INDEPENDENT NATURAL EXTENSION FOR CHOICE FUNCTIONS

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ABSTRACT. We introduce an independence notion for choice functions, which we call 'epistemic independence' following the work by De Cooman et al. [17] for lower previsions, and study it in a multivariate setting. This work is a continuation of earlier work of one of the authors [29], and our results build on the characterization of choice functions in terms of sets of binary preferences recently established by De Bock and De Cooman [11]. We obtain the many-to-one independent natural extension in this framework. Given the generality of choice functions, our expression for the independent natural extension is the most general one we are aware of, and we show how it implies the independent natural extension for sets of desirable gambles, and therefore also for less expressive imprecise-probabilistic models. Once this is in place, we compare this concept of epistemic independence to another independence concept for choice functions proposed by Seidenfeld [28], which De Bock and De Cooman [2] have called S-independence. We show that neither is more general than the other.

1. INTRODUCTION

We study independence concepts for choice functions. Choice functions are a very general imprecise-probabilistic uncertainty model, which has gained increasing interest, instigated by their introduction by Seidenfeld et al. [19, 28]. Rather than working with choice functions directly, we instead work with the completely equivalent framework of sets of desirable gamble sets [11, 12], which inherits its great expressive power. A strict generalization of the more familiar framework of sets of desirable gambles [16, 24, 27, 31], it also allows us to model preferences that are not expressible by mere binary comparisons between options. The dark side of this great power is that sets of desirable gamble sets are typically difficult to work with. They can capture exotic and complicated sorts of opinions and may be computationally demanding. Luckily, it is often manageable to model simple local assessments about variables of interest. In this paper, we investigate the least informative way of combining these assessments into a joint set of desirable gamble sets, assuming that the variables in question are "epistemically independent." We call this the *independent natural extension*. This allows us to turn simple local assessments into a global model.

The first few sections of the paper introduce key formal and notational details and review relevant extant results. We then develop a very general notion of epistemic irrelevance in the sets of desirable gambles sets framework. It specifies when learning a (possibly partial) answer to one question (a nonempty subset of one set of events) provides no relevant information, in the agent's view, to some other question (characterized by a different set of events): epistemic irrelevance for sets of events. In the bulk of the paper, we focus specifically on judgments of epistemic irrelevance between *variables* (learning a partial answer about the values of certain variables provides no relevant information about the values of another variable). Given local assessments about variables, we characterize their independent natural extension and show that this set of desirable gamble *sets* can be obtained using a modified version of the independent natural extension for sets of desirable *gambles*. This is the primary contribution of our paper. We also compare our notion of epistemic independence to another independence notion proposed by Seidenfeld [28], which De Bock and De Cooman [2] have called S-independence. We show that neither is more general than the other. In this part of the paper, we appeal to the more general sets-of-events definition of irrelevance, which has the benefit of not assuming logical independence of the two questions at issue.

This paper is an extended journal version of [30], and contains proofs, a new Section 4, and additional examples throughout. Importantly, Theorem 15 corrects [30, Theorem 20] whose statement was flawed.

2. Choice and Desirability

We start by considering the simple case of binary choice and explain how sets of desirable gambles represent these binary preferences. Then we introduce choice functions (via their dual: rejection functions), and explain how they generalize sets of desirable gambles. Finally, we introduce the main formalism we will be working with for most of the paper: sets of desirable gamble sets, which De Bock and de Cooman [11] have shown are representationally equivalent to choice functions.

Consider a finite set Ω—called *possibility space*—of possible values that a (discrete) uncertain variable X can take. We denote by $\mathcal{L}(\Omega)$ the set of all gambles—real-valued functions—on Ω, often denoted by $\mathcal{L}$ when it is clear from the context what the possibility space is. We interpret a gamble f as an uncertain reward: if the actual outcome turns out to be ω in Ω, then the agent's capital changes by $f(\omega)$. For any two gambles f and g, we write $f \leq g$ when $f(\omega) \leq g(\omega)$ for all ω in Ω, and we write $f < g$ when $f \leq g$ and $f \neq g$. We identify a real constant α with the (constant) gamble that maps every element of Ω to α. We collect all the non-negative gambles—the gambles f for which $f \geq 0$—in the set $\mathcal{L}(\Omega)_{\geq 0}$ (often denoted by $\mathcal{L}_{\geq 0}$), the non-negative and non-zero ones—for which $f > 0$—in $\mathcal{L}(\Omega)_{>0}$ (often denoted by $\mathcal{L}_{>0}$), and the non-positive ones—for which $f \leq 0$—in $\mathcal{L}(\Omega)_{\leq 0}$ (often denoted by $\mathcal{L}_{\leq 0}$). For future reference, we let $A_1 + A_2 \coloneqq \{f + g : f \in A_1, g \in A_2\}$ be the Minkowski addition of two sets $A_1, A_2 \subseteq \mathcal{L}$. Furthermore, for any set $A \subseteq \mathcal{L}$, we let $-A \coloneqq \{-f : f \in A\}$, and define the subtraction $A_1 - A_2 \coloneqq A_1 + (-A_2) = \{f - g : f \in A_1, g \in A_2\}$, for any two sets $A_1, A_2 \subseteq \mathcal{L}$.

We denote by $\mathcal{Q}(\Omega)$ the set of all finite but nonempty subsets of $\mathcal{L}(\Omega)$, also denoted by $\mathcal{Q}$ when it is clear from the context what Ω is. $\mathcal{Q}$ is a subset of the power set $\mathcal{P}(\mathcal{L})$ of $\mathcal{L}$. Elements of $\mathcal{Q}$ are called *gamble sets*.

2.1. Binary Choice: Sets of Desirable Gambles.

Suppose an agent is offered a choice between two gambles—any binary $\{f, g\} \in \mathcal{Q}$. We are not assuming that the agent has enough information to be able to always decide which of these two they prefer. We interpret preference of g over f as a willingness to give up f in exchange for g, which is the same as preferring $g - f$ to the *status quo*; *strict* preference (which rules out indifference) of g over

f also includes the unwillingness to make the opposite trade. The status quo can also be represented by the gamble 0, which results in no change to the agent's net worth. In this way, all preferences between gambles $\{f,g\}$ can also be represented by the two comparisons: 0 vs. $g-f$ and 0 vs. $f-g$. For this reason, any (strict) binary preferences the agent might have, can be represented by a *set of desirable gambles* which collects all the gambles that the agent prefers to the status quo 0. They were introduced by Seidenfeld et al. [27], and have been studied extensively by Walley [31, 32], De Cooman and Quaeghebeur [16], De Cooman and Miranda [15] and Quaeghebeur [24], amongst others. Formally, a set of desirable gambles on Ω is a subset $D \subseteq \mathcal{L}(\Omega)$ of gambles that are preferred over 0. We collect all the sets of desirable gambles in $\mathcal{D} \coloneqq \mathcal{P}(\mathcal{L})$.

Definition 1 (Coherent set of desirable gambles)**.** A set of desirable gambles D is called *coherent* if for all f and g in $\mathcal{L}$, and λ and μ in $\mathbb{R}$:

D_1. $0 \notin D$;
D_2. $\mathcal{L}_{>0} \subseteq D$;
D_3. if $f,g \in D$ and $(\lambda,\mu) > 0$, then $\lambda f + \mu g \in D$.

We collect all the coherent sets of desirable gambles in $\overline{\mathcal{D}}(\Omega)$, also denoted by $\overline{\mathcal{D}}$ when it is clear from the context what the possibility space Ω is.

Before we proceed, let us introduce some notational shortcuts. For any m and n in $\mathbb{N} \cup \{0\}$,[1] we define $m{:}n$ as the set $\{m,\dots,n\}$, which we take to be the empty set when $n < m$. We will use both notations $m{:}n$ and $\{m,\dots,n\}$ throughout. We denote any sequence $(\lambda_1,\dots,\lambda_n)$ by $\lambda_{1:n}$, and define $\lambda_{1:n} > 0 \Leftrightarrow ((\forall j \in \{1,\dots,n\})\lambda_j \geq 0 \text{ and } (\exists j \in \{1,\dots,n\})\lambda_j > 0)$ for any real-valued sequence $\lambda_{1:n}$. In other words, this means that $\lambda_{1:n} > 0 \Leftrightarrow (\lambda_{1:n} \geq 0 \text{ and } \neg(\lambda_{1:n} = 0))$, where we let '$\geq$' and '$=$' work point-wisely on $(\lambda_1,\dots,\lambda_n)$.

This short-hand notation for sequences allows us, for instance, to efficiently introduce the posi operator on $\mathcal{D}$: $\operatorname{posi}(B) \coloneqq \{\sum_{k=1}^{m} \lambda_k f_k : m \in \mathbb{N}, f_{1:m} \in B^m, \lambda_{1:m} > 0\}$ for all $B \subseteq \mathcal{L}$. $\operatorname{posi}(B)$ may be interpreted as the smallest convex cone that includes B.

We call the set of desirable gambles D_1 *at most as informative as* the set of desirable gambles D_2 if $D_1 \subseteq D_2$. The partially ordered set $(\overline{\mathcal{D}},\subseteq)$ of coherent sets of desirable gamble is a complete meet-semilattice. This implies that if a partially specified set $B \subseteq \mathcal{L}$ can be coherently extended—in other words, if $\mathbf{D}(B) \coloneqq \{D \in \overline{\mathcal{D}} : B \subseteq D\} \neq \emptyset$, in which case we will call B *consistent*—there is a unique least informative such extension $\operatorname{cl}_{\overline{\mathcal{D}}}(B) \coloneqq \bigcap \mathbf{D}(B)$:

Theorem 1 ([16, Theorem 1])**.** *Consider any assessment $B \subseteq \mathcal{L}$. Then B is consistent if and only if $\mathcal{L}_{\leq 0} \cap \operatorname{posi}(B) = \emptyset$. If this is the case, then* $\operatorname{cl}_{\overline{\mathcal{D}}}(B) = \operatorname{posi}(\mathcal{L}_{>0} \cup B)$.

Theorem 1 implies that the smallest coherent—called *vacuous*—set of desirable gamble set is $D_{\mathrm{v}} \coloneqq \mathcal{L}_{>0}$.

2.2. **Choice Functions.** For our purposes, it is easiest to motivate choice functions in terms of their duals, *rejection functions*, introduced in an imprecise-probabilistic context by Seidenfeld et al. [28].

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[1] We let $\mathbb{N}$ be the positive natural numbers. We let $\mathbb{R}_{>0} \coloneqq \{x \in \mathbb{R} : x > 0\}$ be the positive real numbers.

Suppose an agent is going to be posed a decision problem consisting of choosing some gamble from among a proffered nonempty, finite subset of gambles—an element of $\mathcal{Q}$. For an agent with imprecise-probabilistic beliefs, there may be no gamble they can select as the best option; even so, it will often be the case that an agent can *reject* certain options as certainly suboptimal. For instance, suppose the agent has a *credal set* $\mathcal{M}$ of probability mass functions on Ω that represent their uncertainty. When offered $A \in \mathcal{Q}$, our agent might choose to reject any gamble g such that $(\forall p \in \mathcal{M})(\exists f \in A) E_p(f) > E_p(g)$. This means that there is some subset of A that the agent knows contains a strictly better choice than g—even if they cannot identify a single gamble that is better. For each possible set of options the agent might be presented with, the agent's *rejection function* collects all the gambles that are judged to be suboptimal in this way.

Definition 2. A rejection function is a function $R: \mathcal{Q} \to \mathcal{Q}: A \mapsto R(A) \subseteq A$.

The idea is that R identifies the *rejected* options from every decision problem posed by $A \in \mathcal{Q}$.

The agent's *choice set* $C(A)$ from a gamble set A collects all of the gambles $A \setminus R(A)$ *not rejected* in this way.

Definition 3. A choice function is a function $C: \mathcal{Q} \to \mathcal{Q}: A \mapsto C(A) \subseteq A$.

Given a rejection function R, the dual choice function C that represents the same preferences is given by $C(A) \coloneqq A \setminus R(A)$, for all A in $\mathcal{Q}$ and, similarly, given a choice function C, the dual rejection function R is given by $R(A) \coloneqq A \setminus C(A)$, for all A in $\mathcal{Q}$. As $A \setminus (A \setminus R(A)) = R(A)$, and similarly for C, we conclude that R and C are equivalent representations of the same preferences, so we can use either.

Choice functions generalize sets of desirable gambles, in the following sense. Given a choice function C, its *binary part* can be summarized by $D_C \coloneqq \{f \in \mathcal{L}: 0 \notin C(\{0, f\})\}$. There will be different choice functions $C_1 \neq C_2$ whose binary parts $D_{C_1} = D_{C_2}$ coincide; see Section 2.4 for more information.

2.3. Sets of Desirable Gamble Sets.

De Bock and de Cooman [11] established another useful equivalent representation to choice functions.

Definition 4 (Set of desirable gamble sets)**.** A set of desirable gamble sets K on Ω is a subset of $\mathcal{Q}(\Omega)$. We collect all the sets of desirable gamble sets in $\mathcal{K} \coloneqq \mathcal{P}(\mathcal{Q})$.

The idea is that the set of desirable gamble sets K collects all the gamble sets that contain at least one gamble that the agent strictly prefers over the status quo 0. A set of desirable gamble sets K is an equivalent representation to a choice function C: given a choice function C, the corresponding set of desirable gamble sets K is given by $K \coloneqq \{A \in \mathcal{Q}: 0 \notin C(\{0\} \cup A)\}$, and therefore $A \in K \Leftrightarrow 0 \notin C(\{0\} \cup A) \Leftrightarrow 0 \in R(\{0\} \cup A)$, for any gamble set A. In other words, K collects all the gamble sets that contain some gamble preferred to 0. Conversely, given a set of desirable gamble sets K, its corresponding rejection function R is given by $R(A) \coloneqq \{f \in A: A - \{f\} \in K\}$—and therefore its corresponding choice function C by $C(A) \coloneqq \{f \in A: A - \{f\} \notin K\}$—for every gamble set A. Under conditions that are implied by *coherence*, which we are about to define for sets of desirable gamble sets in Definition 5 underneath, and for which we refer to [11] for choice functions, both correspondences commute [see [11]]. Moreover, coherence is preserved by the correspondences. Given this connection, all our results in this paper will apply for choice functions as well. We will use sets of desirable gamble sets mainly for practical reasons: they are easier to work with.

De Bock and De Cooman [11] gave an axiomatization of *coherent* sets of desirable gamble sets—sets of desirable gamble sets of rational agents. We refer to their article for a justification of the axioms.

Definition 5 (Coherent set of desirable gamble sets)**.** A set of desirable gamble sets $K \subseteq \mathcal{Q}$ is called *coherent* if for all A, A_1 and A_2 in $\mathcal{Q}$, all $\{\lambda_{f,g}, \mu_{f,g} : f \in A_1, g \in A_2\} \subseteq \mathbb{R}$, and all f in $\mathcal{L}$:

K_0. $\varnothing \notin K$;
K_1. $A \in K \Rightarrow A \setminus \{0\} \in K$;
K_2. $\{f\} \in K$, for all f in $\mathcal{L}_{>0}$;
K_3. if $A_1, A_2 \in K$ and if, for all f in A_1 and g in A_2, $(\lambda_{f,g}, \mu_{f,g}) > 0$, then $\{\lambda_{f,g} f + \mu_{f,g} g : f \in A_1, g \in A_2\} \in K$;
K_4. if $A_1 \in K$ and $A_1 \subseteq A_2$ then $A_2 \in K$.

We collect all the coherent sets of desirable gamble sets in the collection $\overline{\mathcal{K}}(\Omega)$, often simply denoted by $\overline{\mathcal{K}}$.

In item K_3 of this definition we have used the short-hand notation $(\lambda_{f,g}, \mu_{f,g}) > 0$ introduced earlier, which means '$\lambda_{f,g} \geq 0$ and $\mu_{f,g} \geq 0$, with at least one of the real numbers $\lambda_{f,g}$ and $\mu_{f,g}$ strictly positive'.

Given two sets of desirable gamble sets K_1 and K_2, we follow De Bock and De Cooman [11] in calling K_1 *at most as informative as* K_2 if $K_1 \subseteq K_2$. The resulting partially ordered set $(\mathcal{K}, \subseteq)$ is a complete lattice where intersection serves the role of infimum, and union that of supremum. De Bock and De Cooman [11, Theorem 8] furthermore show that the partially ordered set $(\overline{\mathcal{K}}, \subseteq)$ of coherent sets of desirable gamble sets is a complete meet-semilattice: given an arbitrary family $\{K_i : i \in I\} \subseteq \overline{\mathcal{K}}$, its infimum $\inf\{K_i : i \in I\} = \bigcap_{i \in I} K_i$ is a coherent set of desirable gamble sets. This allows for conservative reasoning: it makes it possible to extend a partially specified set of desirable gamble sets to the most conservative—least informative—coherent one that includes it. This procedure is called *natural extension*:

Definition 6 ([11, Definition 9])**.** For any assessment $\mathcal{A} \subseteq \mathcal{Q}$, we let $\mathbf{K}(\mathcal{A}) := \{K \in \overline{\mathcal{K}} : \mathcal{A} \subseteq K\}$. We call the assessment $\mathcal{A}$ *consistent* if $\mathbf{K}(\mathcal{A}) \neq \varnothing$, and we then call $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A}) := \bigcap \mathbf{K}(\mathcal{A})$ the *natural extension* of $\mathcal{A}$.

One of the main results of De Bock and De Cooman [11] is their expression for the natural extension:

Theorem 2 ([11, Theorem 10])**.** *Consider any assessment $\mathcal{A} \subseteq \mathcal{Q}$. Then $\mathcal{A}$ is consistent if and only if $\varnothing \notin \mathcal{A}$ and $\{0\} \notin \mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A})$. If this is the case, then $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A}) = \mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A}))$.*

Here we used the set $\mathcal{L}^{\mathrm{s}}(\Omega)_{>0} \coloneqq \{\{f\}: f \in \mathcal{L}(\Omega)_{>0}\}$—often denoted simply by $\mathcal{L}^{\mathrm{s}}_{>0}$—and the two operations on $\mathcal{K}$ defined by $\mathrm{Rs}(K) \coloneqq \{A \in \mathcal{Q}: (\exists B \in K) B \setminus \mathcal{L}_{\leq 0} \subseteq A\}$ and

$$\mathrm{Posi}(K) \coloneqq \left\{\left\{\sum_{k=1}^{m} \lambda_k^{f_{1:m}} f_k : f_{1:m} \in \bigtimes_{\ell=1}^{m} A_\ell\right\} : m \in \mathbb{N}, A_1, \dots, A_m \in K, \Big(\forall f_{1:m} \in \bigtimes_{\ell=1}^{m} A_\ell\Big) \lambda_{1:m}^{f_{1:m}} > 0\right\}$$

for all K in $\mathcal{K}$. Both Rs and Posi are *closure operators*: they are extensive, monotone and idempotent; this implies in particular that $\mathrm{Rs}(\mathrm{posi}(K)) = K$, for any $K \in \overline{\mathcal{K}}$.

2.4. **Binary Choice and Representation.** A set of desirable gamble sets K collects all the gamble sets A that contain at least one gamble that the agent strictly prefers over 0. For instance, the agent may know that one of $\{f_1, f_2\}$ is preferred over 0, but she may not know which one it is. So K can represent more than binary choice: indeed, she may have no preference in the binary choices $\{0, f_1\}$ and $\{0, f_2\}$, but in the ternary choice $\{0, f_1, f_2\}$ reject 0. In this section we will quickly summarize relevant known results about the binary choices captured by a set of desirable gamble sets.

Given a set of desirable gamble sets K, its binary behavior is summarized in the set of desirable gambles $D_K \coloneqq \{f \in \mathcal{L}: \{f\} \in K\}$; D_K contains the gambles f that form desirable gamble singletons $\{f\} \in K$. Recall that the gamble singletons $\{f\}$ represent a binary choice between f and 0, and therefore between any gambles g and h for which $g - h = f$.

Conversely, given a coherent set of desirable gambles D, there might be multiple coherent K that imply the same binary choices D_K that are reflected in D: the nonempty collection $\{K \in \overline{\mathcal{K}}: D_K = D\}$ may have more than one element. However, it always contains one unique smallest element, which we call $K_D \coloneqq \{A \in \mathcal{Q}: A \cap D \neq \emptyset\}$ [see [29, Proposition 5]]. De Bock and De Cooman [12, Proposition 8] show that K_D is coherent if and only if D is. We call any set of desirable gamble sets K *binary* if there is a set of desirable gambles D such that $K = K_D$. The smallest coherent—called *vacuous*—set of desirable gamble sets is binary, and given by $K_{\mathrm{v}} = K_{D_{\mathrm{v}}} = \{A \in \mathcal{Q}: A \cap \mathcal{L}_{>0} \neq \emptyset\}$.

De Bock and De Cooman [11] establish an important representation result for coherent sets of desirable gamble sets. They show that any coherent set of desirable gamble sets K can be represented by a collection $\mathbf{D}$ of coherent sets of desirable gambles:[2]

Theorem 3 (Representation [12, Theorem 9])**.** *Any set of desirable gamble sets K is coherent if and only if there is a nonempty set $\mathbf{D} \subseteq \overline{\mathcal{D}}$ of coherent sets of desirable gambles such that $K = \bigcap\{K_D : D \in \mathbf{D}\}$. We then say that $\mathbf{D}$* represents *K. Moreover, K's largest representing set is $\mathbf{D}(K) \coloneqq \{D \in \overline{\mathcal{D}}: K \subseteq K_D\}$.*

Note that $\mathbf{D}(K)$ is an *up-set* in $\overline{\mathcal{D}}$: if $D_1 \in \mathbf{D}(K)$ and $D_1 \subseteq D_2$, then $D_2 \in \mathbf{D}(K)$, for any D_1 and D_2 in $\overline{\mathcal{D}}$. Given a coherent set of desirable gambles D, the set $\uparrow D \coloneqq \{D' \in \overline{\mathcal{D}}: D \subseteq D'\}$ is the smallest up-set that contains D. Given a collection $\mathbf{D} \subseteq \overline{\mathcal{D}}$ of coherent sets of desirable gambles, we denote by $\uparrow \mathbf{D} \coloneqq \{D' \in \overline{\mathcal{D}}: (\exists D \in \mathbf{D}) D \subseteq D'\} = \bigcup_{D \in \mathbf{D}} \uparrow D$ the smallest up-set containing $\mathbf{D}$. Since $\mathbf{D}(K)$ is an up-set, we have $\uparrow \mathbf{D}(K) = \mathbf{D}(K)$. It turns out that up-sets are useful in determining whether a given coherent set of desirable gamble sets K is represented by some $\mathbf{D} \subseteq \overline{\mathcal{D}}$.

[2]This theorem first appeared in De Bock and De Cooman [11, Theorem 7], but we prefer their later formulation in [12, Theorem 9].

Proposition 4. *Consider any coherent set of desirable gamble sets K and any $\mathbf{D} \subseteq \overline{\mathcal{D}}$. If $\uparrow\mathbf{D} = \mathbf{D}(K)$, then $\mathbf{D}$ represents K.*

Proof. That $\uparrow\mathbf{D} = \mathbf{D}(K)$ implies that $\mathbf{D} \subseteq \mathbf{D}(K)$, whence $\bigcap_{D\in\mathbf{D}} K_D \supseteq K$, so it remains to show that $\bigcap_{D\in\mathbf{D}} K_D \subseteq K$. To this end, consider any A in $\bigcap_{D\in\mathbf{D}} K_D$. Then $(\forall D \in \mathbf{D})A \cap D \neq \emptyset$, which implies $(\forall D \in \uparrow\mathbf{D})A \cap D \neq \emptyset$ since every element of $\uparrow\mathbf{D}$ is a superset of an element of $\mathbf{D}$, and hence preserves the property of intersecting A. But since $\uparrow\mathbf{D} = \mathbf{D}(K)$, this set represents K, whence, indeed, $A \in K$. □

Note that the reverse implication does not hold.[3]

Example 1. Work with the binary possibility space $\Omega \coloneqq \{H, T\}$ and consider the vacuous set of desirable gamble sets K_{v} on Ω, introduced earlier in this section, where we observed that K_{v} is a binary set of desirable gamble sets $K_{D_{\mathrm{v}}}$ so it is represented by $\{D_{\mathrm{v}}\}$. Its largest representation is $\mathbf{D}(K_{\mathrm{v}}) = \overline{\mathcal{D}}$, the set of all coherent sets of desirable gambles. We will show that $\mathbf{D} \coloneqq \overline{\mathcal{D}} \setminus \{D_{\mathrm{v}}\}$, the collection of all coherent sets of desirable gambles except the vacuous one, represents the same set of desirable gamble sets K_{v}. Note that $\mathbf{D}$ is an up-set, so $\uparrow\mathbf{D} = \mathbf{D} \neq \mathbf{D}(K_{\mathrm{v}})$, and hence this establishes that K_{v} has a representation $\mathbf{D}$ whose up-set differs from $\mathbf{D}(K_{\mathrm{v}})$.

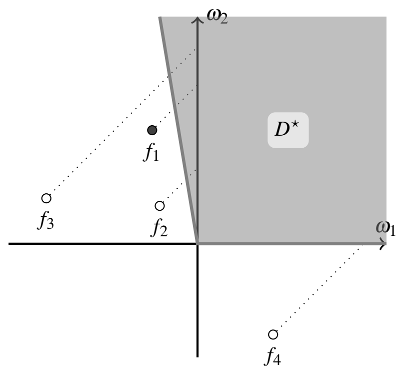
To check that the set of desirable gamble sets $K^\star \coloneqq \bigcap_{D\in\mathbf{D}} K_D$—which is coherent by Theorem 3—equals K_{v}, since $\mathbf{D} \subseteq \mathbf{D}(K_{\mathrm{v}})$ we only need to check $K^\star \subseteq K_{\mathrm{v}}$. To do so, consider any $A \notin K_{\mathrm{v}}$, whence $A \cap \mathcal{L}_{>0} = \emptyset$. If $A \subseteq \mathcal{L}_{\leq 0}$ then for every D in $\mathbf{D}$ we find that $A \cap D = \emptyset$ by D's coherence[4] whence $A \notin K^\star$. So assume that $A \nsubseteq \mathcal{L}_{\leq 0}$. Then $B \coloneqq \left\{\frac{f}{|f(H)|+|f(T)|} : f \in A \setminus \mathcal{L}_{\leq 0}\right\}$ is nonempty, and also $B \cap (\mathcal{L}_{\leq 0} \cup \mathcal{L}_{>0}) = \emptyset$. This implies that every element g of B can be denoted as $g = (g(H), g(T)) = (-\alpha_g, 1-\alpha_g)$ or as $g = (1-\alpha_g, -\alpha_g)$ for some α_g in $(0,1)$. Let $\alpha^\star \coloneqq \min\{\alpha_g : g \in B\}$, and $g^\star$ be an element of B for which this minimum is reached—so it is an element of B such that $g^\star = (-\alpha^\star, 1-\alpha^\star)$ or $g^\star = (1-\alpha^\star, -\alpha^\star)$. Consider $D^\star \coloneqq \operatorname{posi}(\{g^\star + \frac{\alpha^\star}{2}\} \cup \mathcal{L}_{>0})$, which is a coherent set of desirable gambles due to Theorem 1 since $g^\star \notin \mathcal{L}_{\leq 0}$ and therefore also $g^\star + \frac{\alpha^\star}{2} \notin \mathcal{L}_{\leq 0}$. Note that $g^\star + \frac{\alpha^\star}{2}$ either is equal to $(-\frac{\alpha^\star}{2}, 1-\frac{\alpha^\star}{2})$ or to $(1-\frac{\alpha^\star}{2}, -\frac{\alpha^\star}{2})$; in any case $g^\star + \frac{\alpha^\star}{2} \notin \mathcal{L}_{>0}$ so $D^\star \neq D_{\mathrm{v}}$, whence $D^\star \in \mathbf{D}$. An element $(-\alpha, 1-\alpha)$ or $(1-\alpha, -\alpha)$ of B would belong to $D^\star$ only if $\alpha \leq \frac{\alpha^\star}{2}$,

[3] The converse implication does hold for a generalisation of our sets of desirable gamble sets that can contain *infinite* gamble sets, appropriately defined by Jasper De Bock in private communication and ongoing work [1] and also by Campbell-Moore [3]. To see this, consider any $\mathbf{D} \subseteq \overline{\mathcal{D}}$ that represents a given coherent set of desirable gamble sets K: this means that $K = \bigcap_{D\in\mathbf{D}} K_D$, or, in other words, that $A \in K \Leftrightarrow (\forall D \in \mathbf{D})A \cap D \neq \emptyset$, for every gamble set A. Since every element of $\uparrow\mathbf{D}$ is a superset of an element of $\mathbf{D}$, this implies that $\uparrow\mathbf{D}$ represents K, too. But then a representation theorem of De Bock [1, Theorem 1] valid in this context implies that $\uparrow\mathbf{D} \subseteq \mathbf{D}(K)$: this theorem concludes that $\mathbf{D}(K)$ is the largest set of sets of desirable gambles that represents K, so any other set of sets of desirable gambles that represents K must be subset of $\mathbf{D}(K)$. Assume *ex absurdo* that $\uparrow\mathbf{D} \neq \mathbf{D}(K)$, which would imply that $\uparrow\mathbf{D} \subset \mathbf{D}(K)$, so there would be some $D^\star$ in $\mathbf{D}(K)$ such that $D^\star \notin \uparrow\mathbf{D}$, and hence $D^\star \nsupseteq D$ for every D in $\mathbf{D}$. For every D in $\mathbf{D}$, consider any f_D in the nonempty $D \setminus D^\star$, and let $A \coloneqq \{f_D : D \in \mathbf{D}\}$, a possibly infinite gamble set. Then, for every D in $\mathbf{D}$, we would find that f_D belongs to $A \cap D$, whence $A \cap D \neq \emptyset$. Since $\mathbf{D}$ represents K, this would imply that $A \in K$. But $A \cap D^\star = \emptyset$ so $A \notin \bigcap_{D'\in\mathbf{D}(K)} K_{D'}$, implying that $\mathbf{D}(K)$ does not represent K, a contradiction with the conclusions of [1, Theorem 1].

[4] To see this, first note that $0 \notin D$ by Axiom D_1. Moreover, for any gamble $f < 0$ we find that $-f > 0$ so Axiom D_1 implies that $-f \in D$. If also f would belong to D, then by Axiom D_3 with $g \coloneqq -f$ and $(\lambda, \mu) = (1,1)$ we would find that $f - f = 0 \in D$, contradicting Axiom D_1.

which is impossible due to the definition of $\alpha^\star$, whence $B \cap D^\star = \emptyset$. Since A consists of scaled versions of elements of B, together with a subset of $\mathcal{L}_{\leq 0}$, we find that also $A \cap D^\star = \emptyset$. But this implies that, indeed, $A \notin K^\star$.

The figure underneath illustrates the idea of this example. Of the gamble set $A = \{f_1, f_2, f_3, f_4\}$ the gamble f_1 corresponding to $g^\star$ above is indicated with a black dot. This gamble is used to construct $D^\star$ different from D_{v} which does not intersect A.



◊

We end this section by introducing an important class of coherent sets of desirable gamble sets, namely the *E-admissible* sets of desirable gamble sets, which we will use later on.

Example 2. Consider an arbitrary nonempty (possibly non-convex) collection $\mathcal{M} \subseteq \Sigma_\Omega \coloneqq \{p \in \mathcal{L}(\Omega)_{\geq 0} : \sum_{\omega \in \Omega} p(\omega) = 1\}$ of probability mass functions on Ω, called a *credal set*.[5] Let us associate with it the *E-admissible* set of desirable gamble sets

$$K_{\mathcal{M}} \coloneqq \left\{A \in \mathcal{Q} : A \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\forall p \in \mathcal{M})(\exists f \in A) E_p(f) > 0\right\}.$$

$K_{\mathcal{M}}$ collects all the gamble sets A that either contain a non-negative non-zero gamble, or for every probability mass function p in $\mathcal{M}$ contain a gamble with positive p-expectation.

Let us show that $K_{\mathcal{M}}$ is a coherent set of desirable gamble sets. One way to obtain this result is by checking that it satisfies all the rationality requirements from Definition 5, which is a cumbersome task. Thanks to Theorem 3 there is a much more elegant way to obtain this: we claim that $K_{\mathcal{M}}$ is represented by the nonempty $\{D_p : p \in \mathcal{M}\} \subseteq \overline{\mathcal{D}}$, and is therefore coherent. Here $D_p \coloneqq \{f \in \mathcal{L} : f \in \mathcal{L}_{>0} \text{ or } E_p(f) > 0\}$ is the coherent set of desirable gambles that either have a positive p-expectation or are non-negative and non-zero.

Lemma 5. *$K_{\mathcal{M}}$ is represented by $\{D_p : p \in \mathcal{M}\}$.*

Proof. We will show that (i) $K_{\mathcal{M}} \subseteq \bigcap\{K_{D_p} : p \in \mathcal{M}\}$ and (ii) $K_{\mathcal{M}} \supseteq \bigcap\{K_{D_p} : p \in \mathcal{M}\}$. For (i), consider any A in $K_{\mathcal{M}}$, meaning that $A \cap \mathcal{L}_{>0} \neq \emptyset$ or $(\forall p \in \mathcal{M})(\exists f \in A) E_p(f) > 0$. Both cases imply that $A \cap D_p \neq \emptyset$ for every p in $\mathcal{M}$, whence indeed $A \in \bigcap\{K_{D_p} : p \in \mathcal{M}\}$. For

[5] Σ_Ω is called the *simplex* on Ω: it is the collection of all probability mass functions on Ω.

(ii), consider any A in $\bigcap\{K_{D_p}: p \in \mathcal{M}\}$, meaning that $A \cap D_p \neq \emptyset$ for all p in $\mathcal{M}$, and hence indeed $A \in K_{\mathcal{M}}$. □

Therefore, by Theorem 3 and Lemma 5 we conclude that $K_{\mathcal{M}}$ is coherent.

The choice function $C_{\mathcal{M}}$ that corresponds to $K_{\mathcal{M}}$, is given by $C_{\mathcal{M}}(A) = \bigcup_{p\in\mathcal{M}}\{f \in A: (\forall g \in A)(E_p(f) \geq E_p(g) \text{ and } g \ngtr f\})$, for every $A \in \mathcal{Q}$. Indeed, using the correspondence $f \in C_{\mathcal{M}}(A) \Leftrightarrow A - \{f\} \notin K_{\mathcal{M}}$ for every A in $\mathcal{Q}$ and f in A, discussed in Section 2, we find that $f \in C_{\mathcal{M}}(A) \Leftrightarrow ((A - \{f\}) \cap \mathcal{L}_{>0} = \emptyset$ and $(\exists p \in \mathcal{M})(\forall g \in A - \{f\})E_p(g) \leq 0)$. The first conjunct is equivalent to $(\forall g \in A) g \ngtr f$, and the second conjunct to $(\exists p \in \mathcal{M})(\forall g \in A)E_p(f) \geq E_p(g)$, taking into account the linearity of E_p, which establishes the proposed expression for $C_{\mathcal{M}}$. This explains the reason why we decided to call $K_{\mathcal{M}}$ 'E-admissible': $C_{\mathcal{M}}$ is the E-admissible choice function [22, 25].[6] ◊

Jasper De Bock and Gert de Cooman showed us via private communication that Theorem 3 also allows for a simpler expression for the natural extension:

Theorem 6 (Due to De Bock & De Cooman)**.** *An assessment $\mathcal{A} \subseteq \mathcal{Q}$ is consistent if and only if there is some D in $\overline{\mathcal{D}}$ such that $\mathcal{A} \subseteq K_D$. In that case* $\mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A}) = \bigcap\left\{K_D : D \in \overline{\mathcal{D}} \text{ and } \mathcal{A} \subseteq K_D\right\}$.

As a consequence, any consistent assessment $\mathcal{A}$'s natural extension is represented by $\{D \in \overline{\mathcal{D}}: \mathcal{A} \subseteq K_D\} = \{D \in \overline{\mathcal{D}}: (\forall B \in \mathcal{A}) B \cap D \neq \emptyset\}$].

3. Conditioning

Suppose that we have a belief model about the uncertain variable X, be it a coherent set of desirable gamble sets on Ω or a coherent set of desirable gambles on Ω, or—less generally—a set of probability mass functions on Ω. (We can think of a precise probability as a singleton.) When new information becomes available in the form of 'X assumes a value in some (nonempty) subset E of Ω', we can take this into account by conditioning our belief model on E.

We will let any event, except for the (trivially) impossible event $\emptyset$, serve as a conditioning event. We collect the allowed conditioning events in $\mathcal{P}^+(\Omega) \coloneqq \{E \subseteq \Omega : E \neq \emptyset\}$. For any E in $\mathcal{P}^+(\Omega)$ and any gamble f on E, we let its multiplication $\mathbb{I}_E f$ denote the gamble on Ω defined by

$$(\mathbb{I}_E f)(\omega) \coloneqq \begin{cases} f(\omega) & \text{if } \omega \in E \\ 0 & \text{if } \omega \notin E \end{cases} \tag{1}$$

for all ω in Ω. $\mathbb{I}_E f$ is the *called-off* version of f: if E does not occur, the gamble will yield 0. We will often also use *indicator gambles* (which we also sometimes refer to simply as "indicators") $\mathbb{I}_E \coloneqq \mathbb{I}_E 1$ for every $E \subseteq \Omega$; it follows from the definition above that $\mathbb{I}_E$ equals 1 on E and 0 on E^c.

[6] Although Levi's notion of E-admissibility was originally concerned with convex closed sets of probability mass functions [22, Chapter 5], we impose no such requirement here on the set $\mathcal{M}$.

Definition 7 (Conditioning)**.** Given any set of desirable gamble sets K on Ω and any E in $\mathcal{P}^+(\Omega)$, we define the *conditional set of desirable gamble sets* $K\rfloor E$ on $\mathcal{L}(E)$ as $K\rfloor E \coloneqq \{A \in \mathcal{Q}(E) : \mathbb{I}_E A \in K\}$, where for any A in $\mathcal{Q}(E)$ and E in $\mathcal{P}^+(\Omega)$, we let $\mathbb{I}_E A \coloneqq \{\mathbb{I}_E g : g \in A\} \in \mathcal{Q}(\Omega)$ be a set of called-off gambles.

It follows at once that conditioning preserves the order: if $K_1 \subseteq K_2$ then $K_1\rfloor E \subseteq K_2\rfloor E$. This definition coincides with the usual definition for sets of desirable gambles, in the sense that $K_D\rfloor E = K_{D\rfloor E}$, where

$$D\rfloor E \coloneqq \{f \in \mathcal{L}(E) : \mathbb{I}_E f \in D\} \tag{2}$$

is the set of desirable gambles conditional on E (see Van Camp and Miranda [29, Propositions 8]). For more details about conditioning sets of desirable gambles, we refer to [16, 31]. In order to elegantly work with $K\rfloor E$'s representation in terms of sets of desirable gambles, let us define $\mathbf{D}\rfloor E \coloneqq \{D\rfloor E : D \in \mathbf{D}\}$ for any $\mathbf{D} \subseteq \mathcal{D}$.

Proposition 7. *Consider any coherent set of desirable gamble sets K on Ω, any representation $\mathbf{D}$ of K, and any conditioning event E in $\mathcal{P}^+$. Then $K\rfloor E$ is coherent. Furthermore, $K\rfloor E$ is represented by $\mathbf{D}\rfloor E$, meaning that $K\rfloor E = \bigcap\{K_D : D \in \mathbf{D}\rfloor E\}$.*

Proof. The first statement is already established by Van Camp and Miranda [29, Propositions 7], so we limit ourselves to proving the second statement. To this end, consider any A in $\mathcal{Q}(E)$, and infer the following chain of equivalences:

$$A \in K\rfloor E \Leftrightarrow \mathbb{I}_E A \in K \Leftrightarrow (\forall D \in \mathbf{D})\mathbb{I}_E A \cap D \neq \emptyset \Leftrightarrow (\forall D \in \mathbf{D}\rfloor E) A \cap D \neq \emptyset \Leftrightarrow A \in \bigcap_{D \in \mathbf{D}\rfloor E} K_D,$$

where the second equivalence follows from the assumption that $\mathbf{D}$ represents K. Since the choice of A in $\mathcal{Q}(E)$ was arbitrary, this implies that, indeed, $K\rfloor E = \bigcap\{K_D : D \in \mathbf{D}\rfloor E\}$. □

As a consequence, since any coherent K is represented by $\mathbf{D}(K)$, Proposition 7 implies that in particular $K\rfloor E = \bigcap\{K_D : D \in \mathbf{D}(K)\rfloor E\}$.

Example 3. Let us build on Example 2, and condition the E-admissible set of desirable gamble sets $K_{\mathcal{M}}$ for some credal set $\mathcal{M} \subseteq \operatorname{int}(\Sigma_{\mathcal{X}})$,[7] on an event G in $\mathcal{P}^+(\Omega)$. Then, for any A in $\mathcal{Q}(G)$, we have $A \in K_{\mathcal{M}}\rfloor G \Leftrightarrow \mathbb{I}_G A \in K_{\mathcal{M}}$, which is equivalent to the requirement that for any p in $\mathcal{M}$ there is some f in A such that $E_p(\mathbb{I}_G f) > 0$, or $\mathbb{I}_G A \cap \mathcal{L}(\Omega)_{>0} \neq \emptyset$. Consider some p in $\mathcal{M}$. Since $\mathcal{M} \subseteq \operatorname{int}(\Sigma_{\mathcal{X}})$, the p-expectation $E_p(g) > 0$ is positive for every non-negative and non-zero g, so the requirement $\mathbb{I}_G A \cap \mathcal{L}(\Omega)_{>0} \neq \emptyset$ implies that there is some f in A such that $E_p(\mathbb{I}_G f) > 0$. So we find $A \in K_{\mathcal{M}}\rfloor G$ if and only if for every p in $\mathcal{M}$ there is some f in A such that $E_p(\mathbb{I}_G f) > 0$. Since $E_{p|G}(f) = E_p(\mathbb{I}_G f)/E_p(\mathbb{I}_G)$, we have that $E_p(\mathbb{I}_G f) > 0 \Leftrightarrow E_{p|G}(f) > 0$. So the conditional set of desirable gamble sets $K_{\mathcal{M}}\rfloor G$ is equal to the E-admissible set of desirable gamble sets $K_{\{p|G : p \in \mathcal{M}\}}$ obtained by an element-wise application of Bayes's rule on $\mathcal{M}$.

We obtain the same result in a simpler way, using Proposition 7. Note that, by Lemma 5, $K_{\mathcal{M}}$ is represented by $\{D_p : p \in \mathcal{M}\}$, so Proposition 7 tells us that $K_{\mathcal{M}}\rfloor G$ is represented by $\{D_p\rfloor G : p \in \mathcal{M}\}$. Infer for any gamble f in $\mathcal{L}(G)$ that

$$f \in D_p\rfloor G \Leftrightarrow \mathbb{I}_G f \in D_p \Leftrightarrow (E_p(\mathbb{I}_G f) > 0 \text{ or } \mathbb{I}_G f > 0) \Leftrightarrow E_p(\mathbb{I}_G f) > 0$$

[7] We use the (topological) interior $\operatorname{int}(\Sigma_{\mathcal{X}})$ of $\Sigma_{\mathcal{X}}$ to make sure that every outcome in Ω has a (strictly) positive probability for every element of $\mathcal{M}$, and therefore that $E_p(f) > 0$ for every p in $\mathcal{M}$ and f in $\mathcal{L}_{>0}$.

$$\Leftrightarrow E_{p|G}(f) > 0 \Leftrightarrow (E_{p|G}(f) > 0 \text{ or } f > 0) \Leftrightarrow f \in D_{p|G},$$

where the third and fourth equivalences hold because $p \in \operatorname{int}(\Sigma_{\mathcal{X}})$. This implies that $p|G \in \operatorname{int}(\Sigma_G)$, which explains the fifth equivalence. So we find that $D_p\rfloor G = D_{p|G}$, and therefore $K_{\mathcal{M}}\rfloor G$ is represented by $\{D_{p|G}: p \in \mathcal{M}\}$. Again using Lemma 5, this implies that, indeed, $K_{\mathcal{M}}\rfloor G = K_{\{p|G: p\in\mathcal{M}\}}$. ◊

4. Epistemic independence for sets of events

Consider any two nonempty sets of events $\mathcal{E}, \mathcal{F} \subseteq \mathcal{P}^+(\Omega)$. Intuitively, $\mathcal{E}$ and $\mathcal{F}$ represent two *topics of investigation*. We can think of each event E in a set $\mathcal{E}$ as posing a yes-no question: did (will) E occur or not? A complete result of the investigation would be a truth value assignment to every E in $\mathcal{E}$.

Suppose the agent has an assessment given by a set of desirable gamble sets K defined on Ω. We want to ask: when does learning something about the first topic change the agent's beliefs about the second? Since Ω is finite, so is $\mathcal{E}$, and we let $m \coloneqq |\mathcal{E}|$ and $\mathcal{E} = \{E_1, \ldots, E_m\}$.

To develop an answer to this question, we first need to figure out: what do we mean by *learning about* $\mathcal{E}$? And what are the beliefs about $\mathcal{F}$ encoded in K? Our model of learning will be factive and propositional: we mean that the agent updates (conditions) on some proposition about $\mathcal{E}$. So, the first question is: what propositions are about $\mathcal{E}$? Some propositions about $\mathcal{E}$ result from assigning a truth value to every E in $\mathcal{E}$. Every such proposition corresponds to an $\mathcal{E}$-*atom* in the set

$$\mathcal{A}_{\mathcal{E}} \coloneqq \left\{ \bigcap_{k=1}^{m} B_k : (\forall k \in \{1, \ldots, m\} B_k \in \{E_k, E_k^c\}) \right\} \setminus \{\emptyset\}$$

of all $\mathcal{E}$-atoms, which partitions Ω. Any element of the Boolean closure of $\mathcal{E}$ is expressible as the union of some constituents of $\mathcal{E}$.

So, what we mean by learning something about $\mathcal{E}$ is obtaining information that some events in $\mathcal{E}$ do or do not obtain and (exclusive) disjunctions thereof, or in other words, learning that an element of

$$\mathcal{W}_{\mathcal{E}} \coloneqq \left\{ \bigcup G : G \in \mathcal{P}^+(\mathcal{A}_{\mathcal{E}}) \right\} \subseteq \mathcal{P}^+(\Omega)$$

occurs. Whenever $\mathcal{E}$ does not exhaust of Ω, one could for instance learn that none of the events in $\mathcal{E}$ obtain. This corresponds to the event

$$\bigcap_{k=1}^{m} E_k^c \in \mathcal{W}_{\mathcal{E}}.$$

As a trivial example, one could learn nothing about $\mathcal{E}$—by which we mean that there is no event in $\mathcal{E}$ of which we learn anything—which corresponds to the uninformative event $\bigcup \mathcal{A}_{\mathcal{E}} = \Omega$, which we will leave as a degenerate case.

We represent learning by conditioning, as explained in Definition 7. However, in Section 3, conditioning a coherent $K \subseteq \mathcal{Q}(\Omega)$ on an event E in $\mathcal{P}^+(\Omega)$ yields the coherent conditional $K\rfloor E \subseteq \mathcal{Q}(E)$. Because our definition of epistemic irrelevance will turn out cleaner, we will in this section work with a slightly different—but closely related—conditional set of desirable gamble sets:[8]

$$K\|E \coloneqq \{A \in \mathcal{Q}(\Omega) : \mathbb{I}_E A \in K\} \subseteq \mathcal{Q}(\Omega).$$

Next, we need to explain what we mean by the agent's beliefs about $\mathcal{F}$. To do this, we need to relate gambles "about $\mathcal{F}$" to gambles on Ω.

Definition 8 (Cylindrical extension)**.** Given any nonempty set of events $\mathcal{F} \subseteq \mathcal{P}^+(\Omega)$ and any gamble f on $\mathcal{A}_\mathcal{F}$, we let its *cylindrical extension* f^* to Ω be defined by

$$f^*(\omega) \coloneqq f(B) \text{ where } B \text{ is the unique element of } \mathcal{A}_\mathcal{F} \text{ for which } \omega \in B,$$

for all ω in Ω. Similarly, given any set of gambles $A \subseteq \mathcal{L}(\mathcal{A}_\mathcal{F})$, we let its *cylindrical extension* $A^* \subseteq \mathcal{L}(\Omega)$ be defined as $A^* \coloneqq \{f^* : f \in A\}$.

Formally, f belongs to $\mathcal{L}(\mathcal{A}_\mathcal{F})$ while f^* belongs to $\mathcal{L}(\Omega)$. However, f^* is constant on every element of $\mathcal{A}_\mathcal{F}$, so depends only on which element of $\mathcal{A}_\mathcal{F}$ occurs. Therefore, f^* is completely determined by f and *vice versa* and as such, they contain the same information and correspond to the same transaction. They are therefore indistinguishable from a behavioral point of view.

Remark 1. We will frequently use the simplifying device of *identifying* a gamble f on $\mathcal{A}_\mathcal{F}$ with its cylindrical extension f^* on Ω, for any nonempty subset $\mathcal{F} \subseteq \mathcal{P}^+(\Omega)$. This convention allows us, for instance, to identify $\mathcal{L}(\mathcal{A}_\mathcal{F})$ with a subset of $\mathcal{L}(\Omega)$, and, as another example, for any set $A \subseteq \mathcal{L}(\Omega)$, to regard $A \cap \mathcal{L}(\mathcal{A}_\mathcal{F})$ as those gambles in A that are constant on every value of $\mathcal{A}_\mathcal{F}$. Therefore, for any element E in $\mathcal{P}^+(\mathcal{A}_\mathcal{F})$ we can identify the gamble $\mathbb{I}_E$ with $\mathbb{I}_{\bigcup E}$, and hence also the event E with $\bigcup E$.

Suppose we have a set of desirable gamble sets K on Ω modelling an agent's beliefs about Ω. What is the information that K contains about $\mathcal{F}$, where $\mathcal{F} \subseteq \mathcal{P}^+(\Omega)$ is a nonempty set? Marginalization captures this information.

Definition 9 (Marginalization)**.** Given any nonempty subset $\mathcal{F}$ of $\mathcal{P}^+(\Omega)$ and any set of desirable gamble sets K on Ω, its *marginal set of desirable gamble sets* on $\mathcal{A}_\mathcal{F}$ is defined as $\operatorname{marg}_\mathcal{F} K \coloneqq \{A \in \mathcal{Q}(\mathcal{A}_\mathcal{F}) : A \in K\} = K \cap \mathcal{Q}(\mathcal{A}_\mathcal{F})$.

The idea is that $\operatorname{marg}_\mathcal{F} K$ is the subset of K that concerns only gambles about $\mathcal{F}$. It follows at once from Definition 9 that marginalization preserves the order: if $K_1 \subseteq K_2$, then $\operatorname{marg}_\mathcal{F} K_1 \subseteq \operatorname{marg}_\mathcal{F} K_2$. For a set of desirable gambles D, this definition reduces to $\operatorname{marg}_\mathcal{F} D = \{f \in \mathcal{L}(\mathcal{A}_\mathcal{F}) : f \in D\} = D \cap \mathcal{L}(\mathcal{A}_\mathcal{F})$. For notational convenience, we lift the marginalization operator $\operatorname{marg}_\mathcal{F}$ on $\mathcal{D}$ to a version on $\mathcal{P}(\mathcal{D})$ defined by $\operatorname{marg}_\mathcal{F} \mathbf{D} \coloneqq \{\operatorname{marg}_\mathcal{F} D : D \in \mathbf{D}\}$ for any $\mathbf{D} \subseteq \mathcal{D}$.

Proposition 8. *Consider any coherent set of desirable gambles D on Ω, and nonempty subset $\mathcal{F} \subseteq \mathcal{P}^+(\Omega)$. Then* $\operatorname{marg}_\mathcal{F} D$ *is coherent. Moreover, consider any coherent set of desirable gamble sets K on Ω, any representation* $\mathbf{D}$ *of K, and any nonempty subset $\mathcal{F} \subseteq \mathcal{P}^+(\Omega)$. Then* $\operatorname{marg}_\mathcal{F} K$ *is coherent. Furthermore,* $\operatorname{marg}_\mathcal{F} K$ *is represented by* $\operatorname{marg}_\mathcal{F}(\mathbf{D})$, *meaning that* $\operatorname{marg}_\mathcal{F} K = \bigcap\{K_D : D \in \operatorname{marg}_\mathcal{F}(\mathbf{D})\}$.

[8] We use the same notation of '$\|$' used by De Cooman and Quaeghebeur [16] for sets of desirable gambles. Note that, similarly to [16], $K\|E$ is not necessarily coherent as it may not contain all the non-negative and non-zero singletons in $\mathcal{L}^{\mathrm{s}}_{>0}$.

Proof. For the first statement, from $0 \notin D$, we infer $0 \notin \operatorname{marg}_{\mathcal{F}} D = D \cap \mathcal{L}(\mathcal{A}_{\mathcal{F}})$, establishing that $\operatorname{marg}_{\mathcal{F}} D$ satisfies Axiom D_1. To show that $\operatorname{marg}_{\mathcal{F}} D$ satisfies Axiom D_2, consider any f in $\mathcal{L}(\mathcal{A}_{\mathcal{F}})_{>0}$. Then $f \in D$ by D's coherence, whence $f \in \operatorname{marg}_{\mathcal{F}} D$. Since the choice of f in $\mathcal{L}(\mathcal{A}_{\mathcal{F}})_{>0}$ was arbitrary, this implies that, indeed, $\mathcal{L}(\mathcal{A}_{\mathcal{F}})_{>0} \subseteq \operatorname{marg}_{\mathcal{F}} D$. Finally, To show that $\operatorname{marg}_{\mathcal{F}} D$ satisfies Axiom D_3, consider any f and g in $\operatorname{marg}_{\mathcal{F}} D$ and any $(\lambda, \mu) > 0$. Then $\lambda f + \mu g \in D$ by D's coherence, and furthermore $\lambda f + \mu g \in D \in \mathcal{L}(\mathcal{A}_{\mathcal{F}})$, whence, indeed, $\lambda f + \mu g \in \operatorname{marg}_{\mathcal{F}} D$.

We jump directly to the third statement, from which the second will follow using Theorem 3. Consider any A in $\mathcal{Q}(\mathcal{A}_{\mathcal{F}})$, and infer the following chain of equivalences:

$$\begin{aligned} A \in \operatorname{marg}_{\mathcal{F}} K \Leftrightarrow A \in K \Leftrightarrow (\forall D \in \mathbf{D}) A \cap D \neq \varnothing \\ \Leftrightarrow (\forall D \in \operatorname{marg}_{\mathcal{F}} \mathbf{D}) A \cap D \neq \varnothing \Leftrightarrow A \in \bigcap_{D \in \operatorname{marg}_{\mathcal{F}} \mathbf{D}} K_D, \end{aligned}$$

where the second equivalence follows from the assumption that $\mathbf{D}$ represents K. Since the choice of A in $\mathcal{Q}(\mathcal{A}_{\mathcal{F}})$ was arbitrary, this implies that, indeed, $\operatorname{marg}_{\mathcal{F}} K = \bigcap \{K_D : D \in \operatorname{marg}_{\mathcal{F}}(\mathbf{D})\}$. □

Finally, we have the pieces in place to define what it means for an agent to judge that the topic characterized by $\mathcal{E}$ is epistemically irrelevant to the topic represented by $\mathcal{F}$; we will call this *epistemic irrelevance* of $\mathcal{E}$ to $\mathcal{F}$, Epistemic *independence* of $\mathcal{E}$ and $\mathcal{F}$ is just the symmetric version of this property.

Definition 10 (Epistemic irrelevance)**.** Consider any nonempty subsets $\mathcal{E}$ and $\mathcal{F}$ of $\mathcal{P}^+(\Omega)$. We call $\mathcal{E}$ *epistemically (subset) irrelevant* to $\mathcal{F}$ when learning about $\mathcal{E}$ does not influence or change the agent's beliefs about $\mathcal{F}$. A set of desirable gamble sets K on Ω is said to *satisfy epistemic (subset) irrelevance* of $\mathcal{E}$ to $\mathcal{F}$ when

$$\operatorname{marg}_{\mathcal{F}}(K \| E) = \operatorname{marg}_{\mathcal{F}} K \text{ for all } E \text{ in } \mathcal{W}_{\mathcal{E}}. \tag{3}$$

The idea behind this definition is that observing that E in $\mathcal{A}_{\mathcal{E}}$ turns K into the conditioned set of desirable gamble sets $K\|E$ on E. Then requiring that learning that any event E in $\mathcal{W}_{\mathcal{E}}$ obtains does not affect the agent's beliefs about $\mathcal{F}$ amounts to requiring that the marginal models of K and $K\|E$ be equal.

Proposition 9. *Consider any coherent set of desirable gamble sets K that satisfies epistemic irrelevance from the nonempty $\mathcal{E} \subseteq \mathcal{P}^+(\Omega)$ to the nonempty $\mathcal{F} \subseteq \mathcal{P}^+(\Omega)$. Then every B in $\mathcal{A}_{\mathcal{E}}$ intersects every B' in $\mathcal{A}_{\mathcal{F}}$.*

Proof. Consider any B in $\mathcal{A}_{\mathcal{E}}$ and B' in $\mathcal{A}_{\mathcal{F}}$. Since $\{\mathbb{I}_{B'}\}$ belongs to $\mathcal{L}^{\mathrm{s}}(\Omega)_{>0}$, we find $\{\mathbb{I}_{B'}\} \in K$ by K's coherence—more specifically, Axiom K_2. Equation (3) then requires that $\{\mathbb{I}_{B'}\} \in \operatorname{marg}_{\mathcal{F}}(K\|E)$, whence $\{\mathbb{I}_{B'}\} \in K\|E$ and therefore $\mathbb{I}_B\{\mathbb{I}_{B'}\} = \{\mathbb{I}_{B \cap B'}\} \in K$. Use the coherence of K—more specifically, Axioms K_1 and K_0—to infer that then $\mathbb{I}_{B \cap B'} \neq 0$ whence, indeed, $B \cap B' \neq \varnothing$. □

4.1. **Epistemic independence for events.** Let us end this section by showing how independence for events follows from the discussion above as a special case. Roughly speaking, what it means for an event $E \in \mathcal{P}^+(\Omega)$ to be epistemically irrelevant to another event $F \in \mathcal{P}^+(\Omega)$, is that learning that the true outcome $\omega \in \Omega$ belongs to E or to

E^c does not change the inferences about $\{F, F^c\}$, which are described using gambles in $\mathcal{L}(\mathcal{A}_{\{F\}}) = \mathcal{L}(\mathcal{A}_{\{F,F^c\}}) = \{\lambda \mathbb{I}_F + \mu \mathbb{I}_{F^c} : \lambda, \mu \in \mathbb{R}\}$ that depend only on whether or not F occurs.

In this case, $\mathcal{E} \coloneqq \{E, E^c\}$ is the set of events that will be epistemically irrelevant to $\mathcal{F} \coloneqq \{F, F^c\}$. Because both $\mathcal{E}$ and $\mathcal{F}$ are binary partitions, we find that $\mathcal{A}_\mathcal{E} = \{E, E^c\} = \mathcal{E}$ and $\mathcal{A}_\mathcal{F} = \{E, E^c\} = \mathcal{F}$, and therefore also $\mathcal{W}_\mathcal{E} = \{E, E^c, \Omega\}$, $\mathcal{W}_\mathcal{F} = \{F, F^c, \Omega\}$. In other words, when the agent's beliefs are described by the coherent set of desirable gamble sets K on Ω that satisfies epistemic irrelevance of $\mathcal{E}$ to $\mathcal{F}$, then any gamble set A in $\mathcal{Q}(\mathcal{A}_\mathcal{E}) = \mathcal{Q}(\mathcal{E})$ belongs to K if and only if its called-off versions $\mathbb{I}_E A$ and $\mathbb{I}_{E^c} A$ do.

Definition 11 (Epistemic irrelevance for events)**.** Consider any two events E and F in $\mathcal{P}^+(\Omega)$, for which E^c and F^c are also nonempty. We call E *epistemically irrelevant* to F when the partition $\{E, E^c\}$ is epistemically (subset) irrelevant to the partition $\{F, F^c\}$. A set of desirable gamble sets K is said to *satisfy epistemic irrelevance* of E to F if and only if

$$\operatorname{marg}_{\{F,F^c\}}(K \| E) = \operatorname{marg}_{\{F,F^c\}}(K \| E^c) = \operatorname{marg}_{\{F,F^c\}} K.$$

As a result from this definition, K satisfies epistemic irrelevance of E to F if and only if

$$(\forall A \in \mathcal{Q}(\{F, F^c\}))(A \in K \Leftrightarrow \mathbb{I}_E A \in K \Leftrightarrow \mathbb{I}_{E^c} A \in K). \tag{4}$$

Clearly, epistemic irrelevance is closed under arbitrary intersections: if every set of desirable gamble sets in a collection $\mathbf{K}$ satisfies epistemic irrelevance, then so does the set of desirable gamble sets $\bigcap \mathbf{K}$. Example 4 further on shows that this definition is a generalization of the independence concept for probabilities. Also, only *logically independent* events E and F, in the sense that $E \cap F \neq \emptyset$, $E \cap F^c \neq \emptyset$, $E^c \cap F \neq \emptyset$, and $E^c \cap F^c \neq \emptyset$, can ever be epistemically irrelevant with respect to a coherent set of desirable gamble sets K.

Corollary 10. *Consider any coherent set of desirable gamble sets K that satisfies epistemic irrelevance from the event E to the event F. Then E and F are logically independent.*

Proof. This follows at once from Proposition 9 by considering $\mathcal{E} \coloneqq \{E, E^c\}$ and $\mathcal{F} \coloneqq \{F, F^c\}$. □

Let us finish this discussion by showing by means of the following example that the standard independence condition satisfies epistemic independence under the assumption that every outcome in Ω has a positive probability.

Example 4. Consider any credal set $\mathcal{M} \subseteq \operatorname{int}(\Sigma_\Omega)$ that contains only probability mass functions that satisfy independence between two logically independent events E and F in $\mathcal{P}^+$: every p in $\mathcal{M}$ satisfies $\sum_{\omega \in E \cap F} p(\omega) = \left(\sum_{\varpi \in E} p(\varpi)\right)\left(\sum_{\varpi' \in F} p(\varpi')\right)$, or equivalently, its corresponding expectation operator E_p satisfies $E_p(\mathbb{I}_{E\cap F}) = E_p(\mathbb{I}_E) E_p(\mathbb{I}_F)$, and therefore for any $f = \lambda \mathbb{I}_F + \mu \mathbb{I}_{F^c}$ in $\mathcal{L}(\mathcal{A}_{\{F\}})$ also

$$\begin{aligned} E_{p|E}(f) = \frac{E_p(\mathbb{I}_E f)}{E_p(\mathbb{I}_E)} &= \frac{E_p(\lambda \mathbb{I}_{E\cap F} + \mu \mathbb{I}_{E \cap F^c})}{E_p(\mathbb{I}_E)} \\ &= \frac{\lambda E_p(\mathbb{I}_{E\cap F}) + \mu E_p(\mathbb{I}_{E\cap F^c})}{E_p(\mathbb{I}_E)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda E_p(\mathbb{I}_{E\cap F}) + \mu E_p(\mathbb{I}_E) - \mu E_p(\mathbb{I}_{E\cap F})}{E_p(\mathbb{I}_E)} \\
&= \frac{\lambda E_p(\mathbb{I}_E)E_p(\mathbb{I}_F) + \mu E_p(\mathbb{I}_E) - \mu E_p(\mathbb{I}_E)E_p(\mathbb{I}_F)}{E_p(\mathbb{I}_E)} \\
&= \lambda E_p(\mathbb{I}_F) + \mu - \mu E_p(\mathbb{I}_F) = \lambda E_p(\mathbb{I}_F) + \mu E_p(\mathbb{I}_{F^c}) = E_p(f),
\end{aligned}
$$

and hence $E_p(f) > 0 \Leftrightarrow E_p(\mathbb{I}_E f) > 0$. We will use this to study whether the E-admissible set of desirable gamble sets $K_{\mathcal{M}}$ based on $\mathcal{M}$, defined in Example 2, satisfies epistemic independence between E and F. To this end, consider any A in $\mathcal{Q}(\mathcal{A}_{\{F\}})$, and infer the following equivalences:

$$A \cap \mathcal{L}_{>0} \neq \emptyset \Leftrightarrow \mathbb{I}_E A \cap \mathcal{L}_{>0} \neq \emptyset$$

which follows from the logical independence of E and F, and

$$
\begin{aligned}
(\forall p \in \mathcal{M})(\exists f \in A)E_p(f) > 0 &\Leftrightarrow (\forall p \in \mathcal{M})(\exists f \in A)E_p(\mathbb{I}_E f) > 0 \\
&\Leftrightarrow (\forall p \in \mathcal{M})(\exists f \in \mathbb{I}_E A)E_p(f) > 0.
\end{aligned}
$$

Together, this implies that

$$
\begin{aligned}
A \in K_{\mathcal{M}} &\Leftrightarrow A \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\forall p \in \mathcal{M})(\exists f \in A)E_p(f) > 0 \\
&\Leftrightarrow \mathbb{I}_E A \cap \mathcal{L}_{>0} \neq \emptyset \text{ or } (\forall p \in \mathcal{M})(\exists f \in \mathbb{I}_E A)E_p(f) > 0 \Leftrightarrow \mathbb{I}_E A \in K_{\mathcal{M}}.
\end{aligned}
$$

Using an analogous argument where E takes the role of E^c shows that $A \in K_{\mathcal{M}} \Leftrightarrow \mathbb{I}_{E^c} A \in K_{\mathcal{M}}$. Since the choice of A in $\mathcal{Q}(\mathcal{A}_{\{F\}})$ was arbitrary, this guarantees that $K_{\mathcal{M}}$ satisfies epistemic irrelevance of E to F. A completely similar reasoning yields the result that $K_{\mathcal{M}}$ satisfies epistemic irrelevance of F to E, too, and therefore epistemic independence between E and F. $\Diamond$

5. Multivariate Sets of Desirable Gamble Sets

In this section, we briefly present some of the concepts, tools, and notation we will need for analyzing sets of desirable gamble sets in a multivariate context; our exposition closely follows De Cooman and Miranda [15]. In Section 6, we apply Definition 10 in this context to obtain the notions of epistemic independence of variables, and of the independent product. We provide the linear space of gambles, on which we define our sets of desirable gamble sets, with a more complex structure: we consider the vector space of all gambles whose domain is a Cartesian product of a finite number of finite possibility spaces. More specifically, consider n in $\mathbb{N}$ variables $X_1, \dots, X_n$ that assume values in the finite possibility spaces $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. Belief models about these variables $X_1, \dots, X_n$ will be defined using gambles on $\mathcal{X}_1, \dots, \mathcal{X}_n$. We also consider gambles on the Cartesian product $\times_{k=1}^n \mathcal{X}_k$, giving rise to the $\prod_{k=1}^n |\mathcal{X}_k|$-dimensional linear space $\mathcal{L}(\times_{k=1}^n \mathcal{X}_k)$.

5.1. Basic Notation & Cylindrical Extension.

For every nonempty subset $I \subseteq \{1, \dots, n\}$ of indices, we let X_I be the tuple of variables that takes values in $\mathcal{X}_I := \times_{r \in I} \mathcal{X}_r$.[9] We will denote generic elements of $\mathcal{X}_I$ as x_I or z_I, whose components are $x_i := x_I(i)$ and $z_i := z_I(i)$, for all i in I. As we did before, when $I = \{k, \dots, \ell\}$ for some k, ℓ in $\{1, \dots, n\}$ with $k \leq \ell$, we

[9] Note that this guarantees that the variables $X_1, \dots, X_n$ are *logically independent*, meaning that for each nonempty subset I of $\{1, \dots, n\}$, x_I may assume every value in $\mathcal{X}_I$.

will use as a shorthand notation $X_{k:\ell} \coloneqq (X_k,\dots,X_\ell)$, taking values in $\mathcal{X}_{k:\ell}$ and whose generic elements are denoted by $x_{k:\ell} \coloneqq (x_k,\dots,x_\ell)$.

It will be useful for any gamble f on $\mathcal{X}_{1:n}$, any nonempty proper subset I of $\{1,\dots,n\}$ and any x_I in $\mathcal{X}_I$, to interpret the partial map $f(x_I,\bullet)$ as a gamble on $\mathcal{X}_{I^c}$, where $I^c \coloneqq \{1,\dots,n\} \setminus I$. We will need a way to relate gambles on different domains.

Definition 12 (Cylindrical extension[10])**.** Given two disjoint and nonempty subsets I and I' of $\{1,\dots,n\}$ and any gamble f on $\mathcal{X}_I$, we let its *cylindrical extension* f^* to $\mathcal{X}_{I\cup I'}$ be defined by $f^*(x_I,x_{I'}) \coloneqq f(x_I)$ for all x_I in $\mathcal{X}_I$ and $x_{I'}$ in $\mathcal{X}_{I'}$. Similarly, given any set of gambles $A \subseteq \mathcal{L}(\mathcal{X}_I)$, we let its *cylindrical extension* $A^* \subseteq \mathcal{L}(\mathcal{X}_{I\cup I'})$ be defined as $A^* \coloneqq \{f^* : f \in A\}$.

Formally, f^* belongs to $\mathcal{L}(\mathcal{X}_{I\cup I'})$ while f belongs to $\mathcal{L}(\mathcal{X}_I)$. However, f^* is completely determined by f and *vice versa*: they clearly only depend on the value of X_I, and as such, they contain the same information and correspond to the same transaction. They are therefore indistinguishable from a behavioral point of view.

Remark 2. As in [13, 15], we will frequently use the simplifying device of *identifying* a gamble f on $\mathcal{X}_I$ with its cylindrical extension f^* on $\mathcal{X}_{I\cup I'}$, for any disjoint and nonempty subsets I and I' of the index set $\{1,\dots,n\}$. This convention allows us, for instance, to identify $\mathcal{L}(\mathcal{X}_I)$ with a subset of $\mathcal{L}(\mathcal{X}_{1:n})$, and, as another example, for any set $A \subseteq \mathcal{L}(\mathcal{X}_{1:n})$, to regard $A \cap \mathcal{L}(\mathcal{X}_I)$ as those gambles in A that depend on the value of $\mathcal{X}_I$ only. Therefore, for any event E in $\mathcal{P}^+(\mathcal{X}_I)$ we can identify the gamble $\mathbb{I}_E$ with $\mathbb{I}_{E\times\mathcal{X}_{I^c}}$, and hence also the event E with $E\times\mathcal{X}_{I^c}$.

5.2. **Marginalization.** Suppose we have a set of desirable gamble sets K on $\mathcal{X}_{1:n}$ modelling an agent's beliefs about the variable $X_{1:n}$. What is the information that K contains about X_O, where O is some nonempty subset of the index set $\{1,\dots,n\}$? Marginalization captures this information.

Definition 13 (Marginalization[11])**.** Given any nonempty subset O of $\{1,\dots,n\}$ and any set of desirable gamble sets K on $\mathcal{X}_{1:n}$, its *marginal set of desirable gamble sets* $\operatorname{marg}_O K$ on $\mathcal{X}_O$ is defined as $\operatorname{marg}_O K \coloneqq \{A \in \mathcal{Q}(\mathcal{X}_O) : A \in K\} = K \cap \mathcal{Q}(\mathcal{X}_O)$. If O is a singleton $\{\ell\}$, then we denote $\operatorname{marg}_{\{\ell\}}$ by marg_ℓ.

Since Definition 13 is a specialization of Definition 9, we have that, this specialized version of marginalization preserves the order: if $K_1 \subseteq K_2$, then $\operatorname{marg}_O K_1 \subseteq \operatorname{marg}_O K_2$. This definition coincides with the usual definition for sets of desirable gambles, in the sense that $\operatorname{marg}_O K_D = K_{\operatorname{marg}_O D}$, where $\operatorname{marg}_O D \coloneqq \{f \in \mathcal{L}(\mathcal{X}_O) : f \in D\} = D \cap \mathcal{L}(\mathcal{X}_O)$. For notational convenience, we lift the marginalization operator marg_O on $\mathcal{D}$ to a version on $\mathcal{P}(\mathcal{D})$ defined by $\operatorname{marg}_O \mathbf{D} \coloneqq \{\operatorname{marg}_O D : D \in \mathbf{D}\}$ for any $\mathbf{D} \subseteq \mathcal{D}$. The following proposition is an immediate consequence of Proposition 8.

[10]After identifying the set $\mathcal{X}_I$ with $\{\{x_I\}\times\mathcal{X}_{I'} : x_I \in \mathcal{X}_I\}$, this definition may be seen as a specialized version of Definition 8, with $\mathcal{F} \coloneqq \{\{x_I\}\times\mathcal{X}_{I'} : x_I \in \mathcal{X}_I\}$. Note that $\mathcal{F}$'s atoms are $\mathcal{A}_\mathcal{F} = \mathcal{F}$, and a gamble f on $\mathcal{A}_\mathcal{F}$ is cylindrically extended to f^* given by $f^*(x_I,x_{I'}) = f(\{x_I\}\times\mathcal{X}_{I'})$ for every x_I in $\mathcal{X}_I$ and $x_{I'}$ in $\mathcal{X}_{I'}$.

[11]After identifying the set $\mathcal{X}_I$ with $\{\{x_I\}\times\mathcal{X}_{I'} : x_I \in \mathcal{X}_I\}$, this definition may be seen as a specialized version of Definition 9 with $\mathcal{F} \coloneqq \{\{x_I\}\times\mathcal{X}_{I^c} : x_I \in \mathcal{X}_I\}$.

Proposition 11. *Consider any coherent set of desirable gamble sets K on $\mathcal{X}_{1:n}$, any representation $\mathbf{D}$ of K, and any nonempty subset O of $\{1,\dots,n\}$. Then $\mathrm{marg}_O K$ is coherent. Furthermore, $\mathrm{marg}_O K$ is represented by $\mathrm{marg}_O(\mathbf{D})$, meaning that $\mathrm{marg}_O K = \bigcap\{K_D : D \in \mathrm{marg}_O(\mathbf{D})\}$.*

As a consequence, since any coherent K is represented by $\mathbf{D}(K)$, Proposition 11 implies that in particular $\mathrm{marg}_O K = \bigcap\{K_D : D \in \mathrm{marg}_O(\mathbf{D}(K))\}$.

5.3. Conditioning on Variables.

In Section 3 we have seen how we can condition sets of desirable gamble sets on events. Here, we take a closer look at conditioning in a multivariate context.

Suppose we have a set of desirable gamble sets K on $\mathcal{X}_{1:n}$, representing an agent's beliefs about the value of $X_{1:n}$. Assume now that we obtain the information that the I-tuple of variables X_I—where I is a nonempty proper subset of $\{1,\dots,n\}$—assumes a value in a certain nonempty subset E_I of $\mathcal{X}_I$. There is no new information about the other variables X_{I^c}. How can we condition K on this new information?

This is a particular instance of Definition 7, with the following specifications: $\Omega = \mathcal{X}_{1:n}$ and $E = E_I \times \mathcal{X}_{I^c}$. The indicator $\mathbb{I}_E$ of the conditioning event E satisfies $\mathbb{I}_E(x_{1:n}) = \mathbb{I}_{E_I}(x_I)$ for all $x_{1:n}$ in $\mathcal{X}_{1:n}$, and taking Remark 2 into account, therefore $\mathbb{I}_E = \mathbb{I}_{E_I}$. Equation (1) defines the multiplication of a gamble f on $E_I \times \mathcal{X}_{I^c}$ with $\mathbb{I}_{E_I}$ to be a gamble $\mathbb{I}_{E_I} f$ on $\mathcal{X}_{1:n}$, given by, for all $x_{1:n}$ in $\mathcal{X}_{1:n}$:

$$\mathbb{I}_{E_I} f(x_{1:n}) = \begin{cases} f(x_{1:n}) & \text{if } x_I \in E_I \\ 0 & \text{if } x_I \notin E_I \end{cases} \tag{5}$$

and the multiplication of $\mathbb{I}_{E_I}$ with a set A of gambles on $E_I \times \mathcal{X}_{I^c}$ is the set $\mathbb{I}_{E_I} A = \{\mathbb{I}_{E_I} f : f \in A\}$ of gambles on $\mathcal{X}_{1:n}$.

Now that we have instantiated all the relevant aspects of Definition 7, we see that $K\rfloor E_I = \{A \in \mathcal{Q}(E_I \times \mathcal{X}_{I^c}) : \mathbb{I}_{E_I} A \in K\}$. Proposition 7 guarantees that $K\rfloor E_I$ is represented by $\mathbf{D}(K)\rfloor E_I = \{D\rfloor E_I : D \in \mathbf{D}(K)\}$, where in this context $D\rfloor E_I = \{f \in \mathcal{L}(E_I \times \mathcal{X}_{I^c}) : \mathbb{I}_{E_I} f \in D\}$.

The conditional set of desirable gamble sets $K\rfloor E_I$ is defined on gambles on $E_I \times \mathcal{X}_{I^c}$. However, usually—see, for instance, [7, 15]—conditioning on information about X_I results in a model on X_{I^c}. We consider

$$\mathrm{marg}_{I^c}(K\rfloor E_I) = \{A \in \mathcal{Q}(\mathcal{X}_{I^c}) : \mathbb{I}_{E_I} A \in K\}$$

as the set of desirable gamble sets that represents the conditional beliefs about X_{I^c}, given that $X_I \in E_I$. Note that Definition 13 can be applied to $K\rfloor E_I$ by interpreting E_I as the possibility space of an uncertain variable X, so $K\rfloor E_I$ is a set of desirable gamble sets about $X \times X_{I^c}$. Propositions 7 and 11 guarantee the coherence of $\mathrm{marg}_{I^c}(K\rfloor E_I)$, for any coherent K.

6. Independent Natural Extension

Now that the basic operations on multivariate sets of desirable gamble sets—marginalization and conditioning—are in place, we are ready to look at a simple type of structural assessment.

The assessment that we will consider, is a specialized version of *epistemic independence*, which we define to be a symmetrized version of *epistemic irrelevance*.

Definition 14 (Epistemic (subset) irrelevance[12])**.** Consider any disjoint and nonempty subsets I and O of $\{1,\dots,n\}$. We call X_I *epistemically (subset) irrelevant* to X_O when learning about the value of X_I does not influence or change the agent's beliefs about X_O. A set of desirable gamble sets K on $\mathcal{X}_{1:n}$ is said to *satisfy epistemic (subset) irrelevance* of X_I to X_O when

$$\operatorname{marg}_O(K\rfloor E_I) = \operatorname{marg}_O K \text{ for all } E_I \text{ in } \mathcal{P}^+(\mathcal{X}_I). \tag{6}$$

The idea behind this definition is that observing that X_I belongs to E_I turns K into the conditioned set of desirable gamble sets $K\rfloor E_I$ on $E_I \times \mathcal{X}_{I^c}$. Then requiring that learning that X_I belongs to E_I does not affect the agent's beliefs about X_O amounts to requiring that the marginal models of K and $K\rfloor E_I$ be equal.

Definition 14 is a generalization of De Cooman and Miranda [15]'s definition for sets of desirable gambles. Besides their use of the less expressive models of sets of desirable gambles, there is another difference: De Cooman and Miranda [15] consider epistemic *value* irrelevance, which requires the analogue of Equation (6) only for events of the form $E_I = \{x_I\}$, with $x_I \in \mathcal{X}_I$.

De Bock [7, Example 2] shows that the two notions do indeed come apart: he gives a coherent set of desirable gambles that satisfies epistemic value irrelevance of X_1 to X_2, but not epistemic subset irrelevance. Given the connection between sets of desirable gambles and sets of desirable gamble sets, this example establishes that the two notions come apart also in the context of sets of desirable gamble sets. We follow De Bock [7] in considering epistemic subset-irrelevance to be the more natural of the two irrelevance concepts, as it requires all information about the value of X_I to be irrelevant, including partial information of the form $X_I \in E_I$, and not only of the form $X_I = x_I$.[13]

Definition 15 (Epistemic (subset) many-to-one independence)**.** We call $X_1,\dots,X_n$ *epistemically (subset) many-to-one independent* when learning about the values of any of them does not influence or change the agent's beliefs about any other: for any o in $\{1,\dots,n\}$, and any nonempty subset I of $\{1,\dots,n\}\setminus\{o\}$, X_I is epistemically subset irrelevant to X_o. We call a set of desirable gamble sets K on $\mathcal{X}_{1:n}$ *epistemically (subset) many-to-one independent* when

$$\operatorname{marg}_o(K\rfloor E_I) = \operatorname{marg}_o K \text{ for all } E_I \text{ in } \mathcal{P}^+(\mathcal{X}_I) \tag{7}$$

for all o in $\{1,\dots,n\}$ and nonempty subset I of $\{1,\dots,n\}\setminus\{o\}$.

---

[12]Loosely speaking, ignoring the difference between the versions '$\rfloor$' and '$\|$' for conditioning, this definition may be seen as a specialized version of Definition 10, where $\mathcal{E} \coloneqq \{x_I \times \mathcal{X}_{I^c} : x_I \in \mathcal{X}_I\}$ and $\mathcal{F} \coloneqq \{x_O \times \mathcal{X}_{O^c} : x_O \in \mathcal{X}_O\}$. Note that $\mathcal{A}_\mathcal{E} = \mathcal{E}$ and $\mathcal{A}_\mathcal{F} = \mathcal{F}$. The structure of $\mathcal{A}_\mathcal{E}$ and $\mathcal{A}_\mathcal{F}$ make the use of the '$\rfloor$' conditioning version easier than '$\|$'.

[13]Jasper De Bock [9] has moved to referring to what we call epistemic "value independence" as epistemic "atom-independence"; similarly, "event-independence" replaces our "subset independence". We eschew this terminology only to avoid confusion with the notion of "events" on arbitrary outcome spaces that are used elsewhere in this paper. In any case, hopefully the distinction is intuitive: "atom-" or "value" independence considers only learning an element of the outcome space for a variable, whereas "event-" or "subset" irrelevance considers any fact specifiable as a disjunction of outcomes.

Every $E_I \in \mathcal{P}^+(\mathcal{X}_I)$ can be identified with $E_I \times \mathcal{X}_{1:n\setminus(I\cup\{o\})} \in \mathcal{P}^+(\mathcal{X}_{\{o\}^c})$ so the Requirement (7) reduces to

$$\mathrm{marg}_o(K\rfloor E) = \mathrm{marg}_o K \text{ for all } E \text{ in } \mathcal{P}^+(\mathcal{X}_{\{o\}^c}) \tag{8}$$

for all o in $\{1,\dots,n\}$.

Independence assessments are useful in constructing joint sets of desirable gamble sets from local ones. Suppose we have a coherent set K_ℓ of desirable gamble sets on $\mathcal{X}_\ell$, for each ℓ in $\{1,\dots,n\}$, and an assessment that the variables $X_1, \dots, X_n$ are epistemically (subset) many-to-one independent. Then how can we combine the coherent local assessments K_ℓ and this structural independence assessment into a coherent set of desirable gamble sets on $\mathcal{X}_{1:n}$ in a way that is as conservative as possible? If we call any coherent and epistemically many-to-one independent K on $\mathcal{X}_{1:n}$ that marginalizes to K_ℓ for all ℓ in $\{1,\dots,n\}$ a *many-to-one independent product* of $K_1, \dots, K_n$, this means we are looking for the smallest many-to-one independent product, which we will call the *many-to-one independent natural extension of* $K_1, \dots, K_n$, after [15].

6.1. **Many-to-many independence.** Contrast Definition 15 of epistemic many-to-one independence with the stronger requirement of epistemic *many-to-many* independence.

Definition 16 (Epistemic (subset) many-to-many Independence)**.** We call $X_1, \dots, X_n$ *epistemically (subset) independent* when learning about the values of any of them does not influence or change the agent's beliefs about the remaining ones: for any two disjoint nonempty subsets I and O of $\{1,\dots,n\}$, X_I is epistemically subset irrelevant to X_O. We call a set of desirable gamble sets K on $\mathcal{X}_{1:n}$ *epistemically (subset) many-to-many independent* when

$$\mathrm{marg}_O(K\rfloor E_I) = \mathrm{marg}_O K \text{ for all } E_I \text{ in } \mathcal{P}^+(\mathcal{X}_I)$$

for all disjoint nonempty subsets I and O of $\{1,\dots,n\}$.

In Theorem 15 further on we will find the many-to-one independent natural extension, which therefore will be a conservative approximation of the many-to-many independent natural extension—the smallest many-to-many independent product of $X_1, \dots, X_n$. It is an open question whether or not this approximation is exact.

In order to build on the work [15] of independent natural extension for sets of desirable gambles, let us here define the analogous concepts in this context. We call a set of desirable gambles D on $\mathcal{X}_{1:n}$ *epistemically subset many-to-many independent* when

$$\mathrm{marg}_O(D\rfloor E_I) = \mathrm{marg}_O D \text{ for all } E_I \text{ in } \mathcal{P}^+(\mathcal{X}_I) \tag{9}$$

for all disjoint nonempty subsets I and O of $\{1,\dots,n\}$. Because of the connection with [15], we will also need to define the weaker notion of epistemic value independence: We call a set of desirable gambles D on $\mathcal{X}_{1:n}$ *epistemically value many-to-many independent* when

$$\mathrm{marg}_O(D\rfloor\{x_I\}) = \mathrm{marg}_O D \text{ for all } x_I \text{ in } \mathcal{X}_I \tag{10}$$

Suppose we have a coherent set D_ℓ of desirable gambles on $\mathcal{X}_\ell$, for each ℓ in $\{1,\dots,n\}$, and an assessment that the variables $X_1, \dots, X_n$ are epistemically subset (or value) many-to-many independent. Then how can we combine the coherent local assessments D_ℓ and this structural independence assessment into a coherent set of desirable gambles on $\mathcal{X}_{1:n}$ in a way that is as conservative as possible? If we call any coherent and epistemically subset (or value) many-to-many independent D on $\mathcal{X}_{1:n}$ that marginalizes to D_ℓ for all ℓ in $\{1,\dots,n\}$ a *subset (or value) many-to-many independent product* of $D_1, \dots, D_n$, this means we are looking for the smallest many-to-many independent product, called by [15] the *subset (or value) many-to-many independent natural extension of* $D_1, \dots, D_n$.

Theorem 12 ([15, Theorem 19])**.** *The value many-to-many independent natural extension of the n coherent sets of desirable gambles* $D_1 \subseteq \mathcal{L}(\mathcal{X}_1)$*, ..., *$D_n \subseteq \mathcal{L}(\mathcal{X}_n)$ *exists and is given by* $\bigotimes_{k=1}^n D_k \coloneqq \operatorname{posi}(\bigcup_{j=1}^n A_{1:n\setminus\{j\}\to\{j\}} \cup \mathcal{L}(\mathcal{X}_{1:n})_{>0})$*, where*

$$A_{1:n\setminus\{j\}\to\{j\}} \coloneqq \{\mathbb{I}_E f : f \in D_j \text{ and } E \in \mathcal{P}^+(\mathcal{X}_{\{j\}^c})\} \tag{11}$$

for any j *in* $\{1,\dots,n\}$.

As said, we are interested in the *subset* many-to-many independent natural extension, as we find subset irrelevance a more natural concept than value irrelevance. Unfortunately, the project of [15] considers value irrelevance only. However, Jasper De Bock [10] compared the both approaches, in a more general context of independence assessments determined by an underlying directed acyclic graph. In his work, the conclusions of [10, Corollary 12] imply in the special case of many-to-many independence[14] that $\bigotimes_{k=1}^n D_k$ is also a subset many-to-many independent product of $D_1, \dots, D_n$, and therefore, as subset many-to-many independence implies value many-to-many independence, it necessarily is the *smallest* subset many-to-many independent product, or, in other words, it is the subset many-to-many independent natural extension. For future reference, we conclude this result in the following corollary.

Corollary 13 (follows from [10, Corollary 12])**.** *The subset many-to-many independent natural extension of the n coherent sets of desirable gambles* $D_1 \subseteq \mathcal{L}(\mathcal{X}_1)$*, ..., *$D_n \subseteq \mathcal{L}(\mathcal{X}_n)$ *exists and is given by* $\bigotimes_{k=1}^n D_k$.

De Cooman and Miranda [15] do not discuss whether or not $\bigotimes_{k=1}^n D_k$ is also the *many-to-one* independent natural extension of $D_1, \dots, D_n$, which is defined similar to Definition 15 of the corresponding concept for sets of desirable gamble sets. However, this is indeed the case.[15] To see this, let $D^\star$ be the many-to-one independent natural extension of $D_1, \dots, D_n$. As many-to-one independence is a weaker concept than many-to-many independence, it follows that $D^\star \subseteq \bigotimes_{k=1}^n D_k$. On the other hand, $A_{1:n\setminus\{j\}\to\{j\}}$ must belong to $D^\star$ for every j in $\{1,\dots,n\}$ by its definition, and therefore, because posi is a closure operator, indeed also $\bigotimes_{k=1}^n D_k = \operatorname{posi}(\bigcup_{j=1}^n A_{1:n\setminus\{j\}\to\{j\}} \cup \mathcal{L}(\mathcal{X}_{1:n})_{>0}) \subseteq \operatorname{posi}(D^\star \cup \mathcal{L}(\mathcal{X}_{1:n})_{>0}) = D^\star$, where the final equality follows because $D^\star$ is the many-to-one independent natural extension and is therefore coherent.

6.2. **Many-to-one Independent Natural Extension.** We will look for the many-to-one independent natural extension for sets of desirable gamble sets. To this end, consider the following counterpart of Equation (11)

$$\mathcal{A}_{1:n\setminus\{j\}\to\{j\}} \coloneqq \left\{\mathbb{I}_E A : A \in K_j \text{ and } E \in \mathcal{P}^+(\mathcal{X}_{\{j\}^c})\right\} \tag{12}$$

[14] How the result for independent products follows from their results for directed acyclic graphs, is discussed in [10, Section 6]. Even thought the author does not explicitly mention that it holds for many-to-many independent products, it is clear from the discussion in [10, Section 6] that this is indeed the case.

[15] We thank one of the reviewers for the providing us with the proof of this fact.

for any j in $\{1,\dots,n\}$, with which we build the following set of desirable gamble sets:

$$\bigotimes_{j=1}^{n} K_j \coloneqq \mathrm{Rs}\left(\mathrm{Posi}\left(\bigcup_{j=1}^{n} \mathcal{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0}\right)\right).$$

We will show that the independent natural extension—the smallest independent product—of $K_1, \dots, K_n$ is exactly $\bigotimes_{j=1}^{n} K_j$.

To find a representation of $\bigotimes_{j=1}^{n} K_j$, it will turn out useful to introduce the notation

$$\bigotimes_{j=1}^{n} \mathbf{D}_j \coloneqq \overline{\mathcal{D}} \cap \Big\{\mathrm{posi}\big(\bigcup\{\mathbb{I}_E D_{k,E} : k \in \{1,\dots,n\}, E \in \mathcal{P}^+(\mathcal{X}_{\{k\}^c})\} \cup \mathcal{L}(\mathcal{X}_{1:n})_{>0}\big) : \\ (\forall k \in \{1,\dots,n\}, E \in \mathcal{P}^+(\mathcal{X}_{\{k\}^c}))D_{k,E} \in \mathbf{D}_k\Big\} \quad (13)$$

for any nonempty $\mathbf{D}_1 \subseteq \overline{\mathcal{D}}(\mathcal{X}_1), \dots, \mathbf{D}_n \subseteq \overline{\mathcal{D}}(\mathcal{X}_n)$. We know that $\bigotimes_{j=1}^{n} \mathbf{D}_j$ is nonempty, since it contains the coherent

$$\bigotimes_{j=1}^{n} D_j = \mathrm{posi}\Big(\bigcup\big\{\mathbb{I}_E D_k : k \in \{1,\dots,n\}, E \in \mathcal{P}^+(\mathcal{X}_{\{k\}^c})\big\} \cup \mathcal{L}(\mathcal{X}_{1:n})_{>0}\Big)$$

for every D_1 in $\mathbf{D}_1, \dots, D_n$ in $\mathbf{D}_n$.

Before we establish the main result of this paper, let us first prove the following lemma.

Lemma 14. *Consider any coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X}_{1:n})$, any disjoint and nonempty subsets I and O of $\{1,\dots,n\}$, and any E in $\mathcal{P}^+(\mathcal{X}_I)$. Then*

$$\mathbb{I}_E \mathrm{marg}_O(D\rfloor E) \subseteq D.$$

Proof. Consider any f in $\mathbb{I}_E \mathrm{marg}_O(D\rfloor E)$, implying that $f = \mathbb{I}_E g$ for some g in $\mathrm{marg}_O(D\rfloor E)$, which in turns implies that g belongs to $D\rfloor E$ and to $\mathcal{L}(\mathcal{X}_O)$, and therefore $\mathbb{I}_E g \in D$ using Equation (2). Since $f = \mathbb{I}_E g$, this implies that f belongs to D. The choice of f in $\mathbb{I}_E \mathrm{marg}_O(D\rfloor E)$ was arbitrary, whence indeed $\mathbb{I}_E \mathrm{marg}_O(D\rfloor E) \subseteq D$. □

Theorem 15 (Independent many-to-one natural extension)**.** *Consider, for each j in $\{1,\dots,n\}$, a coherent set of desirable gamble sets K_j on $\mathcal{X}_j$. Then the smallest many-to-one independent product of $K_1, \dots, K_n$ is given by $\bigotimes_{j=1}^{n} K_j$. Furthermore, $\bigotimes_{j=1}^{n} K_j$ is represented by $\bigotimes_{j=1}^{n} \mathbf{D}(K_j)$.*

Proof. This proof will consist of five parts: we will subsequently show that (i) $\bigotimes_{j=1}^{n} K_j$ is coherent, (ii) it is represented by $\bigotimes_{j=1}^{n} \mathbf{D}(K_j)$, (iii) $\mathrm{marg}_\ell(\bigotimes_{j=1}^{n} K_j) = K_\ell$ for every ℓ in $\{1,\dots,n\}$, (iv) $\bigotimes_{j=1}^{n} K_j$ satisfies epistemic many-to-one independence, and (v) $\bigotimes_{j=1}^{n} K_j$ is the smallest such set of desirable gamble sets. Then (i), (iii) and (iv) establish that $\bigotimes_{j=1}^{n} K_j$ is an independent many-to-one product of $K_1, \dots, K_n$, which is by (v) the smallest one. (ii) establishes the last claim about $\bigotimes_{j=1}^{n} K_j$'s representation.

For (i), to show that $\bigotimes_{j=1}^{n} K_j$ is coherent, we will regard $\mathcal{A} \coloneqq \bigcup_{j=1}^{n} \mathcal{A}_{1:n\setminus\{j\}\to\{j\}}$ as an assessment on $\mathcal{Q}(\mathcal{X}_{1:n})$. By Theorem 6, showing that $\mathcal{A} \subseteq K_D$ for some coherent set of desirable gambles $D \subseteq \mathcal{L}(\mathcal{X}_{1:n})$ establishes that $\mathcal{A}$ is consistent. Theorem 2 then implies that $\bigotimes_{j=1}^{n} K_j = \mathrm{cl}_{\overline{\mathcal{K}}}(\mathcal{A})$ is coherent.

To this end, use Theorem 3 to note already that $\mathbf{D}(K_1),\dots,\mathbf{D}(K_n)$ all are nonempty since $K_1,\dots,K_n$ are coherent. Consider any D_1 in $\mathbf{D}(K_1),\dots,D_n$ in $\mathbf{D}(K_n)$, and let $D^* := \bigotimes_{j=1}^n D_j$. Then Corollary 13 implies that D^* is a coherent set of desirable gambles on $\mathcal{X}_{1:n}$ that is epistemically independent—by which we mean that $\mathrm{marg}_O D^* = \mathrm{marg}_O(D^* \rfloor E_I)$ for all disjoint nonempty subsets I and O of $\{1,\dots,n\}$ and E_I in $\mathcal{P}^+(\mathcal{X}_I)$—and marginalizes to $D_1,\dots,D_n$. We will show that $\mathcal{A} \subseteq K_{D^*}$. To this end, consider any A in $\mathcal{A}$, meaning that there is an index j in $\{1,\dots,n\}$ such that $A \in \mathcal{A}_{1:n\setminus\{j\}\to\{j\}}$, or, in other words, such that $A = \mathbb{I}_E B$ for some B in K_j and E in $\mathcal{P}^+(\mathcal{X}_{1:n\setminus\{j\}})$. Since D_j belongs to $\mathbf{D}(K_j)$ we have that $K_j \subseteq K_{D_j}$, and therefore $B \in K_{D_j} = K_{\mathrm{marg}_j D^*}$. Since $K_{\mathrm{marg}_j D^*} = \mathrm{marg}_j K_{D^*}$ by Proposition 11, this means that $B \in K_{D^*}$. But D^* is an epistemically independent set of desirable gambles, and it therefore satisfies $\mathrm{marg}_j(D^* \rfloor E) = \mathrm{marg}_j D^*$, or in other words, $f \in D^* \Leftrightarrow \mathbb{I}_E f \in D^*$, for any f in $\mathcal{L}(\mathcal{X}_j)$, and hence also $A = \mathbb{I}_E B \in K_{D^*}$. Since the choice of A in $\mathcal{A}$ was arbitrary, this implies that indeed $\mathcal{A} \subseteq K_{D^*}$, guaranteeing that $\bigotimes_{j=1}^n K_j$ is indeed coherent.

For (ii), we prove that $\mathbf{D}(\bigotimes_{j=1}^n K_j) = \uparrow\bigotimes_{j=1}^n \mathbf{D}(K_j)$: we will show the two set inclusions (a) $\mathbf{D}(\bigotimes_{j=1}^n K_j) \subseteq \uparrow\bigotimes_{j=1}^n \mathbf{D}(K_j)$ and (b) $\mathbf{D}(\bigotimes_{j=1}^n K_j) \supseteq \uparrow\bigotimes_{j=1}^n \mathbf{D}(K_j)$. Then Proposition 4 implies that $\bigotimes_{j=1}^n \mathbf{D}(K_j)$ represents $\bigotimes_{j=1}^n K_j$.

For (a), consider any D in $\mathbf{D}(\bigotimes_{j=1}^n K_j)$. Then Theorem 6 implies that $\bigcup_{j=1}^n \mathcal{A}_{1:n\setminus\{j\}\to\{j\}} \subseteq K_D$, or, in other words,

$$(\forall j \in \{1,\dots,n\}, E \in \mathcal{P}^+(\mathcal{X}_{\{j\}^c}), A \in K_j)\, \mathbb{I}_E A \cap D \neq \emptyset.$$

Consider any j in $\{1,\dots,n\}$, E in $\mathcal{P}^+(\mathcal{X}_{\{j\}^c})$ and A in K_j. Then $\mathbb{I}_E A \cap D \neq \emptyset$, and hence $A \cap D \rfloor E \neq \emptyset$ using Equation (2). Since A belongs to $\mathcal{Q}(\mathcal{X}_j)$, we infer that $A \cap \mathrm{marg}_j(D\rfloor E) \neq \emptyset$, whence $A \in K_{\mathrm{marg}_j(D\rfloor E)}$. But the choice of A in K_j was arbitrary, so $K_j \subseteq K_{\mathrm{marg}_j(D\rfloor E)}$ and hence Theorem 3 tells us that $\mathrm{marg}_j(D\rfloor E)$ belongs to $\mathbf{D}(K_j)$. Using Lemma 14, we find that $\mathbb{I}_E \mathrm{marg}_j(D\rfloor E) \subseteq D$, and hence, since the choices of j in $\{1,\dots,n\}$ and of E in $\mathcal{P}^+(\mathcal{X}_{\{j\}^c})$ were arbitrary, also $\bigcup\{\mathbb{I}_E \mathrm{marg}_j(D\rfloor E) : j \in \{1,\dots,n\}, E \in \mathcal{P}^+(\mathcal{X}_{\{j\}^c})\} \subseteq D$. Recall that posi is a closure operator—in particular, that it is monotonic—whence

$$\begin{aligned} D^* &:= \mathrm{posi}\big(\bigcup\{\mathbb{I}_E \mathrm{marg}_j(D\rfloor E) : j \in \{1,\dots,n\}, E \in \mathcal{P}^+(\mathcal{X}_{\{j\}^c})\} \cup \mathcal{L}(\mathcal{X}_{1:n})_{>0}\big) \\ &\subseteq \mathrm{posi}(D \cup \mathcal{L}(\mathcal{X}_{1:n})_{>0}) = D, \end{aligned}$$

where the equality follows from D's coherence. Note that D^* is coherent—it is the natural extension of a subset of the coherent D—so it belongs to $\bigotimes_{j=1}^n \mathbf{D}(K_j)$, using Equation (13). This implies that, indeed, $D \in \uparrow\bigotimes_{j=1}^n \mathbf{D}(K_j)$

Conversely, for (b), consider any D in $\uparrow\bigotimes_{j=1}^n \mathbf{D}(K_j)$. Then $D \supseteq D'$ for some D' in $\bigotimes_{j=1}^n \mathbf{D}(K_j)$, implying that, for every j in $\{1,\dots,n\}$ and E in $\mathcal{P}^+(\mathcal{X}_{\{j\}^c})$, there are $D_{j,E}$ in $\mathbf{D}(K_j)$, such that $D' = \mathrm{posi}(\bigcup\{\mathbb{I}_E D_{j,E} : j \in \{1,\dots,n\}, E \in \mathcal{P}^+(\mathcal{X}_{\{j\}^c})\} \cup \mathcal{L}_{>0})$. Consider any A_j in K_j, for all j in $\{1,\dots,n\}$. Since $D_{j,E}$ belongs to $\mathbf{D}(K_j)$, we find that $A_j \cap D_{j,E} \neq \emptyset$, and hence also $\mathbb{I}_E A_j \cap \mathbb{I}_E D_{j,E} \neq \emptyset$, for every j in $\{1,\dots,n\}$ and E in $\mathcal{P}^+(\mathcal{X}_{\{j\}^c})$. Note that $D \supseteq D' \supseteq \mathbb{I}_E D_{j,E}$, where in the second set inclusion we used that posi is a closure operator which is therefore extensive—meaning that $B \subseteq \mathrm{posi}(B)$ for every $B \subseteq \mathcal{L}$—so we find that $\mathbb{I}_E A_j \cap D \neq \emptyset$, whence $\mathbb{I}_E A_j \in K_D$, for every j in $\{1,\dots,n\}$ and E in $\mathcal{P}^+(\mathcal{X}_{\{j\}^c})$. Since the choices of A_j in K_j for all j in $\{1,\dots,n\}$ were arbitrary, this implies that $\bigcup_{j=1}^n \mathcal{A}_{1:n\setminus\{j\}\to\{j\}} \subseteq K_D$, whence, indeed, $D \in \mathbf{D}(\bigotimes_{j=1}^n K_j)$ using Theorem 6.

Before we start the proof of (iii), let us first establish the following useful property, which we will need in the proofs of (iii) and (iv): For any ℓ in $\{1,\dots,n\}$ and E in $\mathcal{P}^+(\mathcal{X}_{\{\ell\}^c})$,

$$(\forall A \in \mathcal{Q}(\mathcal{X}_\ell))A \in K_\ell \Leftrightarrow \mathbb{I}_E A \in \bigotimes_{j=1}^n K_j. \tag{14}$$

For necessity, $A \in K_\ell$ implies that $A \cap D_\ell \neq \emptyset$ for all D_ℓ in $\mathbf{D}(K_\ell)$. To show that then $\mathbb{I}_E A \in \otimes_{j=1}^n K_j$ we use $\otimes_{j=1}^n K_j$'s representation $\otimes_{j=1}^n \mathbf{D}(K_j)$, established above in (ii), so by Theorem 3 it suffices to show that $\mathbb{I}_E A \cap D \neq \emptyset$ for every D in $\otimes_{j=1}^n \mathbf{D}(K_j)$. To this end, consider any D in $\otimes_{j=1}^n \mathbf{D}(K_j)$. Then by Equation (13), for every j in $\{1,\dots,n\}$ and G in $\mathcal{P}^+(\mathcal{X}_{\{j\}^c})$ there are $D_{j,G}$ in $\mathbf{D}(K_j)$ for which $D = \operatorname{posi}(\bigcup\{\mathbb{I}_G D_{j,G} : j \in \{1,\dots,n\}, G \in \mathcal{P}^+(\mathcal{X}_{\{j\}^c})\} \cup \mathcal{L}_{>0})$. For the choice $j = \ell$ and $G = E$, we know that $D_{\ell,E}$ belongs to $\mathbf{D}(K_\ell)$—whence $A \cap D_{\ell,E} \neq \emptyset$ and therefore $\mathbb{I}_E A \cap \mathbb{I}_E D_{\ell,E} \neq \emptyset$—and also that the set of desirable gambles $\mathbb{I}_E D_{\ell,E}$ belongs to $\{\mathbb{I}_G D_{j,G} : j \in \{1,\dots,n\}, G \in \mathcal{P}^+(\mathcal{X}_{\{j\}^c})\}$ and is therefore a subset of D, since posi is a closure operator, which is in particular monotonic—whence, indeed, $\mathbb{I}_E A \cap D \neq \emptyset$.

For sufficiency, we use $\otimes_{j=1}^n K_j$'s representation $\otimes_{j=1}^n \mathbf{D}(K_j)$, established above in (ii), to infer using Theorem 3 that $\mathbb{I}_E A \in \otimes_{j=1}^n K_j$ implies that $\mathbb{I}_E A \cap D \neq \emptyset$ for every D in $\otimes_{j=1}^n \mathbf{D}(K_j)$, and therefore in particular that $\mathbb{I}_E A \cap \otimes_{j=1}^n D_j \neq \emptyset$ for every D_1 in $\mathbf{D}(K_1)$, …, D_n in $\mathbf{D}(K_n)$, using Equation (13). Consider any D_1 in $\mathbf{D}(K_1)$, …, D_n in $\mathbf{D}(K_n)$, whence $\mathbb{I}_E A \cap \otimes_{j=1}^n D_j \neq \emptyset$, which implies that $\mathbb{I}_E f \in \otimes_{j=1}^n D_j$ for some f in A. But Theorem 12 tells us that $\otimes_{j=1}^n D_j$ is an independent product of D_1, …, D_n, and therefore $\mathbb{I}_E f \in \otimes_{j=1}^n D_j \Leftrightarrow f \in \otimes_{j=1}^n D_j$. Since f belongs to $\mathcal{L}(\mathcal{X}_\ell)$, we have that $f \in \otimes_{j=1}^n D_j \Leftrightarrow f \in \operatorname{marg}_\ell \otimes_{j=1}^n D_j \Leftrightarrow f \in D_\ell$, where in the second equivalence we again used that $\otimes_{j=1}^n D_j$ is an independent product of D_1, …, D_n. This implies that $A \cap D_\ell \neq \emptyset$, and since the choice of D_ℓ in $\mathbf{D}(K_\ell)$ was arbitrary, by Theorem 3 also that, indeed, $A \in K_\ell$.

For (iii), to show that $\operatorname{marg}_\ell(\otimes_{j=1}^n K_j) = K_\ell$ for every ℓ in $\{1,\dots,n\}$, consider any A in $\mathcal{Q}(\mathcal{X}_\ell)$ and we need to show that $A \in K_\ell \Leftrightarrow A \in \otimes_{j=1}^n K_j$. Use Equation (14) with $E = \mathcal{X}_{\{j\}^c}$, and infer that then $\mathbb{I}_{\mathcal{X}_{\{j\}^c}} A = A$, which establishes this equivalence.

For (iv), by Equation (8) we need to show that $\mathbb{I}_E A \in \otimes_{j=1}^n K_j \Leftrightarrow A \in \otimes_{j=1}^n K_j$ for all o in $\{1,\dots,n\}$, A in $\mathcal{Q}(\mathcal{X}_o)$ and E in $\mathcal{P}^+(\mathcal{X}_{\{o\}^c})$. Since A belongs to $\mathcal{Q}(\mathcal{X}_o)$, we have $A \in \otimes_{j=1}^n K_j \Leftrightarrow A \in \operatorname{marg}_o \otimes_{j=1}^n K_j \Leftrightarrow A \in K_o$, where the final equivalence follows from the earlier established marginalization property that $\operatorname{marg}_o \otimes_{j=1}^n K_j = K_o$. So, consider any o in $\{1,\dots,n\}$, A in $\mathcal{Q}(\mathcal{X}_o)$ and E in $\mathcal{P}^+(\mathcal{X}_{\{o\}^c})$, and it suffices to show that $\mathbb{I}_E A \in \otimes_{j=1}^n K_j \Leftrightarrow A \in K_o$. This follows indeed directly from Equation (14) with $\ell = o$.

Finally, for (v), note that the results (i), (iii) and (iv) above imply that $\otimes_{j=1}^n K_j$ is an independent product of K_1, …, K_n. To show that it also is the *smallest* such set of desirable gamble sets, consider any independent product $K^* \subseteq \mathcal{Q}(\mathcal{X}_{1:n})$ of K_1, …, K_n. Since K^* is epistemically many-to-one independent, we have by Equation (8) in particular, for any o in $\{1,\dots,n\}$ and E in $\mathcal{P}^+(\mathcal{X}_{\{o\}^c})$, that

$$\operatorname{marg}_o(K^* \rfloor E) = \operatorname{marg}_o K^* = K_o,$$

where the first equality holds because K^* is epistemically many-to-one independent, and the second one because K^* marginalizes to K_1, …, K_n. This implies that any A in K_o should belong to $K^* \rfloor E$, and hence that $\mathbb{I}_E A \in K^*$. Since this should hold for any o in $\{1,\dots,n\}$,

A in K_o, and E in $\mathcal{P}^+(\mathcal{X}_{\{o\}^c})$, we have that $\bigcup_{j=1}^n \mathcal{A}_{1:n\setminus\{j\}\to\{j\}} \subseteq K^*$. Since K^* is coherent, also $\operatorname{posi}(\bigcup_{j=1}^n \mathcal{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0}) \subseteq K^*$. But this tells us that $\bigotimes_{j=1}^n K_j \subseteq K^*$, and since the choice of epistemically many-to-one independent product K^* was arbitrary, this establishes that $\bigotimes_{j=1}^n K_j$ indeed is the smallest independent product of $K_1, \ldots, K_n$. □

Interestingly, the many-to-one independent natural extension of binary sets of desirable gambles, is binary itself.

Proposition 16. *Consider, for each j in $\{1,\ldots,n\}$, a coherent set of desirable gambles D_j on $\mathcal{X}_j$. Then $K_{\bigotimes_{j=1}^n D_j} = \bigotimes_{j=1}^n K_{D_j}$.*

Proof. We will subsequently show that (i) $K_{\bigotimes_{j=1}^n D_j} \subseteq \bigotimes_{j=1}^n K_{D_j}$ and (ii) $K_{\bigotimes_{j=1}^n D_j} \supseteq \bigotimes_{j=1}^n K_{D_j}$.

For (i), consider any A in $K_{\bigotimes_{j=1}^n D_j}$. Then $A \cap \bigotimes_{j=1}^n D_j \neq \emptyset$, so let $f \in A$ belong to $\bigotimes_{j=1}^n D_j$. Then $f \in \mathcal{L}(\mathcal{X}_{1:n})_{>0}$—in which case $A \in \bigotimes_{j=1}^n K_{D_j}$ by coherence guaranteed by Theorem 15—or $f \geq \sum_{k=1}^m \lambda_k f_k$ for some m in $\mathbb{N}$, $f_1, \ldots, f_m$ in $\bigcup_{j=1}^n A_{1:n\setminus\{j\}\to\{j\}}$ and m real coefficients $\lambda_{1:m} > 0$. But then, for every k in $\{1,\ldots,m\}$, the gamble set $A_k \coloneqq \{f_k\}$ belongs to $\bigcup_{j=1}^n \mathcal{A}_{1:n\setminus\{j\}\to\{j\}}$ with $K_1 \coloneqq K_{D_1}, \ldots, K_n \coloneqq K_{D_n}$, using Equations (11) and (12). Let furthermore $\lambda_{1:m}^{f_{1:m}} \coloneqq \lambda_{1:m} > 0$ for the unique—and hence all—$f_{1:m}$ in $\times_{k=1}^m A_k$. This implies that the gamble set $\{\sum_{k=1}^m \lambda_k f_k\} = \{\sum_{k=1}^m \lambda_k^{f_{1:m}} f_k : f_{1:m} \in \times_{k=1}^m A_k\}$ belongs to $\operatorname{Posi}(\bigcup_{j=1}^n \mathcal{A}_{1:n\setminus\{j\}\to\{j\}})$ and since $f \geq \sum_{k=1}^m \lambda_k f_k$, we find that also $\{f\} \in \operatorname{Posi}(\bigcup_{j=1}^n \mathcal{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0})$, by [11, Lemma 16]. Since $f \in A$, we have that then indeed $A \in \bigotimes_{j=1}^n K_{D_j}$.

For (ii), consider any A in $\bigotimes_{j=1}^n K_{D_j}$. This means that $A \supseteq B \setminus \mathcal{L}(\mathcal{X}_{1:n})_{\leq 0}$ for some B in $\operatorname{Posi}(\bigcup_{j=1}^n \mathcal{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0})$ with $K_1 \coloneqq K_{D_1}, \ldots, K_n \coloneqq K_{D_n}$, meaning that $B = \{\sum_{k=1}^m \lambda_k^{f_{1:m}} f_k : f_{1:m} \in \times_{k=1}^n B_k\}$ for some m in $\mathbb{N}$, $B_1, \ldots, B_m$ in $\bigcup_{j=1}^n \mathcal{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0}$ and, for every $f_{1:m}$ in $\times_{k=1}^m B_k$, m real coefficients $\lambda_{1:m}^{f_{1:m}} > 0$. For any k in $\{1,\ldots,m\}$ we have that B_k belongs to $\mathcal{L}^{\mathrm{s}}(\mathcal{X}_{1:n})_{>0}$—in which case we call $\{g_k\} \coloneqq B_k$—or $B_k = \mathbb{I}_E B'_k$ for some j in $\{1,\ldots,n\}$, E in $\mathcal{P}^+(\mathcal{X}_{\{j\}^c})$ and B'_k in K_{D_j}, meaning that $B'_k \cap D_j \neq \emptyset$—in which case we let h_k belong to $B'_k \cap D_j$ and define $g_k \coloneqq \mathbb{I}_E h_k \in B_k$. Then the gamble $f \coloneqq \sum_{k=1}^m \lambda_k^{g_{1:m}} g_k$ belongs to B, and all of its terms belong to $\mathcal{L}(\mathcal{X}_{1:m})_{>0}$ or to $\bigcup_{j=1}^n A_{1:n\setminus\{j\}\to\{j\}}$. Then, by definition, $f \in \bigotimes_{j=1}^n D_j$, so that B belongs to $K_{\bigotimes_{j=1}^n D_j}$. Furthermore, because $\bigotimes_{j=1}^n D_j$ is coherent (Theorem 12), $f \notin \mathcal{L}_{\leq 0}$, and so $A \in K_{\bigotimes_{j=1}^n D_j}$. □

6.3. **A Stronger Independence Requirement.** In the discussion following Definition 15, we have seen that a set of desirable gamble sets K is epistemically many-to-one independent when Equation (8) holds, or, in other words, when

$$A \in K \Leftrightarrow \mathbb{I}_E A \in K, \tag{15}$$

for all o in $\{1,\ldots,n\}$, E in $\mathcal{P}^+(\mathcal{X}_{\{o\}^c})$ and A in $\mathcal{Q}(\mathcal{X}_{\{o\}})$. This requires that, if A is a desirable gamble set, then at least one gamble of $\{\mathbb{I}_E f : f \in A\} = \mathbb{I}_E A$ should be preferred to zero, and hence $\mathbb{I}_E A \in K$. But one might argue that independence—or indeed, also irrelevance—should require something stronger, namely that

$$E_A \cdot A \coloneqq \{\mathbb{I}_{E_f} f : f \in A\} \tag{16}$$

should belong to K, for every choice of conditioning events $E_A \coloneqq \{E_f : f \in A\} \subseteq \mathcal{P}^+(\mathcal{X}_{\{o\}^c})$. The gamble set $E_A \cdot A$ is the result of multiplying any gamble f in A with its corresponding indicator $\mathbb{I}_{E_f}$ with $E_f \in E_A$, so $E_A \cdot A$ contains the elements of A multiplied with (indicators of) *different events*, rather than then same event E. This leads to the following stronger independence requirement, which, as we shall see, has a tight connection with epistemic independence: We say that K satisfies the stronger notion of many-to-one independence if

$$A \in K \Leftrightarrow E_A \cdot A \in K, \tag{17}$$

for all o in $\{1,\dots,n\}$, E in $\mathcal{P}^+(\mathcal{X}_{\{o\}^c})$ and A in $\mathcal{Q}(\mathcal{X}_{\{o\}})$. This is a valid generalization of many-to-one independence in the sense that, for any epistemically many-to-one independent set of desirable gambles D, the binary K_D satisfies the requirement in Equation (17), too. We find this independence notion at least as compelling as epistemic independence. As E_A may contain only one event E, in which case $E_A \cdot A = \{\mathbb{I}_E f : f \in A\} = \mathbb{I}_E A$, this requirement implies the usual requirement of Equation (15). This does not prevent the independent many-to-one natural extension from satisfying the stronger requirement, but it is an open question whether or not it does.

We will end this section with an example showing that the *completely independent* E-admissible set of desirable gamble sets does satisfy the stronger independence requirement or Equation (17), and even many-to-many independence.

Example 5. Consider for each j in $\{1,\dots,n\}$ a credal set $\mathcal{M}_j \subseteq \mathrm{int}(\Sigma_{\mathcal{X}_j})$,[16] and consider the *completely independent* [4, 26] credal set on $\mathcal{X}_{1:n}$ given by $\mathcal{M} \coloneqq \{\prod_{j=1}^n p_j : p_1 \in \mathcal{M}_1, \dots, p_n \in \mathcal{M}_n\}$. This credal set $\mathcal{M}$ will generally be non-convex.

Let us consider the E-admissible set of desirable gamble sets $K_{\mathcal{M}}$, defined in Example 2. Since $\mathcal{M}$ marginalizes to $\mathcal{M}_1, \dots, \mathcal{M}_n$, the set of desirable gamble sets $K_{\mathcal{M}}$, too, will marginalize to $K_{\mathcal{M}_1}, \dots, K_{\mathcal{M}_n}$. To see this, note by Lemma 5 that $K_{\mathcal{M}}$ is represented by $\{D_p : p \in \mathcal{M}\}$. Since $\mathrm{marg}_\ell D_p = D_{\mathrm{marg}_\ell p} = D_{p_\ell}$,[17] we find using Proposition 11 that $\mathrm{marg}_\ell K$ is represented by $\{D_p : p \in \mathcal{M}_\ell\}$ and hence $\mathrm{marg}_\ell K = \bigcap_{p \in \mathcal{M}_\ell} D_p = K_{\mathcal{M}_\ell}$, for every ℓ in $\{1,\dots,n\}$. Moreover, Example 3 guarantees that $K_{\mathcal{M}}$ satisfies many-to-one independence, and even many-to-many independence. This tells us that $K_{\mathcal{M}}$ is an independent product of $K_{\mathcal{M}_1}, \dots, K_{\mathcal{M}_n}$.

To see that it indeed does satisfy the alternative independence requirement of Equation (17), consider any A in $\mathcal{Q}(\mathcal{X}_O)$ and $G_A \subseteq \mathcal{P}^+(\mathcal{X}_I)$. Infer for any p in $\mathcal{M}$, f in A and G_f in G_A that $E_p(\mathbb{I}_{G_f} f) = E_p(\mathbb{I}_{G_f}) E_p(f)$, so that the following equivalences hold: $A \in K_{\mathcal{M}} \Leftrightarrow (\forall p \in \mathcal{M})(\exists f \in A) E_p(f) > 0 \Leftrightarrow (\forall p \in \mathcal{M})(\exists f \in A) E_p(\mathbb{I}_{G_f} f) > 0 \Leftrightarrow G_A \cdot A \in K_{\mathcal{M}}$, which implies that $K_{\mathcal{M}}$ indeed satisfies Equation (17). Here we used the observation that, if $A \cap \mathcal{L}_{>0} \neq \emptyset$ then $(\forall p \in \mathcal{M})(\exists f \in A) E_p(f) > 0$, which holds because $\mathcal{M} \subseteq \mathrm{int}(\Sigma_{\mathcal{X}})$, similar to what we have done in Example 3 $\lozenge$

[16]We use the (topological) interior $\mathrm{int}(\Sigma_{\mathcal{X}_j})$ of $\Sigma_{\mathcal{X}_j}$ to make sure that every outcome in $\mathcal{X}_j$ has a (strictly) positive probability for every element of $\mathcal{M}_j$.

[17]Here, $\mathrm{marg}_\ell p$ is defined in the usual way, as $\mathrm{marg}_\ell p(x_\ell) \coloneqq \sum_{x_{1:n\setminus\{\ell\}} \in \mathcal{X}_{1:n\setminus\{\ell\}}} p(x_\ell, x_{1:n\setminus\{\ell\}})$ for every x_ℓ in $\mathcal{X}_\ell$. For any gamble f on $\mathcal{X}_\ell$ its $\mathrm{marg}_\ell p$-expectation is $E_{\mathrm{marg}_\ell p}(f) = \sum_{x_\ell \in \mathcal{X}_\ell} \mathrm{marg}_\ell p(x_\ell) f(x_\ell) = \sum_{x_\ell \in \mathcal{X}_\ell} \sum_{x_{1:n\setminus\{\ell\}} \in \mathcal{X}_{1:n\setminus\{\ell\}}} p(x_\ell, x_{1:n\setminus\{\ell\}}) f(x_\ell) = \sum_{x_{1:n} \in \mathcal{X}_{1:n}} p(x_{1:n}) f(x_\ell) = \sum_{x_{1:n} \in \mathcal{X}_{1:n}} p(x_{1:n}) f(x_{1:n}) = E_p(f)$, where we used the simplifying device of identifying f with its cylindrical extension to $\mathcal{X}_{1:n}$ in the penultimate equality. This readily implies that $D_{\mathrm{marg}_\ell p} = \{f \in \mathcal{L}(\mathcal{X}_\ell) : E_{\mathrm{marg}_\ell p}(f) > 0 \text{ or } f > 0\} = \{f \in \mathcal{L}(\mathcal{X}_\ell) : E_p(f) > 0 \text{ or } f > 0\} = \mathrm{marg}_\ell D_p$, which explains the equality.

7. Contrasting Epistemic Irrelevance with S-Irrelevance

We have chosen to investigate the independent natural extension of sets of desirable gamble sets according to the standard that we have called 'epistemic irrelevance', but there are numerous other notions of irrelevance we might have investigated. One particularly interesting conception of irrelevance is a notion due to Teddy Seidenfeld [28, Section 4] and recently investigated by Jasper De Bock and Gert de Cooman [2]. The basic idea is that one proposition is irrelevant to another if the agent doesn't regard learning about the first proposition as valuable to decisions that depend only on whether the second proposition is true [2, Section 4.1]:

> "When two variables, X and Y, are 'independent' then it is not reasonable to spend resources in order to use the observed value of one of them, say X, to choose between options that depend solely on the value of the other variable, Y."

To translate this into a workable definition, consider any partition $\mathcal{P}$ of X's finite possibility space $\mathcal{X}$, and for every element E of $\mathcal{P}$, a gamble f_E on Y's finite possibility space $\mathcal{Y}$. Then the suggested notion of irrelevance, which De Bock and De Cooman [2] term *S-irrelevance*,[18] is that an agent who judges that X is S-irrelevant to Y will be forced to disprefer the composite gamble $\sum_{E\in\mathcal{P}}\mathbb{I}_E(X)f_E(Y)-\varepsilon$, which is the result of paying ε to find out which E^* in $\mathcal{P}$ occurs—the E^* such that $X\in E^*$—in order to decide to take the gamble $f_{E^*}(Y)$, to at least one of $\{f_E:E\in\mathcal{P}\}$. In other words, $\{f_E-\sum_{G\in\mathcal{P}}\mathbb{I}_Gf_G+\varepsilon:E\in\mathcal{P}\}$ is a desirable gamble set. De Bock and De Cooman show that this is equivalent to the following requirement.

Definition 17 ([2, Definition 9])**.** We say that X is S-irrelevant to Y with respect to a coherent set of desirable gamble sets K if $\{\sum_{G\in\mathcal{P}\setminus\{E\}}\mathbb{I}_G(f_E-f_G)+\varepsilon:E\in\mathcal{P}\}\in K$ for all partitions $\mathcal{P}$ of $\mathcal{X}$, $f_E\in\mathcal{L}(\mathcal{Y})$ for all E in $\mathcal{P}$, and $\varepsilon\in\mathbb{R}_{>0}$. X and Y are called S-independent when X is S-irrelevant to Y and *vice versa*.

When X is a binary variable, meaning that $\mathcal{X}=\{x_1,x_2\}$, irrelevance of X to Y reduces to $\{\mathbb{I}_{\{x_1\}}f+\varepsilon,-\mathbb{I}_{\{x_2\}}f+\varepsilon\}\in K$, for all f in $\mathcal{L}(\mathcal{Y})$ and $\varepsilon\in\mathbb{R}_{>0}$.

S-irrelevance is an intuitively very compelling standard, which raises a natural question: how is our concept of epistemic irrelevance related? It is already clear from De Bock and De Cooman [2]'s analysis of S-irrelevance that it is not entailed by our notion of epistemic irrelevance; as they note [2, Theorem 10], under suitable continuity conditions[19] S-irrelevance has the surprising consequence forcing mixingness on the sets of desirable gamble sets, which loosely speaking implies that the set of desirable gamble sets is represented by a collection of linear previsions. Let us give an explicit example, specialized to our context.

Example 6. Consider the independent natural extension $K_1\otimes K_2$ of two vacuous local models K_1 and K_2 on $\mathcal{X}$ and $\mathcal{Y}$, respectively. We will show that this is the vacuous K_{v}

[18] After 'Seidenfeld'.

[19] They are Archimedeanity and "credible indeterminacy", which implies that there is one event $E\subseteq\mathcal{X}$ whose lower probability is strictly positive and upper probability strictly smaller than 1.

on $\mathcal{X}\times\mathcal{Y}$. Indeed, K_{v} marginalizes to K_1 and K_2, and it also satisfies the independence requirements of Equation (15):

$$A \in K_{\mathrm{v}} \Leftrightarrow (\exists f \in A) f > 0 \Leftrightarrow (\exists f \in A)\mathbb{I}_E f > 0 \Leftrightarrow \mathbb{I}_E A \in K_{\mathrm{v}},$$

for any A in $\mathcal{Q}(\mathcal{X})$ and E in $\mathcal{P}^+(\mathcal{Y})$, or A in $\mathcal{Q}(\mathcal{Y})$ and E in $\mathcal{P}^+(\mathcal{X})$. So K_{v} is an independent product. Since it is the smallest coherent set of desirable gamble sets, it is equal to $K_{\mathrm{v}} = K_1 \otimes K_2$.

Let us show that $K_1 \otimes K_2$ does not satisfy S-irrelevance, and therefore also not S-independence. To this end, assume that both $\mathcal{X} = \{x_1, x_2\}$ and $\mathcal{Y} = \{y_1, y_2\}$ are binary. Consider the gamble $f \coloneqq (f_1(x_1), f_1(x_2)) = (1, -1)$, any y in $\mathcal{Y}$, and $\varepsilon \coloneqq \frac{1}{2} > 0$. The gamble set $\{\mathbb{I}_{\{y\}} f + \varepsilon, -\mathbb{I}_{\{y\}} f + \varepsilon\}$ does not contain a non-negative and non-zero gamble, and therefore does not belong to $K_1 \otimes K_2$. This means that $K_1 \otimes K_2$ does not satisfy S-irrelevance from X to Y. $\lozenge$

As far as we know, however, whether S-irrelevance entails our standard of epistemic irrelevance has not yet been shown. In the following section, we develop an example which shows that it is possible to *satisfy* S-irrelevance while *flouting* (both value and subset) epistemic irrelevance. Thus, neither S- nor epistemic irrelevance entails the other.

7.1. **Violating Epistemic Irrelevance While Satisfying S-Irrelevance.** There are two general ways it is possible to violate epistemic subset irrelevance while satisfying S-irrelevance. A variable X fails to be epistemically irrelevant to a variable Y just in case $A \in K$ but $\mathbb{I}_E A \notin K$, or $\mathbb{I}_E A \in K$ but $A \notin K$, for some E in $\mathcal{P}^+(\mathcal{X})$ and A in $\mathcal{Q}(\mathcal{Y})$.

In essence, there is some proposition about the value of X that the agent can learn which will change their views about the preferences for some gambles that depend only on Y. The question we are interested in is whether there is a way for an agent who knows that learning E will change their views about the preferences between these gambles to not place any *real* monetary value on learning it. There are at least two ways that it occurs to us that this could happen:

- the agent thinks that there is no real value gained by using the informed strategy over merely accepting a wager without learning;
- the agent is certain that the experiment they are (not) paying for will not yield the outcome which would change their beliefs.

In this subsection, we develop an example of the former. The framework of sets of desirable gamble sets (and indeed, the less expressive framework of sets of desirable gambles) is capable of representing an agent as believing that one outcome is infinitesimally more likely than another. In the multivariate arena, this raises the possibility of correlations that generate only infinitesimal change in belief. Information that generates such an infinitesimal change will not have any *real* expected value as long as the gambles that are at issue have only finite value, which is consistent with S-irrelevance; Nonetheless, learning the information *does* make an identifiable change to which gambles the agent finds desirable, and thus subset irrelevance is violated.

Example 7. A factory produces two kinds of coins: coins that are fair (heads and tails are equally likely) and coins that are infinitesimally biased in favor of heads (heads is more likely than tails but not by any definite amount). Consider an agent who knows that a coin produced by this factory is about to be flipped; the above description of the factory is all they know. They have no beliefs about the proportion of coins of each type the factory produces, or any specific reason to believe that the coin in question is of one type or the other.

The agent is offered the following decision problem: they can accept a wager that pays some real payout a if the coin lands heads and $-a$ if tails, they can decline the wager (maintain the status quo, accept the zero gamble), or they can pay some small (real-valued) fee $\varepsilon > 0$ to learn the type of the coin, and then decide whether to accept or reject the $(a,-a)$ wager. Let X take values in $\mathcal{X} \coloneqq \{F,U\}$ representing whether the coin is fair or unfair and let Y take values in $\mathcal{Y} \coloneqq \{H,T\}$ representing whether the coin lands heads or tails. For ease of future reference, we denote $D_1 \coloneqq \{f : f \in \mathcal{L}(\mathcal{Y})$ and $f(H)+f(T) > 0\}$ and $D_2 \coloneqq \{f : f \in \mathcal{L}(\mathcal{Y})$ and $(f(H)+f(T) > 0$ or $(f(H)+f(T) = 0$ and $f(H) > f(T)))\}$. Note that D_1 and D_2 are coherent sets of desirable gambles.

We model the agent's beliefs by $\mathbb{I}_{\{F\}}D_1 \cup \mathbb{I}_{\{U\}}D_2$'s natural extension $D \coloneqq \operatorname{posi}(\mathbb{I}_{\{F\}}D_1 \cup \mathbb{I}_{\{U\}}D_2 \cup \mathcal{L}(\mathcal{X}\times\mathcal{Y})_{>0})$.

Lemma 17. *D is a coherent set of desirable gambles. Moreover, it is no independent product, because it fails epistemic irrelevance of X to Y.*

Proof of Lemma 17. To show that D is coherent, it suffices by Theorem 1 to show that $\mathcal{L}_{\leq 0} \cap \operatorname{posi}(\mathbb{I}_{\{F\}}D_1 \cup \mathbb{I}_{\{U\}}D_2) = \emptyset$. To this end, consider any f in $\operatorname{posi}(\mathbb{I}_{\{F\}}D_1 \cup \mathbb{I}_{\{U\}}D_2)$, meaning that $f = \sum_{k=1}^{m} \lambda_k f_k$ for some m in $\mathbb{N}$, real coefficients $\lambda_{1:m} > 0$, and gambles $f_1, \dots, f_m$ in $\mathbb{I}_{\{F\}}D_1 \cup \mathbb{I}_{\{U\}}D_2$. For every k in $\{1,\dots,m\}$, if f_k belongs to $\mathbb{I}_{\{F\}}D_1$ then $f_k(U,H) = f_k(U,T) = 0$ and $f_k(F,H)+f_k(F,T) > 0$, and if f_k belongs to $\mathbb{I}_{\{U\}}D_2$ then $f_k(F,H) = f_k(F,T) = 0$ and $f_k(U,H)+f_k(U,T) > 0$, or $f_k(U,H)+f_k(U,T) = 0$ but then $f_k(U,H) > f_k(U,T)$. This implies that $f(\bullet,H)+f(\bullet,T) \geq 0$, with equality only if all the f_k with non-zero coefficient λ_k belong to $\mathbb{I}_{\{U\}}D_2$, in which case $f(U,H) > f(U,T)$, whence $f \neq 0$. Therefore indeed $f \notin \mathcal{L}_{\leq 0}$.

To show that it is no independent product, let us show that $\operatorname{marg}_Y D \subset \operatorname{marg}_Y(D\rfloor\{U\})$, so that learning that the coin is unfair, results in a bigger Y-marginal than not learning anything at all. More specifically, we will show that $\operatorname{marg}_Y D = D_1$ and $\operatorname{marg}_Y(D\rfloor\{U\}) = D_2$.

To show that $\operatorname{marg}_Y D \subseteq D_1$, consider any f in $\operatorname{marg}_Y D$. Then $f \in \mathcal{L}(\mathcal{Y})$ and $f \in D$, meaning that $f > 0$—in which case $f \in D_1$ by its coherence—or $f \geq \sum_{k=1}^{m} \lambda_k f_k$ for some m in $\mathbb{N}$, real coefficients $\lambda_{1:m} > 0$, and gambles $f_1, \dots, f_m$ in $\mathbb{I}_{\{F\}}D_1 \cup \mathbb{I}_{\{U\}}D_2$. Note that then $f(y) \geq \sum_{k=1}^{m} \lambda_k f_k(x,y)$ for every x in $\mathcal{X}$ and y in $\mathcal{Y}$, and therefore $2f(y) = \sum_{x\in\mathcal{X}} f(y) \geq \sum_{x\in\mathcal{X}} \sum_{k=1}^{m} \lambda_k f_k(x,y)$, whence $f(y) \geq \frac{1}{2}\sum_{k=1}^{m} \lambda_k \sum_{x\in\mathcal{X}} f_k(x,y)$, for every y in $\mathcal{Y}$. Consider any k in $\{1,\dots,m\}$. If f_k belongs to $\mathbb{I}_{\{F\}}D_1$, then $f_k(U,H) = f_k(U,T) = 0$ and $f_k(F,H) + f_k(F,T) > 0$, whence $\frac{1}{2}\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} f_k(x,y) > 0$. On the other hand, if f_k belongs to $\mathbb{I}_{\{U\}}D_2$, then $f_k(U,H)+f_k(U,T) \geq 0$ and $f_k(F,H) = f_k(F,T) = 0$, whence $\frac{1}{2}\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} f_k(x,y) \geq 0$. As a consequence, if there is some $k^\star$ in $\{1,\dots,m\}$ such that $\lambda_{k^\star} > 0$ and $f_{k^\star}$ belongs to

$\mathbb{I}_{\{F\}}D_1$, then

$$\sum_{y\in\mathcal{Y}} f(y) \geq \frac{1}{2}\sum_{k=1}^{m}\lambda_k \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} f_k(x,y) = \underbrace{\frac{1}{2}\lambda_{k^\star}\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} f_{k^\star}(x,y)}_{>0} + \underbrace{\frac{1}{2}\sum_{k=1,k\neq k^\star}^{m}\lambda_k \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} f_k(x,y)}_{\geq 0} > 0,$$

so f belongs to D_1 by its definition. Otherwise, we infer that $\sum_{k=1}^{n}\lambda_k f_k(F,\bullet) = 0$ and therefore also that $f \geq 0$. Since the coherence of D, which we have established above, disallows $f \neq 0$, it follows that $f > 0$ and therefore, that $f \in D_1$.

That also $\mathrm{marg}_Y D \supseteq D_1$ follows once we realize that $D_1 \subseteq D_2$, whence $D \supseteq \mathrm{posi}(\mathbb{I}_{\{F\}}D_1 \cup \mathbb{I}_{\{U\}}D_1 \cup \mathcal{L}(\mathcal{X}\times\mathcal{Y})_{>0}) \supseteq \mathrm{posi}(D_1 \cup \mathcal{L}(\mathcal{X}\times\mathcal{Y})_{>0})$, which is the smallest coherent set of gambles on $\mathcal{X}\times\mathcal{Y}$ that marginalizes to D_1[20].

To show now that conditioning on $\{U\}$ changes the marginal information $\mathrm{marg}_Y(D\rfloor\{U\})$ to D_2, let us show first that $\mathrm{marg}_Y(D\rfloor\{U\}) \subseteq D_2$. To this end, consider any f in $\mathrm{marg}_Y(D\rfloor\{U\})$, then f in $\mathcal{L}(\mathcal{Y})$ and $\mathbb{I}_{\{U\}}f \in D$, meaning that $\mathbb{I}_{\{U\}}f > 0$—in which case $f > 0$ whence $f \in D_2$ by its coherence—or $\mathbb{I}_{\{U\}}f \geq \sum_{k=1}^{m}\lambda_k f_k$ for some m in $\mathbb{N}$, real coefficients $\lambda_{1:m} > 0$, and gambles $f_1, \ldots, f_m$ in $\mathbb{I}_{\{F\}}D_1 \cup \mathbb{I}_{\{U\}}D_2$. Note that not all f_k with non-zero coefficient λ_k can belong to $\mathbb{I}_{\{F\}}D_1$ since that would imply that $0 = \mathbb{I}_{\{U\}}(F)f \geq \sum_{k=1}^{m}\lambda_k f_k(F,\bullet)$, contradicting the coherence of D_1, so there are some f_k with non-zero coefficient λ_k for which $f_k \in \mathbb{I}_{\{U\}}D_2$. This implies that $f = \mathbb{I}_{\{U\}}(U)f \geq \sum_{k=1}^{m}\lambda_k f_k(U,\bullet)$, which indeed belongs to D_2 by D_2's coherence.

To show, conversely, that $\mathrm{marg}_Y(D\rfloor\{U\}) \supseteq D_2$, consider any f in D_2. This implies that $\mathbb{I}_{\{U\}}f \in \mathbb{I}_{\{U\}}D_2 \subseteq D$. By the conditioning rule for sets of desirable gambles

$$D\rfloor\{U\} := \{f \in \mathcal{L}(\{U\}) : \mathbb{I}_{\{U\}}f \in D\},$$

then $f \in D\rfloor\{U\}$, and since f belongs to $\mathcal{L}(\mathcal{Y})$, indeed $f \in \mathrm{marg}_Y(D\rfloor\{U\})$. □

So D—and therefore K_D, which is coherent since D is—fails to satisfy epistemic value irrelevance, and therefore also epistemic subset irrelevance. However, despite this difference in desirability dependent on learning about X, there is no positive value ε that the agent will be willing to pay to learn the bias before deciding whether to accept or reject a gamble on the outcome of the flip, so a gamble on $\mathcal{Y}$. More generally, there are no gambles f and g on $\mathcal{Y}$ such that the agent would pay to learn the bias of the coin before deciding between f and g.

To show this formally, we consider K_D and show that it satisfies S-irrelevance:

Lemma 18. *X is S-irrelevant to Y with respect to K_D.*

Proof of Lemma 18. Since Y is a binary variable, it suffices to check that

$$\{\mathbb{I}_{\{U\}}f + \varepsilon, -\mathbb{I}_{\{F\}}f + \varepsilon\} \in K_D$$

[20] See De Cooman and Miranda [15, Proposition 7]; they term this the "cylindrical extension", but this terminology would subtly conflict with the way that we've defined the cylindrical extension of a set of gambles – see defintion 12 above.

for all f in $\mathcal{L}(\mathcal{Y})$ and $\varepsilon \in \mathbb{R}_{>0}$, as discussed right after Definition 17. Note already that, if $f = 0$, then $\{\mathbb{I}_{\{U\}}f+\varepsilon, -\mathbb{I}_{\{F\}}f+\varepsilon\} = \{\varepsilon\}$, which belongs to K_D by its coherence. So consider any $f \neq 0$ in $\mathcal{L}(\mathcal{Y})$ and $\varepsilon \in \mathbb{R}_{>0}$; we need to show that then $\mathbb{I}_{\{U\}}f+\varepsilon$ or $-\mathbb{I}_{\{F\}}f+\varepsilon$ belongs to D. We will proceed by considering two exhaustive cases: (i) $f \in D_2$ and (ii) $f \notin D_2$.

For (i) $f \in D_2$ implies that $\mathbb{I}_{\{U\}}f \in D$ by D's definition. But then indeed also $\mathbb{I}_{\{U\}}f+\varepsilon \in D$ by D's coherence.

For (ii), $f \notin D_2$ entails that $f(H)+f(T) \leq 0$. So $(-f(H)+\varepsilon)+(-f(T)+\varepsilon) \geq 2\varepsilon > 0$, and therefore $-f+\varepsilon \in D_1$. Using D's definition, this implies that $\mathbb{I}_{\{F\}}(-f+\varepsilon) \in D$. But $-\mathbb{I}_{\{F\}}f+\varepsilon \geq \mathbb{I}_{\{F\}}(-f+\varepsilon)$ whence, by D's coherence, indeed also $-\mathbb{I}_{\{F\}}f+\varepsilon \in D$. □

So we conclude that K_D satisfies S-irrelevance but no epistemic irrelevance. ◊

Thus, we have a case where X is not epistemically irrelevant to Y, but X is S-irrelevant to Y. The upshot is that there are cases where learning about a variable X makes an identifiable change to which gambles, defined only on $\mathcal{Y}$, an agent prefers, but the agent sees this difference as negligible in (real) value.

7.2. **S-irrelevance Without Epistemic Irrelevance: 2nd Case.** In this subsection, we develop the second kind of case, which involves conditioning on events that had prior probability zero.

Example 8. Consider an agent of whom the following two facts are true:

(1) The agent knows that Wensleydale is a variety of cheese.
(2) The agent assigns probability zero to the Moon being made of any kind of cheese; they do, however, entertain it as a logical possibility.

It seems like our agent should be modelled as entertaining an outcome space that includes at least the following possibilities:

- The Moon is not made of cheese ($\neg C$).
- The Moon is made of Wensleydale ($C \wedge W$).
- The Moon is made of some kind of cheese other than Wensleydale ($C \wedge \neg W$).

These possibilities constitute a partition of logical space, $\Omega = \{\neg C, C \wedge W, C \wedge \neg W\}$, although it's certainly a very coarse partition that could easily be refined. But it will suffice for our present purposes. The question of whether the moon is made of cheese or not is represented by the partition $\mathcal{C} = \{\{\neg C\}, \{C \wedge W, C \wedge \neg W\}\}$, the question of whether the moon is made of Wensleydale is $\mathcal{W} = \{\{\neg C, C \wedge \neg W\}, \{C \wedge W\}\}$.

We have explicitly written the content of these propositions to make it clear that the Moon being made of Wensleydale entails that the Moon is made of cheese. This is also reflected in the fact that there is no cell of the partition representing $\neg C \wedge W$; our agent – correctly! – regards this as an impossible event.

Given that W entails C, it seems obvious that it had better turn out that whether W (whether $\{C \wedge W\}$) is not epistemically irrelevant to whether C (viz., $\{C \wedge W, C \wedge \neg W\}$). What might initially be less obvious is that our agent *will* judge W to be S-irrelevant to C. We'll go through the formal argument below, but the intuitive explanation is this: because the agent gives prior probability zero to the moon being made of any kind of cheese (and hence, Wensleydale), they will not see any value in paying to learn whether the Moon is made of Wensleydale or not before making a decision that depends on whether the Moon is made of cheese; although they know their beliefs about whether the Moon is made of cheese would change *completely* if they learned it was made of Wensleydale, they assign probability zero to learning this from the experiment. And so, they can't see the value in paying for an experiment whose outcome they anticipate with probability 1.

The formal model. Initially, the agent is represented by the set of desirable gambles $D \coloneqq \{g \in \mathcal{L}(\Omega) : g(\neg C) > 0\} \cup \mathcal{L}_{>0}(\Omega)$.

Let Σ_Ω be the set of all probability functions defined on Ω (defined analogously to the definition of $\Sigma_\mathcal{X}$ in Example 2). Believing that $p(\neg C) = 1$ determines a unique probability function $p \in \Sigma_\Omega$, namely $p = (p(C \wedge W), p(C \wedge \neg W), p(\neg C)) = (0, 0, 1)$. Suppose our agent, as is standard, wants to accept every gamble that has positive expected value according to their probability function. It's clear that $E_p(g) = g(\neg C)$, so $E_p(g) > 0$ iff $g(\neg C) > 0$, and therefore $\{g \in \mathcal{L}(\Omega) : E_p(g) > 0\} = \{g \in \mathcal{L}(\Omega) : g(\neg C) > 0\}$. But $D = \{g \in \mathcal{L}(\Omega) : g(\neg C) > 0\}$ is not coherent, because it fails to include gambles that are non-negative and non-zero, but only in virtue of $g(C \wedge W)$ or $g(C \wedge \neg W)$. However, $D = \{g \in \mathcal{L}(\Omega) : g(\neg C) > 0\} \cup \mathcal{L}_{>0}(\Omega)$ is coherent (we give a proof below).[21]

First we show that $\mathcal{W}$ is not epistemically irrelevant to $\mathcal{C}$. We can show this as an immediate consequence of proposition 9: show that D is coherent which entails that K_D is coherent (De Bock and De Cooman [12, Proposition 8]), then K_D cannot satisfy epistemic irrelevance of $\mathcal{W}$ to $\mathcal{C}$, because $\{C \wedge W\} \cap \{\neg C\} = \varnothing$.

To see that D is coherent, check the validity of the three axioms D_1–D_3:

- D_2. By construction, $\mathcal{L}(\Omega)_{>0} \subset D$.
- D_3. Consider any f, g in D and real $(\lambda, \mu) > 0$. Then either $\lambda f + \mu g$ is non-negative and non-zero (hence in D), or $(\lambda f(\neg C), \mu g(\neg C)) > 0$, in which case $\lambda f(\neg C) + \mu g(\neg C) > 0$, hence $\lambda f + \mu g$ in D.
- D_1. If $g(\neg C) > 0$ or $g \in \mathcal{L}(\Omega)_{>0}$, then $g \neq 0$.

Now, the more surprising part: we show that $\mathcal{W}$ *is* S-irrelevant to $\mathcal{C}$. Because $\mathcal{W}$ is a binary partition ($\{\{C \wedge W\}, \{\neg C, C \wedge \neg W\}\}$), we can use the simpler version of S-irrelevance: K_D

[21] Is the fact that the standard definition of coherence for sets of desirable gambles requires $D \supseteq \mathcal{L}_{>0}$ even if the agent assigns probability 0 to attaining the positive payouts a problem? We've had several discussions about this, and what we think is: no, as long as the outcomes are *genuinely possible*. Then it still makes sense to think that these expected-value-0, non-negative, non-zero gambles *weakly dominate* the status quo: there is a possible outcome where you win money and no outcome where you lose. We are intending this to be a case where the agent does entertain the Moon being made of cheese as a logical possibility, just one that they would bet at any odds is false.

represents $\mathcal{W}$ as S-irrelevant to $\mathcal{C}$ if and only if

$$(\forall f \in \mathcal{L}(\mathcal{C}), \varepsilon \in \mathbb{R}_{>0})\{\mathbb{I}_{\{C\wedge W\}} f + \varepsilon, -\mathbb{I}_{\{\neg C, C\wedge\neg W\}} f + \varepsilon\} \in K_D.$$

And given that our agent is representable by a set of desirable gambles (merely binary comparisons suffice), this further collapses to: $(\forall f \in \mathcal{L}(\mathcal{C}), \varepsilon \in \mathbb{R}_{>0})(\mathbb{I}_{\{C\wedge W\}} f + \varepsilon \in D \vee -\mathbb{I}_{\{\neg C, C\wedge\neg W\}} f + \varepsilon \in D)$.

Lemma 19. *$\mathcal{W}$ is S-irrelevant to $\mathcal{C}$ with respect to K_D.*

Proof. In what follows, we will use the notation $g = (g(C\wedge W), g(C\wedge\neg W), g(\neg C))$ for a gamble g on Ω.

Any gamble $f \in \mathcal{L}(\mathcal{C})$ has the form $f = (a,a,b)$, with $a,b \in \mathbb{R}$; thus, $\mathbb{I}_{\{C\wedge W\}} f + \varepsilon = (a+\varepsilon, \varepsilon, \varepsilon)$. Note that $\mathbb{I}_{\{C\wedge W\}}(\neg C) f(\neg C) + \varepsilon = \varepsilon > 0$. By construction, D contains any gamble g such that $g(\neg C) > 0$, so $\mathbb{I}_{\{C\wedge W\}} f + \varepsilon \in D$, for any $f \in \mathcal{L}(\mathcal{C})$. □

So here too, we have found a coherent set of desirable gamble sets K_D that does not satisfy epistemic irrelevance of $\mathcal{W}$ to $\mathcal{C}$, but does satisfy S-irrelevance of $\mathcal{W}$ to $\mathcal{C}$. In contrast with Example 7, here the agent does believe that there is potentially a real value to be gained by using the informed strategy over merely accepting a wager without learning. ◊

8. DISCUSSION

Independence is an interesting concept. When we model uncertainty with precise probabilities, it seems univocal. But when we model uncertainty with imprecise probabilities, it fractures into a multiplicity of distinct concepts, including:

- complete independence for sets of probabilities [4, 26];
- independence in selection for lower previsions [14];
- strong independence for lower previsions and sets of desirable gambles [15];
- epistemic independence (value and subset) for sets of desirable gambles [23];
- epistemic h-independence for lower previsions and credal sets [7];
- S-independence for choice functions [2];

These concepts collapse in the limit, when applied to precise probabilities, but come apart in general.

Independence is also an *important* concept. For example, many have thought that when pooling expert opinions we ought to preserve unanimous judgments of independence [18, 20, 21]. Take another example: causal modelling. Causal Bayesian networks consist of a directed acyclic graph together with an appropriate probability distribution. They are popular formal tools for modelling causal relationships. Independence judgments play a key role in constructing causal Bayes nets. Missing edges between variables in the graph of a causal Bayes net indicate that those variables are causally independent of one another.

In this paper we investigated epistemic independence in the general framework of sets of desirable gamble sets. We developed a very general notion of epistemic independence that subsumes the standard notion for uncertain variables in a multivariate setting [7]. We then characterized the independent natural extension and showed that it can be represented by a collection of sets of desirable gambles that can be obtained based on the local representations.

In addition, we took some initial steps to compare epistemic independence with another attractive notion of independence proposed by Teddy Seidenfeld: S-independence. Recently, Jasper De Bock and Gert de Cooman [2, Corollary 1] showed that if an Archimedean set of desirable gamble sets renders a variable X "credibly indeterminate", then judging that X is S-irrelevant to Y forces you to choose between gambles on Y using E-admissibility. Judgments of S-irrelevance, it turns out, are much more informative than they appear at first glance. An interesting next question is a characterisation of the conditions under which S-irrelevance implies epistemic irrelevance. For instance, [2, Theorem 10] implies that, for finite possibility spaces, Archimedeanity and credible indeterminacy, S-irrelevance implies complete independence, which by Example 5 also satisfies epistemic irrelevance.

There are still a number of open questions about epistemic independence for sets of desirable gamble sets. For example, Cozman and Seidenfeld [6] explore the notion of *layer independence* for full conditional probability measures. Cozman [5] shows that the only extant concept of independence for (non-convex) sets of probabilities that has a range of graphoid properties is element-wise layer independence. Whether or not these graphoids properties are all desirable in an imprecise-probabilistic context is questionable; see [7, Section 6.5.3] and [8, Section 8]. It is an open question what additional structural constraints on coherent sets of desirable gamble sets are necessary and sufficient to secure the relevant graphoid properties.

A main open problem is to find an expression for the many-to-many independent natural extension, which is arguably a more useful concept than the many-to-one independent natural extension we have established here. We suspect that looking at sets of desirable gamble sets with infinite gamble sets will be useful, because in such a context Proposition 4 becomes an equivalence. In fact, in retrospect we are now convinced that such sets of "possibly infinite" desirable gamble sets are more promising. One saving grace for the current notion, is that the largest representation $\mathbf{D}(K)$ of a coherent set of desirable gamble sets K contains its minimal elements $\min\mathbf{D}(K)$ in the poset $(\mathbf{D}(K),\subseteq)$, and these minimal elements also represent K. We have a proof for this, but the result that hinged on this was removed due to an insightful comment by one of the reviewers. Since this now no longer has a direct use in this paper, we have decided to omit this result.

As sets of desirable gamble sets generalize many of the extant imprecise-probabilistic uncertainty models, including sets of desirable gambles, lower previsions, and sets of probability mass functions, they may be expressive enough to unify some of the aforementioned independence concepts. We intend to investigate these connections, with the hope to obtain a unifying theory.

ACKNOWLEDGEMENTS

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under grant agreement No 852677. We would also like to thank two anonymous reviewers for their extremely helpful feedback and extraordinary effort for spotting many inconsistencies, some of which turned out to be crucial. We would like to thank Catrin Campbell-Moore, Arne Decadt, Gert de Cooman, Enrique Miranda, Teddy Seidenfeld and, especially, Jasper De Bock for helpful discussion.

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