S_1 \cap S_2} D_1 = \operatorname{marg}_{S_1 \cap S_2} D_2$, and m coherent sets of desirable gambles $D_\ell \subseteq \mathcal{Q}(\mathcal{X}_{S_\ell})$, $\ell \in \{1, \dots, m\}$, are called *pairwise compatible* if any two of them are pairwise compatible.

In order to conclude that K_1 and K_2 are pairwise compatible, it suffices that the marginalisations of their representations coincide.

Proposition 6. *Consider two coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$ and $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$. If*

$$\operatorname{marg}_{S_1 \cap S_2} \mathcal{D}_{K_1} = \operatorname{marg}_{S_1 \cap S_2} \mathcal{D}_{K_2}$$

then they are pairwise compatible.

Proof. Use Proposition 4 to infer that $\operatorname{marg}_{S_1 \cap S_2} \mathcal{D}_{K_1}$ represents $\operatorname{Marg}_{S_1 \cap S_2} K_1$, and, similarly, that $\operatorname{marg}_{S_1 \cap S_2} \mathcal{D}_{K_2}$ represents $\operatorname{Marg}_{S_1 \cap S_2} K_2$. Since these two representations coincide, necessarily also $\operatorname{Marg}_{S_1 \cap S_2} K_1 = \operatorname{Marg}_{S_1 \cap S_2} K_2$, so that K_1 and K_2 are pairwise compatible, indeed. □

Moreover, it turns out that for coherent SoDGSes with a finite representation, this sufficient condition is also necessary, making it an equivalent property to pairwise compatibility.

Proposition 7. *Consider two coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$ and $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$ with a finite representation. Then they are pairwise compatible if and only if*

$$\operatorname{marg}_{S_1\cap S_2} \mathcal{D}_{K_1} = \operatorname{marg}_{S_1\cap S_2} \mathcal{D}_{K_2}.$$

Proof. Sufficiency is, in a more general context, established in Proposition 6, so we focus on necessity. Use Proposition 5 to infer that $\operatorname{Marg}_{S_1\cap S_2} K_1$ and $\operatorname{Marg}_{S_1\cap S_2} K_2$ have a finite representation. Using the pairwise compatibility of K_1 and K_2 we infer that $\operatorname{marg}_{S_1\cap S_2} K_1 = \operatorname{marg}_{S_1\cap S_2} K_2$ and therefore $\mathcal{D}_{\operatorname{Marg}_{S_1\cap S_2} K_1} = \mathcal{D}_{\operatorname{Marg}_{S_1\cap S_2} K_2}$. Since $\operatorname{Marg}_{S_1\cap S_2} K_1 = \operatorname{Marg}_{S_1\cap S_2} K_2$ has a finite representation, we can use Proposition 5 to infer that then, indeed, $\operatorname{marg}_{S_1\cap S_2} \mathcal{D}_{K_1} = \operatorname{marg}_{S_1\cap S_2} \mathcal{D}_{K_2}$. □

Another way to interpret Proposition 7 is that K_1 and K_2 are pairwise compatible if and only if every D_1 in $\mathcal{D}_{K_1}$ has a pairwise compatible D_2 in $\mathcal{D}_{K_2}$, and every D_2 in $\mathcal{D}_{K_2}$ has a pairwise compatible D_1 in $\mathcal{D}_{K_1}$.

Running example 5. *Consider finite possibility spaces $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$, and two coherent SoDGSes $K_{\mathcal{M}}$ and $K_{\mathcal{M}'}$, where $\mathcal{M}$ is a finite set of pmfs on $\mathcal{X}_1 \times \mathcal{X}_2$, and $\mathcal{M}'$ is a finite set of pmfs on $\mathcal{X}_2 \times \mathcal{X}_3$. By Proposition 7, $K_{\mathcal{M}}$ and $K_{\mathcal{M}'}$ are pairwise compatible if and only if*

$$\operatorname{marg}_2(\uparrow\{D_p : p \in \mathcal{M}\}) = \operatorname{marg}_2(\uparrow\{D_q : q \in \mathcal{M}'\}),$$

where we recalled the largest representations of $K_{\mathcal{M}}$ and $K_{\mathcal{M}'}$ found in Running Example 1. Taking into account Running Example 4, we find that this is equivalent to $\uparrow\{D_p : p \in \operatorname{marg}_2 \mathcal{M}\} = \uparrow\{D_q : q \in \operatorname{marg}_2 \mathcal{M}'\}$, and therefore also to

$$\operatorname{marg}_2 \mathcal{M} = \operatorname{marg}_2 \mathcal{M}'.$$

This tells us that pairwise compatibility of E-admissible SoDGSes is equivalent to the requirement that their representing sets $\mathcal{M}$ and $\mathcal{M}'$ of pmfs marginalise to the same set $\operatorname{marg}_2 \mathcal{M} = \operatorname{marg}_2 \mathcal{M}'$.

We also need a global version of compatibility, which is the requirement in the marginal problem.

Definition 7 (Compatibility)**.** The $m \geq 1$ coherent SoDGSes $K_\ell \subseteq \mathcal{Q}(\mathcal{X}_{S_\ell})$, $\ell \in \{1,\dots,m\}$, are called *compatible* if there is a coherent SoDGS $K \subseteq \mathcal{Q}(\mathcal{X})$ that is pairwise compatible with each of them, in the sense that $\operatorname{Marg}_{S_\ell} K = K_\ell$ for every ℓ in $\{1,\dots,m\}$. We then also call K *compatible with* K_1, …, K_m. Similarly, the m coherent sets of desirable gambles $D_\ell \subseteq \mathcal{L}(\mathcal{X}_{S_\ell})$, $\ell \in \{1,\dots,m\}$, are called *compatible* if there is a coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X})$ that is pairwise compatible with each of them, in the sense that $\operatorname{marg}_{S_\ell} D = D_\ell$ for every ℓ in $\{1,\dots,m\}$. We then also call D *compatible with* D_1, …, D_m.

The following result is directly inspired by [3][Proposition 1], and generalises it to SoDGSes. It provides an equivalent condition to compatibility.

Proposition 8. *Consider any coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, …, $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$. Then they are compatible if and only if the natural extension $K := \operatorname{cl}_{\overline{\mathcal{K}}}\big(\bigcup_{\ell\leq m} K_\ell\big)$ is compatible with them, or, in other words, if and only if $\operatorname{Marg}_{S_i} K = K_i$ for every i in $\{1,\dots,m\}$. As a consequence, if K_1, …, K_m are compatible, then K is coherent.*

Proof. We start with the first statement. For necessity, consider any i in $\{1,\dots,m\}$. Since K_1, …, K_m are compatible, there is a coherent SoDGS $K^\star \subseteq \mathcal{Q}(\mathcal{X})$ such that, for all $\ell \in \{1,\dots,m\}$, $\operatorname{Marg}_{S_\ell} K^\star = K_\ell$, or equivalently, $K^\star \cap \mathcal{Q}(\mathcal{X}_{S_\ell}) = K_\ell$. Hence $\bigcup_{\ell\leq m} K_\ell \subseteq K^\star$, whence also $K_i \subseteq \operatorname{cl}_{\overline{\mathcal{K}}}\big(\bigcup_{\ell\leq m} K_\ell\big) = K \subseteq \operatorname{cl}_{\overline{\mathcal{K}}}(K^\star) = K^\star$, where we used the facts that $\operatorname{cl}_{\overline{\mathcal{K}}}$ is a closure operator and that $K^\star$ is coherent. Taking into account that $\operatorname{Marg}_{S_i}$ preserves the order, we obtain that $K_i = \operatorname{Marg}_{S_i} K_i \subseteq \operatorname{Marg}_{S_i} K \subseteq \operatorname{Marg}_{S_i} K^\star = K_i$, whence, indeed, $\operatorname{Marg}_{S_i} K = K_i$.

For sufficiency, and for the second statement, it suffices to show that K is coherent, because then it serves as a coherent SoDGS on $\mathcal{X}$ that is compatible with K_1, …, K_m. To this end, taking into account Theorem 2 with $\mathcal{A} = K$, it suffices to show that $\emptyset \notin K$ and $\{0\} \notin K$. For every i in $\{1,\dots,m\}$, note that $\emptyset \notin K_i$ and $\{0\} \notin K_i$ by K_i's coherence. Since $\emptyset$ and $\{0\}$ belong to $\mathcal{Q}(\mathcal{X}_{S_i})$ for every i in $\{1,\dots,m\}$, we find by the compatibility of K with K_1, …, K_m that, indeed, $\emptyset \notin K$ and $\{0\} \notin K$. □

Consider any coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, …, $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$ that are compatible. Proposition 8 then tells us that the *smallest* coherent SoDGS on $\mathcal{X}$ that is compatible with them is their natural extension $K := \operatorname{cl}_{\overline{\mathcal{K}}}\big(\bigcup_{\ell\leq m} K_\ell\big)$: indeed, any coherent SoDGS $\hat{K} \subseteq \mathcal{Q}(\mathcal{X})$ that is compatible with K_1, …, K_m must contain $\bigcup_{\ell\leq m} K_\ell$, and therefore $K = \operatorname{cl}_{\overline{\mathcal{K}}}\big(\bigcup_{\ell\leq m} K_\ell\big) \subseteq \operatorname{cl}_{\overline{\mathcal{K}}}(\hat{K}) = \hat{K}$, taking into account that $\operatorname{cl}_{\overline{\mathcal{K}}}$ is a closure operator, and that $\hat{K}$ is coherent. This coherent SoDGSD K on $\mathcal{X}$ will be important in what follows, and we call it the *smallest compatible joint (of K_1, …, K_m)*.

Proposition 8 can be rephrased in terms of the largest representations $\mathcal{D}_{K_\ell}$ of the given SoDGSes K_ℓ, with ℓ in $\{1,\dots,m\}$.

Proposition 9. *Consider any coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, …, $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$. Then they are compatible if and only if there is a coherent $K \subseteq \mathcal{Q}(\mathcal{X})$ such that $\operatorname{marg}_{S_\ell} \mathcal{D}_K = \mathcal{D}_{K_\ell}$ for every ℓ in $\{1,\dots,m\}$.*

Proof. Infer the following equivalences:

$$\begin{aligned} K_1,\dots,K_m \text{ are compatible} &\Leftrightarrow (\exists K \in \overline{\mathcal{K}}(\mathcal{X}))(\forall \ell \in \{1,\dots,m\}) \operatorname{Marg}_{S_\ell} K = K_\ell \\ &\Leftrightarrow (\exists K \in \overline{\mathcal{K}}(\mathcal{X}))(\forall \ell \in \{1,\dots,m\}) \mathcal{D}_{\operatorname{Marg}_{S_\ell} K} = \mathcal{D}_{K_\ell} \\ &\Leftrightarrow (\exists K \in \overline{\mathcal{K}}(\mathcal{X}))(\forall \ell \in \{1,\dots,m\}) \operatorname{marg}_{S_\ell} \mathcal{D}_K = \mathcal{D}_{K_\ell}, \end{aligned}$$

where we used Definition 7 in the first equivalence, Theorem 3 in the second one, and Proposition 4 in the last one. □

Proposition 9 implies that the largest representations $\mathcal{D}_{K_\ell}$ of the compatible SoDGSes K_ℓ, with ℓ in $\{1,\dots,m\}$, must be compatible in the sense that there is a coherent $K \subseteq \mathcal{Q}(\mathcal{X})$ that produces each of them, in that $\operatorname{marg}_{S_\ell} \mathcal{D}_K = \mathcal{D}_{K_\ell}$.

5.2. A sufficient condition for compatibility: the running intersection property

We are now in a position to establish the main result of this paper. When we are given coherent SoDGSes that are pairwise compatible, under what condition are they compatible? In the special case of probability measures, Beeri et al. [36] showed that a sufficient condition is that the index sets satisfy the *running intersection property* [cf. also] [37]. Miranda and Zaffalon [3][Theorem 2] have established that this is also sufficient for the compatibility of sets of desirable gambles, and it will play an important role in the present setting, too.

Definition 8 (Running intersection property [3][Definition 11]). The index sets S_1, …, S_m satisfy the *running intersection property* when

$$(\forall \ell \in \{2,\dots,m\})(\exists i^\star < \ell) S_\ell \cap S_{i^\star} = S_\ell \cap \textstyle\bigcup_{i<\ell} S_i. \tag{RIP}$$

It may happen that S_1, …, S_m do not satisfy the running intersection property, while a reordering $S_{\sigma(1)}$, …, $S_{\sigma(m)}$, with σ a permutation of $\{1,\dots,m\}$, does. To keep our exposition simple, we assume from here on that the index sets S_1, …, S_m are ordered in such a way that they satisfy the running intersection property, whenever this is possible. This is not a substantial constraint: if they are not in this order, then simply reorder them, and the following results will continue to hold. Note that if $m = 2$, then the index sets S_1 and S_2 satisfy the running intersection property trivially.

The running intersection property can be interpreted as follows. Suppose the m index sets S_1, …, S_m satisfy (RIP), and that we have pieces of information about the uncertain variables X_{S_ℓ}, with ℓ in $\{1,\dots,m\}$ – be it in the form of probabilities, sets of desirable gambles, sets of desirable gamble sets, or another type of belief model – that are *pairwise compatible*. (RIP) then tells us that the piece of information about X_{S_ℓ} that is shared with the pieces of information about the uncertain variables X_{S_i}, with $i < \ell$, is already contained in a single piece of information about $X_{S_{i^\star}}$, for some $i^\star < \ell$. Using the pairwise compatibility of the pieces of information about X_{S_ℓ} and $X_{S_{i^\star}}$, our proof will then ensure that the piece of information about X_{S_ℓ} is compatible with all pieces of information about the uncertain variables X_{S_i}, with $i < \ell$.

Running example 6. *Let $S_1 = \{1,2\}$, $S_2 = \{2,3\}$ and $S_3 = \{2,3,4\}$, then $S_3 \cap (S_1 \cup S_2) = \{2,3\} = S_3 \cap S_2$, so the index sets S_1, S_2 and S_3 satisfy* (RIP).

On the other hand, let $S_1' = \{1,2\}$, $S_2' = \{2,3\}$ and $S_3' = \{1,3\}$, then $S_3' \cap (S_1' \cup S_2') = \{1,3\}$, $S_2' \cap (S_1' \cup S_3') = \{2,3\}$ and $S_1' \cap (S_2' \cup S_3') = \{1,2\}$ while all intersections of two index sets contain only one element – so there is no ordering for which the index sets S_1', S_2' and S_3' satisfy (RIP).

Theorem 6. *[3][Theorem 2] Consider any coherent sets of desirable gambles $D_1 \subseteq \mathcal{L}(\mathcal{X}_{S_1})$, $D_2 \subseteq \mathcal{L}(\mathcal{X}_{S_2})$, …, $D_m \subseteq \mathcal{L}(\mathcal{X}_{S_m})$. If S_1, …, S_m satisfy* (RIP) *and D_1, …, D_m are pairwise compatible, then D_1, …, D_m are compatible.*

We lift the content of Theorem 6 from sets of desirable gambles to SoDGSes with a finite representation, and show that the running intersection property is a sufficient condition for the current context, too.

Theorem 7. *Consider any coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, …, $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$ with a finite representation. If S_1, …, S_m satisfy* (RIP) *and K_1, …, K_m are pairwise compatible, then K_1, …, K_m are compatible.*

Proof. For any $\ell \in \{1,\dots,m\}$ let $\hat{K}_\ell := \operatorname{cl}_{\overline{\mathcal{K}}}\big(\bigcup_{j\le\ell} K_j\big)$. If $\ell \ge 2$, note that, because $\operatorname{cl}_{\overline{\mathcal{K}}}$ is a closure operator, we have that $\hat{K}_\ell = \operatorname{cl}_{\overline{\mathcal{K}}}\big(\bigcup_{j\le\ell} K_j\big) \subseteq \operatorname{cl}_{\overline{\mathcal{K}}}\big(\operatorname{cl}_{\overline{\mathcal{K}}}(\bigcup_{j\le\ell-1} K_j) \cup K_\ell\big) = \operatorname{cl}_{\overline{\mathcal{K}}}\big(\hat{K}_{\ell-1} \cup K_\ell\big) \subseteq \operatorname{cl}_{\overline{\mathcal{K}}}\big(\operatorname{cl}_{\overline{\mathcal{K}}}(\bigcup_{j\le\ell} K_j)\big) = \hat{K}_\ell$, so $\hat{K}_\ell = \operatorname{cl}_{\overline{\mathcal{K}}}\big(\hat{K}_{\ell-1} \cup K_\ell\big)$. We will show, using induction on ℓ, that $\hat{K}_m$ is compatible with K_1, …, K_m. Proposition 8 then guarantees that K_1, …, K_m are compatible.

For the base case, start with $\ell = 1$. Note that $\hat{K}_1 = K_1$, so it is trivially compatible with K_1.

For the induction step, consider any $\ell \in \{2,\dots,m\}$ and assume that $\hat{K}_{\ell-1}$ is compatible with K_1, …, $K_{\ell-1}$. We will show that then $\hat{K}_\ell = \operatorname{cl}_{\overline{\mathcal{K}}}\big(\bigcup_{i\le\ell} K_i\big) = \operatorname{cl}_{\overline{\mathcal{K}}}(\hat{K}_{\ell-1} \cup K_\ell)$ is compatible with $\hat{K}_{\ell-1}$ and K_ℓ. Using the compatibility of $\hat{K}_{\ell-1}$ with K_1, …, $K_{\ell-1}$, the desired result follows, because Eq. (1) then tells us that $\operatorname{Marg}_{S_j} \hat{K}_\ell = \operatorname{Marg}_{S_j} \operatorname{Marg}_{\bigcup_{i\le\ell-1} S_i} \hat{K}_\ell = \operatorname{Marg}_{S_j} \hat{K}_{\ell-1} = K_j$ for any j in $\{1,\dots,\ell-1\}$. Let

$$\hat{\mathcal{D}}_\ell := \{\operatorname{cl}_{\overline{\mathcal{D}}}(D_{\ell-1} \cup D_\ell) : D_{\ell-1} \in \mathcal{D}_{\hat{K}_{\ell-1}}, D_\ell \in \mathcal{D}_{K_\ell}, D_{\ell-1}, D_\ell \text{ pairwise compatible}\},$$

and we will show (i) that $K_{\hat{\mathcal{D}}_\ell}$ is compatible with $\hat{K}_{\ell-1}$ and K_ℓ, and (ii) that $\hat{K}_\ell = K_{\hat{\mathcal{D}}_\ell}$.

For (i), we will first show that $\operatorname{marg}_{S_\ell} \hat{\mathcal{D}}_\ell = \mathcal{D}_{K_\ell}$. To this end, consider any D_ℓ in $\mathcal{D}_{K_\ell}$. By (RIP) there is some $i^\star < \ell$ such that $S_\ell \cap S_{i^\star} = S_\ell \cap \bigcup_{i<\ell} S_i$. By the pairwise compatibility of K_ℓ and $K_{i^\star}$, there is some $D_{i^\star}$ in $\mathcal{D}_{K_{i^\star}}$ that is pairwise compatible with D_ℓ, taking into account Proposition 7. But by the pairwise compatibility of $K_{i^\star}$ with any K_i (with $i \in \{1,\dots,\ell-1\} \setminus \{i^\star\}$) there is some $D_i \in \mathcal{D}_{K_i}$ that is pairwise compatible with $D_{i^\star}$. Since $S_\ell \cap S_{i^\star} \supseteq S_\ell \cap S_i$, this implies that

$$\begin{aligned}
\operatorname{marg}_{S_\ell\cap S_i} D_\ell &= \operatorname{marg}_{S_\ell\cap S_i\cap S_{i^\star}} D_\ell \\
&= \operatorname{marg}_{S_\ell\cap S_i\cap S_{i^\star}} D_{i^\star} \\
&= \operatorname{marg}_{S_\ell\cap S_i\cap S_{i^\star}} D_i = \operatorname{marg}_{S_\ell\cap S_i} D_i
\end{aligned}$$

so we infer that D_ℓ and D_i are pairwise compatible, and therefore, so is D_ℓ with $\hat{D}_{\ell-1} := \mathrm{cl}_{\overline{D}}(\bigcup_{i<\ell} D_i)$, which is coherent and compatible with $D_1, \ldots, D_{\ell-1}$ due to Theorem 6 and [3][Proposition 1], again taking (RIP) into account.

Since $D_i \in \mathcal{D}_{K_i}$, we find that $K_i \subseteq K_{D_i}$, and moreover $K_{D_i} \subseteq K_{\hat{D}_{\ell-1}}$ because $\hat{D}_{\ell-1} \supseteq D_i$, so we find that $K_i \subseteq K_{\hat{D}_{\ell-1}}$ for any i in $\{1, \ldots, \ell-1\}$. Therefore $\bigcup_{i<\ell} K_i \subseteq K_{\hat{D}_{\ell-1}}$. Taking $K_{\hat{D}_{\ell-1}}$'s coherence into account, we infer that then also $\hat{K}_{\ell-1} = \mathrm{cl}_{\overline{K}}(\bigcup_{i<\ell} K_i) \subseteq \mathrm{cl}_{\overline{K}}(K_{\hat{D}_{\ell-1}}) = K_{\hat{D}_{\ell-1}}$, so $\hat{D}_{\ell-1}$ belongs to $\mathcal{D}_{\hat{K}_{\ell-1}}$. Consider now $\mathrm{cl}_{\overline{D}}(\hat{D}_{\ell-1} \cup D_\ell)$, which belongs to $\hat{\mathcal{D}}_\ell$ because $\hat{D}_{\ell-1}$ and D_ℓ are pairwise compatible, $\hat{D}_{\ell-1}$ belongs to $\mathcal{D}_{\hat{K}_{\ell-1}}$, and D_ℓ to $\mathcal{D}_{K_\ell}$. Then, again invoking the pairwise compatibility of $\hat{D}_{\ell-1}$ and D_ℓ, we find that $\mathrm{marg}_{S_{\ell-1}} \mathrm{cl}_{\overline{D}}(\hat{D}_\ell \cup D_\ell) = D_\ell$, so D_ℓ belongs to $\mathrm{marg}_{S_\ell} \hat{\mathcal{D}}_\ell$, indeed. This establishes that $\mathcal{D}_{K_\ell} \subseteq \mathrm{marg}_{S_\ell} \hat{\mathcal{D}}_\ell$.

The converse set inclusion follows since $\mathrm{marg}_{S_\ell} \mathrm{cl}_{\overline{D}}(D_{\ell-1} \cup D_\ell) \supseteq D_\ell$ for any D_ℓ, and hence the former belongs to $\mathcal{D}_{K_\ell}$, too. Using Proposition 4, we infer that $\mathrm{Marg}_{S_\ell} K_{\hat{\mathcal{D}}_\ell} = K_\ell$.

Next, we will show that $\mathrm{marg}_{\hat{S}_{\ell-1}} \hat{\mathcal{D}}_\ell = \mathcal{D}_{\hat{K}_{\ell-1}}$, where $\hat{S}_{\ell-1} := \bigcup_{i<\ell} S_i$. To this end, consider any $\hat{D}_{\ell-1}$ in $\mathcal{D}_{\hat{K}_{\ell-1}}$. Then $\mathrm{marg}_{S_1} \hat{D}_{\ell-1}$, …, $\mathrm{marg}_{S_{\ell-1}} \hat{D}_{\ell-1}$ are pairwise compatible by construction, and belong to $\mathcal{D}_{K_1}, \ldots, \mathcal{D}_{K_{\ell-1}}$, respectively, taking Proposition 4 into account. By (RIP) there is some $i^\star$ such that $S_\ell \cap S_{i^\star} = S_\ell \cap \hat{S}_{\ell-1}$. Taking into account the pairwise compatibility of $K_{i^\star}$ and K_ℓ, we find using that Proposition 7 that there is a D_ℓ in $\mathcal{D}_{K_\ell}$ pairwise compatible with $\mathrm{marg}_{S_{i^\star}} \hat{D}_{\ell-1}$. But then

$$
\begin{aligned}
\mathrm{marg}_{\hat{S}_{\ell-1} \cap S_\ell} \hat{D}_{\ell-1} &= \mathrm{marg}_{S_{i^\star} \cap S_\ell} \hat{D}_{\ell-1} \\
&= \mathrm{marg}_{S_{i^\star} \cap S_\ell} \mathrm{marg}_{S_{i^\star}} \hat{D}_{\ell-1} \\
&= \mathrm{marg}_{S_{i^\star} \cap S_\ell} D_\ell = \mathrm{marg}_{\hat{S}_{\ell-1} \cap S_\ell} D_\ell,
\end{aligned}
$$

so $\hat{D}_{\ell-1}$ and D_ℓ are pairwise compatible. We find that $\mathrm{cl}_{\overline{D}}(\hat{D}_{\ell-1} \cup D_\ell)$ belongs to $\hat{\mathcal{D}}_\ell$, and, taking into account Theorem 6, we find that $\mathrm{marg}_{\hat{S}_{\ell-1}} \mathrm{cl}_{\overline{D}}(\hat{D}_{\ell-1} \cup D_\ell) = \hat{D}_{\ell-1}$, so that $\hat{D}_{\ell-1} \in \mathrm{marg}_{\hat{S}_{\ell-1}} \hat{\mathcal{D}}_\ell$. Since the choice of $\hat{D}_{\ell-1}$ in $\mathcal{D}_{\hat{K}_{\ell-1}}$ was arbitrary, this implies that $\mathcal{D}_{\hat{K}_{\ell-1}} \subseteq \mathrm{marg}_{\hat{S}_{\ell-1}} \hat{\mathcal{D}}_\ell$.

The converse set inclusion follows since $D_{\ell-1} \subseteq \mathrm{marg}_{S_{\ell-1}} \mathrm{cl}_{\overline{D}}(D_{\ell-1} \cup D_\ell)$ for any $D_{\ell-1}$, and taking into account that $\mathcal{D}(K_{\ell-1})$ is an upset, the latter belongs to it as well. Using Proposition 4 again, we infer that $\mathrm{Marg}_{\hat{S}_{\ell-1}} K_{\hat{\mathcal{D}}_\ell} = \hat{K}_{\ell-1}$, so that $K_{\hat{\mathcal{D}}_\ell}$ is compatible with $\hat{K}_{\ell-1}$ and K_ℓ, as desired.

To finish the proof, for (ii), we show that $\hat{K}_\ell = K_{\hat{\mathcal{D}}_\ell}$. Since $K_{\hat{\mathcal{D}}_\ell}$ is compatible with $\hat{K}_{\ell-1}$ and K_ℓ, we know that $\hat{K}_{\ell-1} \cup K_\ell \subseteq K_{\hat{\mathcal{D}}_\ell}$, and therefore $\hat{K}_\ell = \mathrm{cl}_{\overline{K}}(\hat{K}_{\ell-1} \cup K_\ell) \subseteq K_{\hat{\mathcal{D}}_\ell}$ since $K_{\hat{\mathcal{D}}_\ell}$ is coherent. So it suffices to prove that $K_{\hat{\mathcal{D}}_\ell} \subseteq \hat{K}_\ell$, with $\hat{K}_\ell = \bigcap\{K_D : D \in \overline{\mathcal{D}}, \hat{K}_{\ell-1} \cup K_\ell \subseteq K_D\}$ taking into account Theorem 4, and $K_{\hat{\mathcal{D}}_\ell} = \bigcap\{K_D : D \in \hat{\mathcal{D}}_\ell\}$. In turn, it suffices to show that $\{D \in \overline{\mathcal{D}} : \hat{K}_{\ell-1} \cup K_\ell \subseteq K_D\} \subseteq \uparrow\hat{\mathcal{D}}_\ell$, using [8][Proposition 4]. To this end, consider any $D \in \overline{\mathcal{D}}$ such that $\hat{K}_{\ell-1} \subseteq K_D$ and $K_\ell \subseteq K_D$, implying that $\hat{K}_{\ell-1} \subseteq K_{\mathrm{marg}_{\hat{S}_{\ell-1}} D}$ and $K_\ell \subseteq K_{\mathrm{marg}_{S_\ell} D}$ taking into account Proposition 4. But $\mathrm{marg}_{\hat{S}_{\ell-1}} D$ and $\mathrm{marg}_{S_\ell} D$ are pairwise compatible because they are derived from the same joint, and $D \supseteq \mathrm{marg}_{\hat{S}_{\ell-1}} D \cup \mathrm{marg}_{S_\ell} D$, so $D \in \uparrow\hat{\mathcal{D}}_\ell$, indeed. This establishes that $\hat{K}_\ell = K_{\hat{\mathcal{D}}_\ell}$, which is therefore compatible with $\hat{K}_{\ell-1}$ and K_ℓ. $\square$

Theorem 7 establishes that (RIP) and pairwise compatibility of coherent SoDGSes with a finite representation imply compatibility, while Proposition 8 shows that $K_1, \ldots, K_m$ are compatible if and only if their natural extension is compatible with them. Combining these two results, we find that coherent SoDGSes $K_1, \ldots, K_m$ with a finite representation that are pairwise compatible and satisfy (RIP) have a (unique) smallest compatible joint $\mathrm{cl}_{\overline{K}}(\bigcup_{\ell \le m} K_\ell)$. It is this SoDGS $\mathrm{cl}_{\overline{K}}(\bigcup_{\ell \le m} K_\ell)$ that summarises the information contained in the partial assessments $K_1, \ldots, K_m$, making it an important SoDGS for inferences. In the next section, we study some of its properties, with a focus on its representation.

5.3. Representations of the smallest compatible joint

We start our study of the smallest compatible joint $\mathrm{cl}_{\overline{K}}(\bigcup_{\ell \le m} K_\ell)$ and its representation by considering the setting from above: suppose that we have coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, …, $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$ that are compatible. Can we find a representation for the smallest compatible joint $K = \mathrm{cl}_{\overline{K}}(\bigcup_{\ell \le m} K_\ell)$ using their largest representations $\mathcal{D}_{K_1}, \mathcal{D}_{K_2}, \ldots, \mathcal{D}_{K_m}$?

Proposition 10. *Consider coherent and compatible SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, …, $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$. Then the smallest compatible joint $K := \mathrm{cl}_{\overline{K}}(\bigcup_{\ell \le m} K_\ell)$ is represented by*

$$\mathcal{D}' := \{\mathrm{cl}_{\overline{D}}(D_1 \cup \cdots \cup D_m) : D_1 \in \mathcal{D}_{K_1}, \ldots, D_m \in \mathcal{D}_{K_m}, D_1 \cup \cdots \cup D_m \text{ consistent}\}.$$

Proof. We first show that $\mathcal{D}' \neq \emptyset$. Use Proposition 9 to infer that there is some coherent $K' \subseteq \mathcal{Q}(\mathcal{X})$ such that $\mathrm{marg}_{S_\ell} \mathcal{D}_{K'} = \mathcal{D}_{K_\ell}$ for every ℓ in $\{1, \ldots, m\}$. Since K' is coherent, its largest representation $\mathcal{D}_{K'}$ is non-empty, by Theorem 3, so there is some $D \in \mathcal{D}_{K'}$. But then $\mathrm{marg}_{S_j} D \in \mathcal{D}_{K_j}$ for every j in $\{1, \ldots, m\}$, and since $D \supseteq \bigcup_{j \le m} \mathrm{marg}_{S_j} D$, we infer that $\bigcup_{j \le m} \mathrm{marg}_{S_j} D$ is consistent. So we have found elements $\mathrm{marg}_{S_j} D \in \mathcal{D}_{K_j}$ for every j in $\{1, \ldots, m\}$ whose union is consistent, and therefore $\mathcal{D}' \neq \emptyset$, indeed.

Since $K_1, \ldots, K_m$ are compatible, they are derived from a joint coherent $\hat{K}$, which then necessarily is a superset of K. This implies that $\hat{K} := \bigcup_{\ell \le m} K_\ell$ is consistent, guaranteeing K's coherence by Theorem 2.

We first show that $K_{\mathcal{D}_{\hat{K}}} \subseteq K_{\mathcal{D}'}$. To this end, consider any D in $\mathcal{D}'$. Then $D = \mathrm{cl}_{\overline{D}}(D_1 \cup \cdots \cup D_m)$ for some $D_1 \in \mathcal{D}_{K_1}, \ldots, D_m \in \mathcal{D}_{K_m}$, whence $K_\ell \subseteq K_{D_\ell} \subseteq K_D$ for every ℓ in $\{1, \ldots, m\}$, so indeed $\hat{K} \subseteq K_D$ and hence $D \in \mathcal{D}_{\hat{K}}$. Since the choice of D in $\mathcal{D}'$ was arbitrary, this implies that $\mathcal{D}' \subseteq \mathcal{D}_{\hat{K}}$, which in turn implies that $K_{\mathcal{D}_{\hat{K}}} \subseteq K_{\mathcal{D}'}$.

Next, we show that $K_{\mathcal{D}'} \subseteq K_{\mathcal{D}_{\hat{K}}}$. To this end, consider any F in $\mathcal{Q}$ such that $F \notin K_{\mathcal{D}_{\hat{K}}}$. This implies that there is some D in $\mathcal{D}_{\hat{K}}$ such that $F \cap D = \emptyset$. Consider any ℓ in $\{1, \dots, m\}$. That $D \in \mathcal{D}_{\hat{K}}$ implies that $\hat{K} \subseteq K_D$, and therefore

$$K_\ell = \mathrm{Marg}_{S_\ell} \hat{K} \subseteq \mathrm{Marg}_{S_\ell} K_D = K_{\mathrm{marg}_{S_\ell} D},$$

so $\mathrm{marg}_{S_\ell} D \in \mathcal{D}_{K_\ell}$. Since $D \supseteq \mathrm{cl}_{\overline{\mathcal{D}}}(\mathrm{marg}_{S_1} D \cup \dots \cup \mathrm{marg}_{S_m} D)$, we infer from $F \cap D = \emptyset$ that $F \cap \mathrm{cl}_{\overline{\mathcal{D}}}(\mathrm{marg}_{S_1} D \cup \dots \cup \mathrm{marg}_{S_m} D) = \emptyset$, But $\mathrm{marg}_{S_1} D$, …, $\mathrm{marg}_{S_m} D$ are compatible because they are derived from the common coherent joint D, so $F \notin K_{\mathcal{D}'}$. Since the choice of F was arbitrary, together with the result established in the paragraph above, this implies that $K_{\mathcal{D}'} = K_{\mathcal{D}_{\hat{K}}}$.

Taking into account Theorem 4, we have that $K_{\mathcal{D}_{\hat{K}}} = \mathrm{cl}_{\overline{\mathcal{K}}}(\hat{K}) = K$, and therefore $K_{\mathcal{D}'} = K$, so $\mathcal{D}'$ represents K, indeed. □

Running example 7. *Consider E-admissible SoDGSes $K_{\mathcal{M}_1}$, $K_{\mathcal{M}_2}$, …, $K_{\mathcal{M}_m}$ based on finite sets of pmfs $\mathcal{M}_1$, $\mathcal{M}_2$, …, $\mathcal{M}_m$, respectively. From Running Example 1 we know that their largest representations are $\uparrow\{D_p : p \in \mathcal{M}_\ell\}$ for ℓ in $\{1, 2, \dots, m\}$, so we cannot guarantee that these representations are finite – and thus cannot construct the smallest compatible joint iteratively, using the method from the proof of Theorem 7. This motivates us to show that the representation of the smallest compatible joint can be constructed also from minimal elements of the largest representations $\min \mathcal{D}_{K_{\mathcal{M}_\ell}}$ for ℓ in $\{1, 2, \dots, m\}$, which we know from Running Example 2 to be finite and equal to $\{D_p : p \in \mathcal{M}_\ell\}$.*

The set $\mathcal{D}'$ in Proposition 10 is a representation of the smallest compatible joint $\mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell \le m} K_\ell)$, constructed from the largest representations $\mathcal{D}_{K_1}$, $\mathcal{D}_{K_2}$, …, $\mathcal{D}_{K_m}$ of the coherent SoDGSes K_1, …, K_m. The proof of Proposition 10 does not require the assumption that the marginal SoDGSes K_1, …, K_m have finite representations.

However, $\mathcal{D}'$ is not necessarily the largest representation of $\mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell \le m} K_\ell)$. To find the largest representation of $\mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell \le m} K_\ell)$, we use the marginalisation properties of the smallest compatible joint.

Proposition 11. *Consider coherent and compatible SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, …, $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$. Then the set*

$$\mathcal{D} := \uparrow\{\mathrm{cl}_{\overline{\mathcal{D}}}(D_1 \cup \dots \cup D_m) : D_1 \in \mathcal{D}_{K_1}, \dots, D_m \in \mathcal{D}_{K_m}, D_1 \cup \dots \cup D_m \text{ consistent}\}.$$

is the (unique) largest representation of the smallest compatible joint $K := \mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell \le m} K_\ell)$.

Proof. It is clear from their definitions that $\mathcal{D} = \uparrow\mathcal{D}'$, with $\mathcal{D}'$ as in Proposition 10, so $\mathcal{D}$ represents K because $\mathcal{D}'$ does. Therefore $\mathcal{D} \subseteq \mathcal{D}_K$, and the proof is finished once we establish that $\mathcal{D}_K \subseteq \mathcal{D}$.

To this end, consider any D in $\mathcal{D}_K$, then $K = \mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell \le m} K_\ell) \subseteq K_D$, so in particular $K_\ell = \mathrm{Marg}_{S_\ell} K \subseteq \mathrm{Marg}_{S_\ell} K_D = K_{\mathrm{marg}_{S_\ell} D}$ for every ℓ in $\{1, \dots, m\}$, taking into account that K is pairwise compatible with K_ℓ in the first equality, and Proposition 4 in the second equality. This implies that $\mathrm{marg}_{S_\ell} D \in \mathcal{D}_{K_\ell}$. Since $D \supseteq \mathrm{cl}_{\overline{\mathcal{D}}}(\mathrm{marg}_{S_1} D \cup \dots \cup \mathrm{marg}_{S_m} D)$, we infer that $\mathrm{marg}_{S_1} D \cup \dots \cup \mathrm{marg}_{S_m} D$ is consistent. So we have found $\mathrm{marg}_{S_1} D \in \mathcal{D}_{K_1}$, …, $\mathrm{marg}_{S_m} D \in \mathcal{D}_{K_m}$ such that $\mathrm{marg}_{S_1} D \cup \dots \cup \mathrm{marg}_{S_m} D$ and $D \supseteq \mathrm{cl}_{\overline{\mathcal{D}}}(\mathrm{marg}_{S_1} D \cup \dots \cup \mathrm{marg}_{S_m} D)$, and therefore, $D \in \mathcal{D}$. Since the choice of $D \in \mathcal{D}_K$ was arbitrary, this implies that $\mathcal{D}_K \subseteq \mathcal{D}$, indeed. □

Proposition 11 tells us that there is a convenient way to obtain the largest representation of the smallest compatible joint from the largest representations of the coherent SoDGSes K_1, K_2, …, K_m, using the representation $\mathcal{D}'$ found in Proposition 10. If we can guarantee that the compatible joint has a finite representation whenever K_1, K_2, …, K_m do, then we can use this to solve the marginal problem *iteratively*: for given marginal SoDGSes with finite representations, their smallest compatible joint – and its representation – can be updated with a new marginal compatible SoDGS with finite representation.

Proposition 12. *Consider coherent and compatible SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1})$, $K_2 \subseteq \mathcal{Q}(\mathcal{X}_{S_2})$, …, $K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$ with a finite representation. Then the smallest compatible joint $K := \mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell \le m} K_\ell)$ has a finite representation as well.*

Proof. It suffices to show that

$$\begin{aligned}\mathcal{D} &:= \uparrow\{\mathrm{cl}_{\overline{\mathcal{D}}}(D_1 \cup \dots \cup D_m) : D_1 \in \mathcal{D}_{K_1}, \dots, D_m \in \mathcal{D}_{K_m}, D_1 \cup \dots \cup D_m \text{ consistent}\}\\ &= \uparrow\{\mathrm{cl}_{\overline{\mathcal{D}}}(D_1 \cup \dots \cup D_m) : D_1 \in \min \mathcal{D}_{K_1}, \dots, D_m \in \min \mathcal{D}_{K_m}, D_1 \cup \dots \cup D_m \text{ consistent}\},\end{aligned}$$

because then Proposition 1 guarantees us that K is represented by $\{\mathrm{cl}_{\overline{\mathcal{D}}}(D_1 \cup \dots \cup D_m) : D_1 \in \min \mathcal{D}_{K_1}, \dots, D_m \in \min \mathcal{D}_{K_m}, D_1 \cup \dots \cup D_m \text{ consistent}\}$, which is a finite set because all the $\min \mathcal{D}_\ell$ are guaranteed to be finite by Proposition 2.

We show that $\mathcal{D} = \mathcal{D}''$, with

$$\mathcal{D}'' := \uparrow\{\mathrm{cl}_{\overline{\mathcal{D}}}(D_1 \cup \dots \cup D_m) : D_1 \in \min \mathcal{D}_{K_1}, \dots, D_m \in \min \mathcal{D}_{K_m}, D_1 \cup \dots \cup D_m \text{ consistent}\}.$$

By construction $\mathcal{D} \supseteq \mathcal{D}''$, so we show that $\mathcal{D} \subseteq \mathcal{D}''$. To this end, consider any D in $\mathcal{D}$, so $D \supseteq \mathrm{cl}_{\overline{\mathcal{D}}}(D_1 \cup \dots \cup D_m)$ for some $D_1 \in \mathcal{D}_{K_1}$, …, $D_m \in \mathcal{D}_{K_m}$ such that $D_1 \cup \dots \cup D_m$ is consistent. Since each K_ℓ has a finite representation, Theorem 5 guarantees us that each $\mathcal{D}_{K_\ell} = \uparrow\min \mathcal{D}_{K_\ell}$, so there are $D'_1 \in \min \mathcal{D}_{K_1}$, …, $D'_m \in \min \mathcal{D}_{K_m}$ such that $D_1 \supseteq D'_1$, …, $D_m \supseteq D'_m$. This implies that $D \supseteq \mathrm{cl}_{\overline{\mathcal{D}}}(D'_1 \cup \dots \cup D'_m)$, and that $D'_1 \cup \dots \cup D'_m$ is consistent as well, so we find that $D \in \mathcal{D}''$, indeed. □

Running example 8. *Consider the same index sets $S_1 = \{1, 2\}$, $S_2 = \{2, 3\}$ and $S_3 = \{2, 3, 4\}$ as in Running Example 6, so we already know that they satisfy (RIP), and let $\mathcal{X} := \times_{\ell=1}^{4} \mathcal{X}_\ell$. Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be finite sets of pmfs on $\mathcal{X}_{S_1}, \mathcal{X}_{S_2}$ and $\mathcal{X}_{S_3}$, respectively, and let $K_{\mathcal{M}_1}$, $K_{\mathcal{M}_2}$ and $K_{\mathcal{M}_3}$ be the coherent SoDGSes based on these sets using E-admissibility, as in Running Example 1. Assume that $K_{\mathcal{M}_1}$, $K_{\mathcal{M}_2}$ and $K_{\mathcal{M}_3}$ are pairwise compatible, so that we can apply apply Theorem 7 to infer that they are compatible. As we know from Running Example 2 that each $K_{\mathcal{M}_\ell}$ has a finite representation $\{D_p : p \in \mathcal{M}_\ell\}$ (for all $\ell \in \{1, 2, 3\}$), Proposition 12 guarantees that their smallest compatible joint has a finite representation. We will find such a finite representation in this instalment.*

Running Example 2 guarantees that $\min \mathcal{D}_{K_{\mathcal{M}_\ell}} = \{D_p \colon p \in \mathcal{M}_\ell\}$ *for all* $\ell \in \{1,2,3\}$*, so the proof of Proposition 12 then tells us that a finite representation of the smallest compatible joint* $K = \mathrm{cl}_{\overline{\mathcal{K}}}(\bigcup_{\ell=1}^{3} K_{\mathcal{M}_\ell})$ *is given by*

$$\mathcal{D} := \{\mathrm{cl}_{\overline{\mathcal{D}}}(D_{p_1} \cup D_{p_2} \cup D_{p_3}) \colon p_1 \in \mathcal{M}_1, p_2 \in \mathcal{M}_2, p_3 \in \mathcal{M}_3, D_{p_1} \cup D_{p_2} \cup D_{p_3} \text{ consistent}\}.$$

Note that $D_{p_1} \cup D_{p_2} \cup D_{p_3}$ *is consistent if and only if the* p_1*,* p_2*,* p_3 *can be derived from a common joint; in other words, if and only if* p_1*,* p_2 *and* p_3 *are* compatible*. We therefore find that*

$$\mathcal{D} = \{\mathrm{cl}_{\overline{\mathcal{D}}}(D_{p_1} \cup D_{p_2} \cup D_{p_3}) \colon p_1 \in \mathcal{M}_1, p_2 \in \mathcal{M}_2, p_3 \in \mathcal{M}_3, p_1, p_2, p_3 \text{ compatible}\}$$

is a finite representation of K*. Because of the specific sets of indices* S_1*,* S_2 *and* S_3*, we can simplify this representation further. Consider any* p_3 *in* $\mathcal{M}_3$ *and compatible* $p_2 \in \mathcal{M}_2$*. Then, for any* $f \in \mathcal{L}(\mathcal{X}_{S_2})$*, we find that* f *belongs to* D_{p_2} *if and only if*

$$E_{p_2}(f) = \sum_{x_2 \in \mathcal{X}_2, x_3 \in \mathcal{X}_3} p_2(x_2, x_3) f(x_2, x_3) > 0 \quad \text{or} \quad f > 0$$

which implies

$$E_{p_3}(f) = \sum_{x_2 \in \mathcal{X}_2, x_3 \in \mathcal{X}_3, x_4 \in \mathcal{X}_4} p_3(x_2, x_3, x_4) f(x_2, x_3) > 0 \quad \text{or} \quad f > 0,$$

which in turn is equivalent to $f \in D_{p_3}$*. This implies that* $D_{p_2} \subseteq D_{p_3}$*, taking into account cylindrical extension. This simplifies the expression for the finite representation of* K *to*

$$\mathcal{D} = \{\mathrm{cl}_{\overline{\mathcal{D}}}(D_{p_1} \cup D_{p_3}) \colon p_1 \in \mathcal{M}_1, p_3 \in \mathcal{M}_3, p_1, p_3 \text{ compatible}\}.$$

By combining Theorem 7 and Proposition 12 we find that for pairwise compatible coherent SoDGSes $K_1 \subseteq \mathcal{Q}(\mathcal{X}_{S_1}), \dots, K_m \subseteq \mathcal{Q}(\mathcal{X}_{S_m})$ with a finite representation, whose index sets satisfy (RIP), the smallest compatible joint also has a finite representation. Moreover, the representation of the smallest compatible joint can be constructed *iteratively*: if we have constructed the largest representation $\mathcal{D}_{m-1}$ for the smallest compatible joint $\mathrm{cl}_{\overline{\mathcal{K}}}(K_1 \cup \dots \cup K_{m-1})$, then Proposition 12 guarantees that this joint has a finite representation as well, so by Theorem 7, we can combine $\mathcal{D}_{m-1}$ and $\mathcal{D}_{K_m}$ to obtain the largest representation $\mathcal{D}_m$ of the smallest compatible joint $\mathrm{cl}_{\overline{\mathcal{K}}}(K_1 \cup \dots \cup K_m)$.

This is shown directly in the induction step of the proof of Theorem 7, which specifies how to update the smallest compatible joint when a new marginal compatible SoDGS is added, as well as how to update its representation.

6. Conclusions

We have established in Theorem 7 that coherent SoDGSes with a finite representation that are pairwise compatible, and whose index sets satisfy the running intersection property, are compatible. In order to do so, we had to study the representation of coherent SoDGSes in more detail. We have established in Theorem 5 that the representation $\mathcal{D}_K$ of a coherent SoDGS K is determined by its minimal elements $\min \mathcal{D}_K$, which by Proposition 2 is a finite set whenever K admits a finite representation. Moreover, Proposition 3 establishes that the coherent SoDGSes that admit a finite representation are *uniquely* determined by finite antichains of coherent sets of desirable gambles.

Our solution to the marginal problem in Theorem 7 generalises the result for sets of desirable gambles by [3][Theorem 2]. Importantly, they note [3][Appendix A.4] that coherent sets of desirable gambles, with the standard definition of marginalisation, can be interpreted as an *information algebra* [38]. As made explicit by [13,35], the structure of an information algebra is rich enough to prove Theorem 7 directly. Our approach is based on describing an iterative method for computing the smallest compatible joint and its finite representation: the proof of Theorem 7 makes this iteration explicit. This expressiveness, in our opinion, justifies approaching the marginal problem for coherent SoDGSes directly, without using the information algebra framework.

Preliminary work indicates that the coherent SoDGSes, with our definition of marginalisation in Definition 5, also form a information algebra. As a consequence, Theorem 7 will come for free, *even for coherent SoDGSes that do not necessarily admit a finite representation*. However, an information algebra does not allow us to derive a representation in the sense of Proposition 10, at least not as far as we see. It is an interesting open question whether the representation Theorem 3 can be incorporated in the framework of information algebras, and we intend to pursue this in the future. Answering this would be useful for the study of compatibility. This paper – specifically Propositions 5 and 7 – establishes foundations for this project.

An alternative to overcome the limitation to coherent SoDGSes with a *finite representation* in Theorem 7, is to consider SoDGSes with *arbitrary* (not necessarily finite) gamble sets. In their pioneering work of imprecise-probabilistic choice functions, Seidenfeld et al. [5] have allowed for infinite (but closed) gamble sets. De Bock and De Cooman [28], De Bock [39] and [4] have continued this line, and studied SoDGSes with arbitrary gamble sets. They also developed representation and natural extension, and working with them is sometimes easier. For instance in Proposition 5, the proof would go through even without the requirement that K has a finite representation, because the requirement that the gamble set F be finite becomes moot now. For such SoDGSes, however, there is no counterpart for Theorem 5, so representation may become more cumbersome in some situations, which is why we opted for SoDGSes with finite gamble sets in this paper.

Another direction for future work, is to study the compatibility of conditional SoDGSes. Miranda and Zaffalon [3] have investigated this problem, which they call the ‘compatibility problem’, for sets of desirable gambles in quite some detail [3][Section 3]. This can be done best using information algebras, so incorporating representation in the framework of information algebras would be useful for this project, too.

The new interesting subclass of coherent SoDGSes that we have introduced in this paper, namely SoDGSes with a finite representation, served its main purpose in the proofs of Lemma 1 and Proposition 3. We have indicated in Example 1 and Running Example 1 that this subclass is sufficiently broad: it contains all the SoDGSes based on Sen–Walley maximality, and also based on E-admissibility with a finite set of probabilities. However, preliminary work indicates that mixing Archimedean SoDGSes [40] also have the property described in Proposition 3, *even when they do not admit a finite representation*; we intend to report on this elsewhere. An interesting open and foundational question is what property on SoDGSes is *necessary* for the conclusion of Proposition 3 to hold.

CRediT authorship contribution statement

Justyna Dąbrowska: Writing – review & editing, Writing – original draft, Project administration, Methodology, Investigation, Formal analysis, Conceptualization; **Arthur Van Camp:** Writing – review & editing, Writing – original draft, Project administration, Methodology, Investigation, Formal analysis, Conceptualization; **Erik Quaeghebeur:** Writing – review & editing, Supervision, Investigation, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

References

[1] G. Boole, The Laws of Thought, Dover Publications, New York, New York, 1847, reprinted in 1961.
[2] R. Féron, Sur les tableaux de corrélation dont les marges sont données. cas de l'espace à trois dimensions, Ann. ISUP V (1) (1956) 3–12. https://hal.science/hal-04095296.
[3] E. Miranda, M. Zaffalon, Compatibility, desirability, and the running intersection property, Artif. Intell. 283 (2020) 103274. https://doi.org/10.1016/j.artint.2020.103274
[4] G. De Cooman, A. Van Camp, J. De Bock, The logic behind desirable sets of things, and its filter representation, Int. J. Approx. Reason. 172 (2024) 109241. https://doi.org/10.1016/j.ijar.2024.109241
[5] T. Seidenfeld, M.J. Schervish, J.B. Kadane, Coherent choice functions under uncertainty, Synthese 172 (1) (2010) 157–176. https://doi.org/10.1007/s11229-009-9470-7
[6] J.B. Kadane, M.J. Schervish, T. Seidenfeld, A rubinesque theory of decision, in: A. DasGupta (Ed.), A Festschrift for Herman Rubin, 45 of *Institute of Mathematical Statistics Lecture Notes – Monograph Series*, 2004, pp. 45–55. https://doi.org/10.1214/lnms/1196285378
[7] G. De Cooman, Coherent and Archimedean choice in general Banach spaces, Int. J. Approx. Reason. 140 (2022) 255–281. https://doi.org/10.1016/j.ijar.2021.09.005
[8] A. Van Camp, K. Blackwell, J. Konek, et al., Independent natural extension for choice functions, Int. J. Approx. Reason. 152 (2023) 390–413. https://doi.org/10.1016/j.ijar.2022.11.003
[9] A. Van Camp, E. Miranda, Modelling epistemic irrelevance with choice functions, Int. J. Approx. Reason. 125 (2020) 49–72. https://doi.org/10.1016/j.ijar.2020.06.010
[10] A. Decadt, A. Erreygers, J. De Bock, Extending choice assessments to choice functions: an algorithm for computing the natural extension, Int. J. Approx. Reason. 178 (2025) 109331. https://doi.org/10.1016/j.ijar.2024.109331
[11] A. Benavoli, D. Azzimonti, A tutorial on learning from preferences and choices with Gaussian Processes, (2024). https://arxiv.org/abs/2403.11782.
[12] J. De Bock, G. De Cooman, On a notion of independence proposed by Teddy Seidenfeld, in: T. Augustin, F.G. Cozman, G. Wheeler (Eds.), Reflections on the Foundations of Probability and Statistics: Essays in Honor of Teddy Seidenfeld, Springer, 2023, pp. 243–284. https://doi.org/10.1007/978-3-031-15436-2_11
[13] A. Casanova, J. Kohlas, M. Zaffalon, Information algebras in the theory of imprecise probabilities, Int. J. Approx. Reason. 142 (2022) 383–416. https://doi.org/10.1016/j.ijar.2021.12.017
[14] A. Del Pia, A. Khajavirad, The running intersection relaxation of the multilinear polytope, Math. Oper. Res. 46 (3) (2021) 1008–1037. https://doi.org/10.1287/moor.2021.1121
[15] M. Kojima, M. Muramatsu, A note on sparse SOS and SDP relaxations for polynomial optimization problems over symmetric cones, Comput. Optim. Appl. 42 (2009) 31–41. https://doi.org/10.1007/s10589-007-9112-2
[16] J.-B. Lasserre, Convergent SDP relaxations in polynomial optimization with sparsity, SIAM J. Optim. 17 (2006) 822–843. https://doi.org/10.1137/05064504X
[17] X. Doan, X. Li, K. Natarajan, Robustness to dependency in portfolio optimization using overlapping marginals, Oper. Res. 63 (2015). https://doi.org/10.1287/opre.2015.1424

[18] J. Dąbrowska, A. Van Camp, E. Quaeghebeur, The marginal problem for sets of desirable gamble sets, in: S. Destercke, A. Erreygers, M. Nendel, F. Riedel, M.C.M. Troffaes (Eds.), Proceedings of the Fourteenth International Symposium on Imprecise Probabilities: Theories and Applications, 290 of *Proceedings of Machine Learning Research*, PMLR, 2025, pp. 103–114. https://proceedings.mlr.press/v290/dabrowska25a.html.
[19] P.M. Williams, Notes on conditional previsions, Int. J. Approx. Reason. 44 (2007) 366–383. Revised journal version of [williams1975]. https://doi.org/https://doi.org/10.1016/j.ijar.2006.07.019
[20] T. Seidenfeld, M.J. Schervish, J.B. Kadane, Decisions without ordering, in: W. Sieg (Ed.), Acting and Reflecting: The Interdisciplinary Turn in Philosophy, 211 of *Synthese Library*, Kluwer, Dordrecht, 1990, pp. 143–170. Reprinted in [seidenfeld1999], pp. 40–68. https://doi.org/10.1007/978-94-009-2476-5_11
[21] P. Walley, Statistical Reasoning with Imprecise Probabilities, Chapman and Hall, London, London, 1991.
[22] P. Walley, Towards a unified theory of imprecise probability, Int. J. Approx. Reason. 24 (2000) 125–148. https://doi.org/10.1016/S0888-613X(00)00031-1
[23] G. De Cooman, E. Quaeghebeur, Exchangeability and sets of desirable gambles, Int. J. Approx. Reason. 53 (3) (2012) 363–395. Special issue in honour of Henry E. Kyburg, Jr. https://doi.org/10.1016/j.ijar.2010.12.002
[24] G. De Cooman, E. Miranda, Irrelevant and independent natural extension for sets of desirable gambles, J. Artif. Intell. Res. 45 (2012) 601–640. http://www.jair.org/vol/vol45.html.
[25] E. Quaeghebeur, Introduction to Imprecise Probabilities, John Wiley & Sons, 2014. https://doi.org/10.1002/9781118763117.ch1
[26] G. De Cooman, Belief models: an order-theoretic investigation, Ann. Math. Artif. Intell. 45 (2005) 5–34. https://doi.org/10.1007/s10472-005-9006-x
[27] J. De Bock, G. De Cooman, A desirability-based axiomatisation for coherent choice functions, in: Uncertainty Modelling in Data Science (Proceedings of SMPS 2018), 2018, pp. 46–53. https://doi.org/10.1007/978-3-319-97547-4_7
[28] J. De Bock, G. De Cooman, Interpreting, axiomatising and representing coherent choice functions in terms of desirability, Proc. Mach. Learn. Res. 103 (2019) 125–134. ISIPTA 2019 — Proceedings of the Eleventh International Symposium on Imprecise Probability: Theories and Applications, https://proceedings.mlr.press/v103/de-bock19b.html.
[29] J. von Neumann, O. Morgenstern, Theory of Games and Economic Behavior, Princeton, first edition, 1944.
[30] K.J. Arrow, Social Choice and Individual Values, Cowles Foundation Monographs Series, Yale University Press, 1951.
[31] H. Rubin, A weak system of axioms for “rational” behavior and the nonseparability of utility from prior, Stat. Risk Model. 5 (1-2) (1987) 47–58. https://doi.org/10.1524/strm.1987.5.12.47
[32] E. Miranda, A. Van Camp, The law of iterated expectation and imprecise probabilities, Fuzzy Sets Syst. 504 (2025) 109258. https://doi.org/10.1016/j.fss.2024.109258
[33] M.C.M. Troffaes, Decision making under uncertainty using imprecise probabilities, Int. J. Approx. Reason. 45 (1) (2007) 17–29. https://doi.org/10.1016/j.ijar.2006.06.001
[34] I. Levi, The Enterprise of Knowledge, MIT Press, London, London, 1980.
[35] A. Casanova, Rationality and Desirability – A Foundational Study, Ph.D. thesis, Università della Svizzera Italiana, 2023. https://susi.usi.ch/usi/documents/326570.
[36] C. Beeri, R. Fagin, D. Maier, M. Yannakakis, On the desirability of acyclic database schemes, J. ACM 30 (3) (1983) 479-513. https://doi.org/10.1145/2402.322389
[37] F.M. Malvestuto, Existence of extensions and product extensions for discrete probability distributions, Discrete Math. 69 (1) (1988) 61–77. https://doi.org/10.1016/0012-365X(88)90178-1
[38] J. Kohlas, Information Algebras: Generic Structures for Inference, Springer-Verlag, Berlin, Heidelberg, Berlin, Heidelberg, 2003. https://doi.org/10.1007/978-1-4471-0009-6
[39] J. De Bock, A theory of desirable things, arXiv e-prints (2023). arXiv:2302.07412
[40] G. de Cooman, Coherent and Archimedean choice in general Banach spaces, Int. J. Approx. Reason. 140 (2022) 255–281. https://doi.org/10.1016/j.ijar.2021.09.005